arXiv:1702.07135 [hep-th]

Integrality of Framing and Geometric Origin of 2-functions (with algebraic coefficients)

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Abstract

We say that a formal power series $\sum a_n z^n$ with rational coefficients is a 2-function if the numerator of the fraction $a_{n/p} - p^2 a_n$ is divisible by p^2 for every prime number p. One can prove that 2-functions with rational coefficients appear as building block of BPS generating functions in topological string theory. Using the Frobenius map we define 2-functions with coefficients in algebraic number fields. We establish two results pertaining to these functions. First, we show that the class of 2-functions is closed under the so-called framing operation (related to compositional inverse of power series). Second, we show that 2-functions arise naturally in geometry as q-expansion of the truncated normal function associated with an algebraic cycle extending a degenerating family of Calabi-Yau 3-folds.

February 2017

Contents

1 Introductions

[1] The classical "mirror principle" as developed in the early 1990's, states that the Gromov-Witten theory of a Calabi-Yau threefold X can be encoded in Hodge theoretic data of a mirror manifold $\mathcal{Y} \to B$, which is a family of Calabi-Yau threefolds, expanded around a maximal degeneration point $0 \in \overline{B} \setminus B$. The physicist's intuition behind this statement is the equivalence of the effective physical theories obtained by compactifying string theory on the two different manifolds.

Beginning in the late 1990's, developments based on other physical dualities (involving M-theory) have shown that Gromov-Witten theory can be rewritten in terms of mathematical invariants enumerating stable objects in D-brane categories that can be attached to either manifold. These invariants capture the physical notion of "degeneracy of BPS states" in the effective theory.

An important feature of this reformulation is that while Gromov-Witten invariants are a priori rational numbers, they can in fact be expressed as linear combination (with fixed denominator) of integers, which moreover have the interpretation as (graded) dimensions of vector spaces (the physical Hilbert space of BPS states). One might say, the invariants are automatically "categorified".

On the Gromov-Witten side (the A-model of mirror symmetry), many of these reformulations have been elevated to mathematical theorems in the recent years. From the point of view of the mirror manifold (the B-model), the integrality underlying Gromov-Witten theory is a rather non-trivial property of the Hodge theoretic expansion around the maximal degeneration point.

Before most of the A-model proofs were available, it had been shown in [1, 2, 3] that integrality can be established independently in the B-model by passing through the world of p-adic Hodge theory. The basic idea is to show that, for any given prime number p, the reformulated invariants (which are a priori rational numbers) have denominators not divisible by p. In other words, one establishes certain mod$p$ congruences between the expansion coefficients of the periods. If these congruences hold for all primes p, then the reformulated invariants themselves have to be integral.

The relevance of p-adic methods is quite intriguing as it connects the physics of Calabi-Yau manifolds to a number of interesting topics in arithmetic geometry and number theory. On the one hand, the method is naively rather unnatural from the

[1]The main results of this paper are stated in subsections 2.1 (Theorem 8) and 5.4 (Theorem 22).

physical point of view. (The idea that our finite experience of the physical world can be accounted for in integers is old and well-known, but prime numbers do not normally play a role in it.) On the other hand, it is not immediately clear how the number theoretic methods mesh with "categorification", what the underlying integers are counting in the B-model, and whether they are naturally dimensions of some vector spaces. Filling these gaps in the current understanding clearly is an opportunity to bridge between the two subjects, supersymmetric quantum theory, and number theory.

In another recent development [5], it was pointed out that a certain class of extensions of the Hodge theoretic situation, that is very natural from both the physical and mathematical point of view, generically leads to expansion coefficients that are no longer rational, but instead take values in an algebraic number field, fixed for each such situation. This raises the intriguing question whether it is possible to interpret such irrational invariants as "enumerative" in a generalized sense or whether some other assumption has broken down. To us, the categorical equivalence (which, at least for the quintic, has now been proven [6]) and the extensive experience in many other situations (most closely related to ours are [9, 7, 10]) suggest that the mirror principle is of very general validity. Therefore, we believe that a suitably applied Gromov-Witten theory should explain or otherwise accomodate the irrationality of the invariants. It is clear that the relevant A-model situation involves the enumerative geometry of generic objects of the Fukaya category, but the details are unknown.[2]

Perhaps the strongest evidence that such an explanation should exist is the fact that the expansion displays an integrality that is a generalization of that underlying the rational B-model (and proven by the p-adic methods in [1, 2, 3, 4]) to the situation with algebraic expansion coefficients. An important feature of the general setup is that (when the Galois group is non-abelian), one needs to invoke p-adic considerations to even *formulate* the statement of integrality (and of course, also in the proof, see below). We then see two possibilities for relaxing the tension with enumerative geometry. Either the physics (or A-model) explanation does depend on the notion of a prime number as well, or it knows implicitly how to eliminate (or "integrate out") p in a way that is so

[2]Some speculations were offered in [5], and in various talks given by the third-named author. See also section 1.1. An interesting possibility, advocated by C. Vafa, is that we are not working around the mirror of a fully classical regime, and that rationality in the B-model will be restored by a further degeneration. We expect that a combination of HMS and SYZ will eventually shed light on this mystery.

far unknown to mathematicians. Either resolution would be very interesting.

This paper is a result of combining the p-adic proofs of integrality of instanton numbers [1, 2, 3] and of integrality of the number of holomorphic disks [4] with the recent observations [5] about the irrationality of the Hodge theoretic expansion in the generic extended situation. Namely, we will prove the integrality statement of [5]. We hope that eventually these results and the method of proof will help to clarify the A-model interpretation of the irrationality (as well as the integrality), and perhaps point to a deeper physical and mathematical message. At a preliminary stage, we were led to introduce and study, independently of the geometric context, a certain class of power series that we dub "2-functions" (where, more generally, 2 could also be replaced by some other positive integer s). In particular, we show that the class of 2-functions (but not general s-functions) is closed under the framing operation known from local open string mirror symmetry [7] (where it is mirror to the framing of knots in 3-manifolds, hence the name). This part is a generalization of the previous paper [8] to the situation with arbitrary algebraic coefficients. (In fact, the proof immediately generalizes to a completely abstract situation, for which however we have no use at the moment.)

Thus, the paper is naturally divided in two parts which are logically independent from each other. The main results are stated in section 2 (integrality of framing with algebraic coefficients) and in section 5 (geometric origin of 2-functions). In the rest of this somewhat lengthy introductory section, we offer some mathematical and physical motivation which we expect to provide a part of the bigger picture.[3]

1.1 Motivation for Physicists

In this subsection, we give a quick review of a few basic notions from algebraic number theory, and explain some reasons we think they might play a role in physics.

To begin with, we recall that an *algebraic number*, x, is simply a root of a (non-constant) polynomial with integer coefficients. In other words $P(x) = 0$ where $P = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ with $a_k \in \mathbb{Z}$, $a_n \neq 0$. The field of all algebraic numbers is denoted $\overline{\mathbb{Q}}$. Given $x \in \overline{\mathbb{Q}}$, the polynomial P of smallest degree of which x is a root (which is unique if we require the coefficients to be co-prime) is known as the minimal polynomial of x. By adjoining x to $\mathbb{Q}$, we obtain an *algebraic number field* (a finite extension of the field $\mathbb{Q}$ of rational numbers), $K = \mathbb{Q}(x) = \mathbb{Q}[x]/P$.

[3] This material, useful for exposition to the mixed readership, is very elementary, but only partly self-contained. It is hardly necessary for an understanding of the technical content of the paper.

Physicists might be used to thinking of algebraic numbers simply as complex numbers. The more abstract definition however does not specify which of the n roots of P to call x, and nothing in the algebra depends on this choice (if P is irreducible). In physics language, one might say that picking one of the roots (to "embed" K into $\mathbb{C}$) amounts to *breaking the symmetries* of the problem.

More formally, given an algebraic number field K generated by an algebraic number x, which we think of as any one of the roots of polynomial P, it is of interest to ask whether K contains any other roots of P. If K contains all roots of P, then K is said to be Galois over $\mathbb{Q}$. This is equivalent to the statement that if we denote by $\mathrm{Gal}(K/\mathbb{Q})$ the (Galois) group of automorphisms of K that leave $\mathbb{Q}$ invariant then $\mathbb{Q}$ is the fixed field of $\mathrm{Gal}(K/\mathbb{Q})$. If $K = \mathbb{Q}(x)$ is not Galois, we may Galois-close the field by adjoining all the other roots of the minimal polynomial. The resulting field, known as the splitting field of P, is generically of higher degree.

Thinking of all roots of P on equal footing respects the Galois symmetries. A fundamental observation is that the more generic the polynomial, the larger the Galois group of its splitting field.

For example, if $x^2 + 3 = 0$, $\mathbb{Q}(x) = \mathbb{Q}(\sqrt{-3}) \cong \mathbb{Q}(e^{2\pi i/3})$ with Galois group $\mathbb{Z}/2$ generated by $\sqrt{-3} \mapsto -\sqrt{-3}$. For a different example, $\mathbb{Q}(5^{1/3})$ is *not* a Galois extension. This is because the other two roots of the minimal polynomial $x^3 - 5$, which are of course $e^{2\pi i/3}5^{1/3}$ and $e^{4\pi i/3}5^{1/3}$ cannot be expressed algebraically in terms of $5^{1/3}$. This is resolved by adjoining $\sqrt{-3}$, and so we learn that the Galois closure is $\mathbb{Q}(5^{1/3}, \sqrt{-3})$, with Galois group S_3. This means simply that the three roots are algebraically on equal footing, and is the generic situation with a cubic polynomial. An example of a cubic extension that is Galois is provided by $x^3 + x^2 - 2x - 1$. In that case, the other two roots can be written in terms of x alone, as $x^2 - 2$, and $1 - x - x^2$. These algebraic relations between the roots break the Galois group from S_3 down to $\mathbb{Z}/3$.

We record two more elementary definitions. First, among all algebraic numbers, those whose minimal polynomial P has leading coefficient $a_n = 1$ are known as *algebraic integers*. They play a similar role in K as the ordinary integers $\mathbb{Z}$ play in $\mathbb{Q}$. In particular, the algebraic integers form a ring, which we denote by $\mathcal{O}_K$ or simply $\mathcal{O}$ if K is clear from the context. Second, the *discriminant* of the extension $D(K/\mathbb{Q}) \in \mathbb{Z}$, is a (rational) integer which gives a measure of the size of $\mathcal{O}$ relative to $\mathbb{Z}$. We won't define it precisely here but note that it divides the discriminant of the minimal polynomial P of an integral generator x (and sometimes $D(K/\mathbb{Q})$ is equal to the discriminant of P).

Now let us ask: How might any of this be relevant to (mathematical) physics? It is a familiar fact that supersymmetry constrains configuration spaces of supersymmetric field and string theories to be complex (Kahler, super-) manifolds. This is true for both the space of continuous off-shell fields, as well as the on-shell spaces of supersymmetric vacua. It is equally familiar that many physical questions about these theories can be answered by viewing the relevant spaces more abstractly as algebraic varieties, and using methods from algebraic geometry.

The results of [5] and of the present paper, however, reveal that to describe the kinematics (and some dynamics) of certain situations involving (close to) minimal supersymmetry, it is essential to understand the field of definition of the underlying spaces, and to separate the algebraic properties from the complex analytic ones. Our physical interpretation is that in these situations, the breaking of supersymmetry should generally be thought of as an "extension of algebraic structure", and that the minimal amount of structure in the vacuum is the field of definition (or more precisely, the "semi-classical residue field"). The Galois group then quite literally acts on the vacua, as well as (more conjecturally) on the space of physical states.

To explain this in more detail, we recall that in supersymmetric field theories with 4 supercharges (corresponding to $\mathcal{N} = 1$ in 4 dimensions), the dynamics of chiral fields Φ (whose lowest component is a complex scalar field) are governed by two types of terms in the supersymmetric Lagrangian: The Kahler potential $\mathcal{K}(\Phi, \bar{\Phi})$ that determines the kinetic terms in the bosonic Lagrangian, and the superpotential $\mathcal{W}(\Phi)$ that determines the potential terms. While the Kahler potential is quite flexible, the superpotential has to be holomorphic (as well as being constrained by any global and local symmetries that might be present). Therefore, if our goal is to connect the physics of $\mathcal{N} = 1$ supersymmetric field theories with algebra and algebraic geometry (say we want to elucidate the physical content of an $\mathcal{N} = 1$ supersymmetric compactification of string theory on an algebraic variety), it is natural to focus on the superpotential as one of the exactly calculable quantities.

But how much invariant physical information is really contained in the superpotential alone, even assuming we could calculate it exactly? Clearly, we should be looking at supersymmetric vacua, in other words, expand around a critical point of the superpotential. However, even in supersymmetric vacua, statements about physical masses and about Yukawa and higher order interactions depend on the proper normalization of the kinetic terms, hence the Kahler potential. The simplest quantity that does not depend on $\mathcal{K}$ is the constant term in the expansion, in other words, the critical value of the superpotential,

$$\mathrm{Crit}(\mathcal{W}) = \{\mathcal{W}|_{\partial\mathcal{W}=0}\} \tag{1.1}$$

More precisely, since (in the absence of gravity) $\mathcal{W}$ is defined only up to an additive constant, the truly invariant quantities are the *differences of the critical values.* These differences are known, by one of the most elementary BPS bounds, to give the tension (or masses, in 2 space-time dimensions) of supersymmetric domain walls (BPS solitons) interpolating between the various supersymmetric vacua. If $\Phi^{(i)}$ and $\Phi^{(j)}$ are two critical points of $\mathcal{W}$, with critical values $\mathcal{W}^{(i)}$ and $\mathcal{W}^{(j)}$, co-dimension one BPS defects interpolating between $\Phi^{(i)}$ and $\Phi^{(j)}$ have tension

$$m_{ij} = |\mathcal{T}_{ij}| := |\mathcal{W}^{(j)} - \mathcal{W}^{(i)}|\,, \tag{1.2}$$

while $\alpha_{ij} := \arg(\mathcal{T}_{ij})$ measures the linear combination of supersymmetries preserved by the defect (assuming that $\mathcal{T}_{ij} \neq 0$). As a secondary quantity, it is of interest to consider the *degeneracy* of such BPS defects, which is the dimension of the corresponding Hilbert space $\mathcal{H}_{ij}^{\mathrm{BPS}}$.

To connect this with algebra and field extensions, let us *assume* that for some a priori reasons, *the superpotential is constrained to be polynomial with integer coefficients.* This will likely sound like a strong assumption, and we have no control over the class of situations in which it holds. What matters for us in the end is that the assumption seems to be satisfied in the examples coming from D-branes on Calabi-Yau manifolds (see section 5 or ref. [5, 8]). Temporarily, one can think of a superpotential that is generated by instantons (counted by integer coefficients), of which only a finite number are relevant for finding the critical points (so that it is polynomial). In a more general version, we like to think that the underlying integral structure comes from a bulk theory with extended supersymmetry into which our 4-supercharge theory is embedded.

In any event, if $\mathcal{W} \in \mathbb{Z}[\Phi]$, it is easy to see that the critical values (1.1) will be algebraic numbers, *i.e.*, they will be roots of some (other!) polynomial P with integral coefficients. We emphasize that although P is of course determined by $\mathcal{W}$, the two polynomials are conceptually and algebraically distinct. For instance, it is not immediately clear whether any P can appear as we vary $\mathcal{W}$, *i.e.*, whether any algebraic number can be obtained as the critical value of a polynomial with rational coefficients.

In thinking about this situation, one is naturally led to wonder whether the Galois symmetries of (the splitting field of) P have any physical import.[4] At first sight, the appearance of the absolute value (the Archimedean norm) in (1.2) looks like evidence that the physically relevant geometry is just that of the complex plane. But again, if we accept that we only want to look at the *algebraic properties*, we ought to not separate $\mathcal{T}_{ij}$ into m_{ij} and α_{ij}, and the vacua of the theory are indeed related by the Galois symmetry of the polynomial P. We propose that this symmetry carries interesting physical information about the theory. More specifically, we expect that $\mathrm{Gal}(K/\mathbb{Q})$ will act on the space of BPS states, $\oplus_{i,j}\mathcal{H}_{ij}^{\mathrm{BPS}}$.

The present paper constitutes some indirect evidence for this proposal. One of the reasons that we are not able to state a more precise conjecture is that the formulation of our results involves one more ingredient for which we presently have no physical interpretation at all: This ingredient is the notion of a prime number p. Whether such primes admit a physical interpretation, or whether it is possible to eliminate the primes from the mathematical formulation, remains to be seen. Either outcome would be very interesting.

To conclude this subsection, we recall why prime numbers are useful for elucidating the structure of algebraic number fields. The main idea (which has no physical counterpart) is to treat a prime p as a "small parameter", and to study $\mathrm{Gal}(K/\mathbb{Q})$ p-adically, *i.e.*, in an expansion in this small parameter.

We recall Fermat's Little Theorem, which states that if p is prime, and $a \in \mathbb{Z}$ any integer, then

$$a^p \equiv a \bmod p \tag{1.3}$$

As a consequence, if $P = a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ and $P(x) = 0$, then,

$$P(x^p) = \sum a_k x^{kp} \equiv \sum (a_k x^k)^p \equiv \left(\sum a_k x^k\right)^p \equiv 0 \bmod p \tag{1.4}$$

Thus, given a prime p, we can obtain a "first approximation" to another root of P by simply raising x to the p-th power. This *Frobenius operation* is of finite order mod$p$ and can be used to identify certain interesting subgroups of the Galois group. We defer precise definitions to section 2, and here only point out the important dichotomy that arises between abelian and non-abelian Galois group. In the former case, the mod p Frobenius element of $\mathrm{Gal}(K/\mathbb{Q})$ is canonically determined and the Frobenius elements

[4] We are aware that similar ideas have been formulated in [12, 13].

for different primes all commute with one another. In the latter, non-abelian case, the Frobenius element only defines a conjugacy class in $\mathrm{Gal}(K/\mathbb{Q})$, which moreover varies in a poorly controlled way with p.

To give some examples for practice, in the field $\mathbb{Q}(\sqrt{-3})$ (of Galois group $\mathbb{Z}/2$), one finds that for $p > 3$, $\sqrt{-3}^p \equiv \sqrt{-3} \bmod p$ if $p \equiv 1 \bmod 3$, while $\sqrt{-3}^p \equiv -\sqrt{-3} \bmod p$ if $p \equiv 2 \bmod 3$. This regularity is a consequence of Gauss' quadratic reciprocity, and the Frobenius elements are the trivial or non-trivial element of the Galois group, respectively. On the other hand, let us consider $\mathbb{Q}(5^{1/3})$, which is not a Galois extension. One might well show that for $5 < p \equiv 1 \bmod 3$, $(5^{1/3})^{p-1}$ is always a cube root of unity $\bmod p$, but it is not possible to predict whether it will be a non-trivial cube root or not. For instance $(5^{1/3})^{7-1} = 4 \bmod 7$ and $4^3 = 1 \bmod 7$ (the Frobenius element generates a $\mathbb{Z}/3$ subgroup of S_3), while $(5^{1/3})^{13-1} = 1 \bmod 13$ (and the Frobenius is trivial). The Frobenius elements at primes p with $p \equiv 2 \bmod 3$ generate the odd permutations in S_3.

Beginning in the next subsection, we will inquire about ways to "go to next order in p", *i.e.*, to find roots of $P \bmod p^2$. It will be seen that this is easy to do as long as we fix p, but that the non-commutativity of the Frobenius elements presents a obstacle for eliminating p from the mathematical formalism.

1.2 Motivation for Mathematicians

The main character of this paper are what we call 2-functions, certain (formal) power series with properties given in section 2. The background in physics and mirror symmetry is explained elsewhere. In this subsection, we explain in an informal way what these definitions can achieve for mathematics.

As above, we let $x \in \mathcal{O}_{\overline{\mathbb{Q}}}$ be an algebraic integer, and $K = \mathbb{Q}(x)$ be the number field generated by x. We denote by $P \in \mathbb{Z}[x]$ the minimal polynomial of x. More generally, for $y \in \mathcal{O}_K$, the ring of integers in K, we'll let $P_y \in \mathbb{Z}[y]$ be the minimal polynomial of y.

Fixing an embedding $K \hookrightarrow \mathbb{C}$, we can think of x as a complex number. Let us consider, for z in a neighborhood of $0 \in \mathbb{C}$, the Mercator series

$$-\log(1 - xz) = \sum_{k=1}^{\infty} \frac{\rho_k(x)}{k} z^k \tag{1.5}$$

where $\rho_k(y) := y^k$. The expansion of course converges in a neigborhood of 0 (depending on the chosen embedding $K \hookrightarrow \mathbb{C}$), and the function can be analytically continued throughout some slit complex plane. The coefficients $\rho_k(x)$ possess the following properties:[5]

(i) For $k = p$ prime, $P_x(\rho_p(x)) = 0 \bmod p$.

(ii) When $k = p^r$ is a prime power, we have $P_{\rho_{p^{r-1}}(x)}(\rho_{p^r}(x)) = 0 \bmod p^r$.

(iii) $\rho_{k_1 k_2}(x) = \rho_{k_1}(\rho_{k_2}(x)) = \rho_{k_2}(\rho_{k_1}(x))$.

To be sure: (i) follows from Fermat's little theorem eq. (1.3), see eq. (1.4). Similarly, (ii) (of which (i) is a special case) follows from Euler's generalization of Fermat's theorem: If $a = b \bmod p^{r-1}$, then $a^p = b^p \bmod p^r$. And while the multiplicativity (iii) is of course trivial, we list it here because of the generalizations below. More precisely, the property we generalize is the following somewhat less trivial-looking corollary,

(iii') If $k = p^r k'$ with k' not divisible by p, then

$$P_{\rho_{k/p}(x)}(\rho_k(x)) = 0 \bmod p^r$$

We can qualitatively summarize these properties by saying that, as we vary p, the power series (1.5) bundles together information about $\bmod p$ arithmetic in the number field $K = \mathbb{Q}(x)$. The results about 2-functions that we obtain in the present paper make the following question seem like a possible starting point to motivate their study:

> Is it possible to "integrate" (1.5) in such a way that the properties of ρ_k are lifted modulo higher powers of k?

We illustrate what we mean in the first non-trivial instance, which is an improvement of the above conditions from holding mod$p$ to modp^2: Given $x \in \mathcal{O}_K$, we are looking for a collection of coefficients

$$\sigma_k(x) \in \mathcal{O}_K \tag{1.6}$$

such that

(i)$_2$ For $k = p$ (unramified) prime, $\sigma_p(x) = x^p \bmod p$, and

$$P_x(\sigma_p(x)) = 0 \bmod p^2 \tag{1.7}$$

(ii)$_2$ For $k = p^r$ a prime power, $\sigma_{p^r}(x) = (\sigma_{p^{r-1}}(x))^p \bmod p$, and

$$P_{\sigma_{p^{r-1}}(x)}(\sigma_{p^r}(x)) = 0 \bmod p^{2r} \tag{1.8}$$

[5] In all statements below, we shall assume that k is co-prime with the discriminant $D(K/\mathbb{Q})$.

(iii)_2 For general k (co-prime with discriminant of $K/\mathbb{Q}$), and any $p|k$, we have $\sigma_k(x) = (\sigma_{k/p}(x))^p \bmod p$, and letting $e_p = \mathrm{ord}_p(k)$ be the largest power of p dividing k,

$$P_{\sigma_{k/p}(x)}(\sigma_k(x)) = 0 \bmod p^{2e_p} \tag{1.9}$$

We remark that (iii)_2 is the natural lift of (iii) in the sense that the ρ_k satisfy its analogue modp^{e_p} (see (iii')), but the σ_k (as maps $\mathcal{O}_K \to \mathcal{O}_K$) cannot be strictly multiplicative in general. In this formulation, of course (i)_2 and (ii)_2 are just special cases of (iii)_2.

Given such a collection of $\sigma_k(x)$, we would like to combine them into a generating series—*It is such series that we will identify as 2-functions below*—

$$L_D(x; z) = \sum_{k=1}^{\infty} \frac{\sigma_k(x)}{k^2} z^k \tag{1.10}$$

and study possible analytic properties of $L_D(x; z)$ as a function of z.

It is in fact not hard to find $\sigma_k(x)$ that satisfy these conditions, and these solutions can also be lifted modulo higher powers of p. The idea is the following: If $k = p$ is prime, and $P'(x^p) \not\equiv 0 \bmod p$, we may use Newton's formula and define

$$\sigma_p(x) = x^p - \frac{P(x^p)}{P'(x^p)} \tag{1.11}$$

which satisfies (1.7) after expansion in p. Moreover, iteration of (1.11) leads to higher-order solutions. (Of course, this solution is not unique. Also note that it is necessary in general that p be unramified for this formula to make sense. A more conceptual explanation is subsumed in the technical part of the paper.) For general k, we may define $\sigma_k(x)$ recursively by similar formulas, assuming $\sigma_{k/p}(x)$ has been defined for all p dividing k.

The crux however, is that this solution is far from unique (any modification of (1.11) by a multiple of p^2 is allowed), and it is far from obvious that the generating function (1.10) will be anything but a formal power series. Therefore, a more meaningful question is whether there is a choice of the $\sigma_k(x)$ such that $L_D(x; z)$ will have some nice analytic properties.

One extreme case is when $x \in \mathbb{Z}$, for we may then simply take $\sigma_k(x) = x$ for all k! Then $L_D(x; z) = x \cdot \mathrm{Li}_2(z)$, where

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{1}{k^2} z^k \tag{1.12}$$

is the series defining the ordinary di-logarithm.

Another simple case is when $x = \zeta$ is a root of unity, where we may take $L_D(\zeta; z) = \mathrm{Li}_2(\zeta z)$. Given this, the Kronecker-Weber theorem will provide a natural solution to our problem for any x such that $K = \mathbb{Q}(x)$ has abelian Galois group over $\mathbb{Q}$ (see section 2.3). As a mathematical problem, the question then becomes non-trivial when $\mathrm{Gal}(K/\mathbb{Q})$ is *non-abelian*.

Not surprisingly, the difficulties with finding a natural simple solution in the general case (see section 2.2), can be traced back to the fact that there is no natural global lift of the Frobenius endomorphism at each prime of K, and that moreover, these Frobenius endomorphisms to not commute amongst each other.

Without the physics, of course, mathematics knows how to circumvent these difficulties. If our goal is to form an analytic function that encodes the global behaviour (over all primes) of the Galois group, we may pick a finite-dimensional representation $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{End}(V)$, for some complex vector space V, and consider the associated (Artin) L-function $L(s; \rho)$, which is built out of characteristic polynomials of the representation. These L-functions are of course much studied.

What the attachment of strings suggests is that, certainly up to $s = 2$, there *exists a different way* of producing an interesting analytic function involving similar data. According to the ideas of section 1.1, the physical setup will involve a vector space acted upon by the Galois group (this might not quite be a representation, but should be closely related). Moreover, to the extent that the physical setup has a geometric origin[6], it will produce a 2-function with the requisite properties.

Our main evidence for this claim is simply that the geometric setup (see section 5) in some sense *already provides a solution* to the above problem. Indeed, we will prove in section 7 that certain Hodge theoretic invariants associated to algebraic cycles on Calabi-Yau 3-folds, when expanded in the appropriate coordinates, satisfy congruence relations of exactly the above type. Moreover, these functions by construction have sensible analytic properties. Therefore, the above conditions are not impossible to satisfy.

From the abstract point of view, the geometric origin of the solutions is not entirely satisfactory. For one thing, the number field K is dictated by the geometry, so we cannot produce a solution for arbitrary choice of K. This also means that the expansion

[6] We do not explain the connection between physics and geometry in any detail in this paper, referring instead to [10, 11, 8]

coefficients carry information that it not intrinsic to the arithmetic of K.

We hope that a better understanding of the physics will allow us to lift these limitations. We find it particularly encouraging that although we do not fully understand the physics implementation of the Galois symmetries, we see no way that physics cares about the distinction between the Galois group being Abelian or non-Abelian. So conceivably, a better understanding of the physics could lead to a solution also in the non-Abelian case. (Again, the alternative would be that physics does care about the nature of the Galois group, which would be at least as interesting.)

The other main result of our paper is a piece of evidence for the idea that among all possible strengthenings of the congruences (i), (ii), (iii), *i.e.*, replacing $2 \mapsto s$ in $(\mathrm{i})_2$, $(\mathrm{ii})_2$, and $(\mathrm{iii})_2$, the initial non-trivial choice, $s = 2$ is distinguished by the existence of the framing operation. We now turn to these formal developments.

2 2-functions and their framing

2.1 Definitions and Results

Preliminaries. Let K be a finite field extension of $\mathbb{Q}$.[7] We denote by $\mathcal{O}$ the ring of integers in K, and by $\mathcal{O}_D$ the ring of elements of K that are integral outside the discriminant of $K/\mathbb{Q}$. For a rational prime p, unramified in the extension $K/\mathbb{Q}$, we consider the p-adic completion of $\mathcal{O}$,

$$\mathcal{O}_p = \varprojlim_n \mathcal{O}/(p^n\mathcal{O}) \tag{2.1}$$

Unless p is inert (*i.e.*, unless $(p) = p\mathcal{O}$ is a prime ideal in $\mathcal{O}$), $\mathcal{O}_p$ is not an integral domain. ($\mathcal{O}/(p)$ is not a field.) In general, we have a factorization

$$(p) = \prod_{i=1}^{r} \mathfrak{p}_i \tag{2.2}$$

into r distinct prime ideals of $\mathcal{O}$ (we are assuming that p is unramified), and $\mathcal{O}/(p) = \prod_i \mathcal{O}/\mathfrak{p}_i$. In fact, by the Chinese Remainder Theorem, for all $n \geq 1$, there is a canonical isomorphism

$$\mathcal{O}/(p^n) \cong \prod_{i=1}^{r} \mathcal{O}/\mathfrak{p}_i^n \tag{2.3}$$

[7] We do not assume that K is Galois over $\mathbb{Q}$.

and so

$$\mathcal{O}_p = \varprojlim_n \prod_{i=1}^r \mathcal{O}/\mathfrak{p}_i^n = \prod_{i=1}^r \varprojlim \mathcal{O}/\mathfrak{p}_i^n = \prod_{i=1}^r \mathcal{O}_{\mathfrak{p}_i} \tag{2.4}$$

where $\mathcal{O}_{\mathfrak{p}_i}$ are the (more) standard rings of $\mathfrak{p}_i$-adic integers. The $\mathcal{O}_{\mathfrak{p}_i}$ are integral domains and their field of fractions, $K_{\mathfrak{p}_i} = (\mathcal{O}_{\mathfrak{p}_i} \setminus \{0\})^{-1}\mathcal{O}_{\mathfrak{p}_i}$, is the $\mathfrak{p}_i$-adic completion of K. It is also true (though perhaps less canonical) that

$$K_{\mathfrak{p}_i} = (\mathbb{Z} \setminus \{0\})^{-1}\mathcal{O}_{\mathfrak{p}_i} \tag{2.5}$$

So in view of (2.4), we define

$$K_p := (\mathbb{Z} \setminus \{0\})^{-1}\mathcal{O}_p \tag{2.6}$$

We have

Lemma 1.

$$K_p = \prod_{i=1}^r K_{\mathfrak{p}_i} \tag{2.7}$$

□

The point of defining K_p via (2.4) (instead of directly as a product of fields) is that it makes the following construction of the Frobenius endomorphism more natural (to us). In particular, it is independent of Galois theory in $K_{\mathfrak{p}_i}$, allowing for several generalizations of our construction (and, in particular, of Theorem 8).

We also note that K is canonically embedded in K_p (namely, diagonally in the product (2.7)).

Frobenius. In $\mathcal{O}/(p)$, we have the endomorphism

$$\mathrm{Frob}_p : \mathcal{O}/(p) \to \mathcal{O}/(p)\,, \qquad x \mapsto x^p \tag{2.8}$$

which under the isomorphism $\mathcal{O}/(p) \cong \prod_i \mathcal{O}/\mathfrak{p}_i$ is identified with the standard Frobenius element in the Galois group of each local field extension $(\mathcal{O}/\mathfrak{p}_i)/(\mathbb{Z}/(p))$. By Hensel's Lemma (or more simply, Newton's formula, (1.11)), Frob_p has a canonical lift to $\mathcal{O}_p$ of (2.4), and can then be extended to K_p by noting that $\mathrm{Frob}_p|_{\mathbb{Z}} = \mathrm{id}$. We denote these by the same symbol, and note that the result of the definition coincides with the Frobenius element $\mathrm{Frob}_{\mathfrak{p}_i/p}$ in each local extension $K_{\mathfrak{p}_i}/\mathbb{Q}_p$.

Power series. Letting z be a (formal) independent variable, we consider the ring of formal power series $K[[z]]$, with obvious embeddings $K[[z]] \hookrightarrow K_p[[z]]$, $K[[z]] \hookrightarrow K_{\mathfrak{p}_i}[[z]]$ and a (compatible) morphism $K_p[[z]] \to K_{\mathfrak{p}_i}[[z]]$ for each prime $\mathfrak{p}_i$ over $p \in \mathbb{Z}$. Given $V \in K[[z]]$, we denote its image in $K_p[[z]]$ and $K_{\mathfrak{p}_i}[[z]]$ by V_p and $V_{\mathfrak{p}_i}$, respectively. We use similar notation for integral coefficients. For instance, $\mathcal{O}_D[[z]]$ is the ring of formal power series with coefficients that are integral outside the discriminant. We extend Frob_p to an endomorphism of $K_p[[z]]$ by declaring

$$\mathrm{Frob}_p(z) = z^p$$

We also introduce the logarithmic derivative

$$\begin{aligned} \delta := \delta_z : K[[z]] \to & zK[[z]] \subset K[[z]] \\ \delta_z(V) := & z\frac{dV}{dz} \end{aligned} \tag{2.9}$$

and its (partial) inverse

$$\int : zK[[z]] \to zK[[z]] \tag{2.10}$$

Explicitly, if

$$V = \sum_{k=1}^{\infty} a_k z^k$$

then

$$\delta V = \sum_{k=1}^{\infty} k a_k z^k\,, \qquad \int V = \sum_{k=1}^{\infty} \frac{a_k}{k} z^k$$

An important observation is

Lemma 2.

$$\begin{aligned} \delta \circ \mathrm{Frob}_p &= p\,\mathrm{Frob}_p \circ \delta \\ \frac{1}{p}\int \circ \mathrm{Frob}_p &= \mathrm{Frob}_p \circ \int \end{aligned}$$

□

In particular, we have $\delta(\mathcal{O}[[z]]) \subset \mathcal{O}[[z]]$, but $\int$ does not preserve integrality in general.

s-functions. Let s be a non-negative integer. A formal power series $V \in zK[[z]]$ is called an *s-function with coefficients in K* if for every unramified prime $p \nmid D(K/\mathbb{Q})$, we have

$$\frac{1}{p^s}\mathrm{Frob}_p V_p - V_p \in z\mathcal{O}_p[[z]] \tag{2.11}$$

Lemma 3. *If $s > 0$ and $V \in K[[z]]$ is an s-function, then δV is an $(s-1)$-function.*

Proof. By Lemma 2,

$$\frac{1}{p^{s-1}} \operatorname{Frob}_p \delta V_p - \delta V_p = \delta\big(\frac{1}{p^s} \operatorname{Frob}_p V_p - V_p\big) \in \delta\big(z\mathcal{O}_p[[z]]\big) \subset z\mathcal{O}_p[[z]]$$

$\square$

We will sometimes find it convenient to verify the s-function property at the level of the coefficients of the power series. (The following lemma formalizes conditions $(\mathrm{i})_s$, $(\mathrm{ii})_s$, and $(\mathrm{iii})_s$ from the introduction.)

Lemma 4. *If $V \in zK[[z]]$ is an s-function, then $\delta^s V \in \mathcal{O}_D[[z]]$, so writing*

$$V = \sum_{k=1}^{\infty} \frac{a_k}{k^s} z^k \tag{2.12}$$

we have that all $a_k \in \mathcal{O}_D$. Moreover, letting for fixed k and p prime, $\alpha = \operatorname{ord}_p(k)$, we have

$$\operatorname{Frob}_p(a_{k/p}) - a_k = 0 \bmod p^{s\alpha}\mathcal{O}_p \tag{2.13}$$

(with the understanding that $a_{k/p} = 0$ if $p \nmid k$). Conversely, if this condition holds for every k and unramified prime p, then V is an s-function.

Proof. Plugging (2.12) into (2.11), the coefficient of z^k gives the condition

$$\frac{1}{p^s} \operatorname{Frob}_p \frac{a_{k/p}}{(k/p)^s} - \frac{a_k}{k^s} \in \mathcal{O}_p$$

Multiplying with $p^{s\alpha}$, and given that $\frac{p^\alpha}{k} \in \mathcal{O}_p$, this is equivalent to (2.13). $\square$

Finally, we note that thanks to Lemma 1, we can equivalently characterize s-functions by the behaviour at the primes of K.

Lemma 5. *$V \in K[[z]]$ is an s-function if and only if for every prime ideal $\mathfrak{p}$ of $\mathcal{O}$ that is not a branch point of $\operatorname{Spec}(\mathcal{O}) \to \operatorname{Spec}(\mathbb{Z})$, we have*

$$\frac{1}{p^s} \operatorname{Frob}_{\mathfrak{p}/p} V_{\mathfrak{p}} - V_{\mathfrak{p}} \in z\mathcal{O}_{\mathfrak{p}}[[z]]$$

where $(p) = \mathfrak{p} \cap \mathbb{Z}$.

$\square$

Remark 6. We have here defined s-functions as power series without constant term. In applications, they are often accompagnied by a non-zero constant term, but the properties of that term depend on the context. For instance, the constant term might take values in a transcendental extension of K, with a rather different action of the Galois group. For a different example, with an algebraic constant term, see section 4. We also emphasize explicitly that we do not impose any condition at the ramified primes, although we suspect that it would be interesting to do so.

1-functions We will now show that 1-functions (*i.e.*, s-functions with $s = 1$) are simply linear combinations of ordinary logarithms. Specifically, we claim that for any 1-function $V \in zK[[z]]$ there exists a sequence $(b_d) \subset \mathcal{O}_D$ such that (as formal power series)

$$V = -\sum_{d=1}^{\infty} \log(1 - b_d z^d) \tag{2.14}$$

Conversely, any power series of this form is a 1-function. As a result, we obtain a version of the celebrated "Dwork integrality lemma"

Proposition 7. *Let $V \in zK[[z]]$ and $Y \in 1 + zK[[z]]$ be related by $V = \log Y$, $Y = \exp(V)$. Then the following are equivalent:*
(i) V is a 1-function
(ii) For every (unramified) prime p,

$$\frac{\mathrm{Frob}_p Y_p}{(Y_p)^p} \in 1 + zp\mathcal{O}_p[[z]]. \tag{2.15}$$

(iii) $Y \in 1 + z\mathcal{O}_D[[z]]$

Proof. We begin with (2.14). Writing

$$V = \sum_{d=1}^{\infty} \frac{a_d}{d} z^d = \sum_{d,k=1}^{\infty} \frac{(b_d z^d)^k}{k}$$

and comparing coefficients, we obtain

$$\frac{a_d}{d} = \sum_{k|d} \frac{(b_{d/k})^k}{k} \tag{2.16}$$

By Lemma 4, what we have to show is that $b_d \in \mathcal{O}_D$ for all d iff $\frac{1}{d}(\mathrm{Frob}_p\, a_{d/p} - a_d) \in \mathcal{O}_p$ for all d, and prime p. The key observation is that if $b_{d/k} \in \mathcal{O}_D$ then, by Euler's theorem, for all p,

$$(b_{d/k})^{kp} = \mathrm{Frob}_p(b_{d/k})^k \bmod p^{\mathrm{ord}_p(k)+1}\mathcal{O}_p \tag{2.17}$$

Therefore, assuming $b_d \in \mathcal{O}_D$ for all d, we have

$$\begin{aligned}
\frac{\operatorname{Frob}_p a_{d/p}}{d} = \frac{1}{p} \operatorname{Frob}_p\Big(\frac{a_{d/p}}{d/p}\Big) &= \sum_{k|\frac{d}{p}} \frac{\operatorname{Frob}_p(b_{d/kp})^k}{kp} \\
&= \sum_{k|\frac{d}{p}} \frac{(b_{d/kp})^{kp}}{kp} \bmod \mathcal{O}_p \\
&= \sum_{\substack{k|d \\ p|k}} \frac{(b_{d/k})^k}{k} + \sum_{\substack{k|d \\ p\nmid k}} \frac{(b_{d/k})^k}{k} \bmod \mathcal{O}_p \\
&= \frac{a_d}{d} \bmod \mathcal{O}_p
\end{aligned} \tag{2.18}$$

For the converse, we first note that by eq. (2.16), $b_1 = a_1 \in \mathcal{O}_D$ in any case. Then, by way of induction, we assume that for some $d > 1$, we have established $b_{d/k} \in \mathcal{O}_D$ for all $k|d$. For any p, with $\alpha = \operatorname{ord}_p(k)$, we have from (2.16)

$$\begin{aligned}
\frac{a_d}{d} &= \sum_{\substack{k|d \\ p\nmid k}} \frac{1}{k} \sum_{i=0}^{\alpha} \frac{(b_{d/kp^i})^{kp^i}}{p^i} \\
&= \sum_{\substack{k|d \\ p\nmid k}} \frac{(b_{d/k})^k}{k} + \sum_{\substack{k|d \\ p\nmid k}} \sum_{i=0}^{\alpha-1} \frac{(b_{d/kp^{i+1}})^{kp^{i+1}}}{kp^{i+1}} \\
&= \sum_{\substack{k|d \\ p\nmid k}} \frac{(b_{d/k})^k}{k} + \sum_{\substack{k|d \\ p\nmid k}} \sum_{i=0}^{\alpha-1} \frac{\operatorname{Frob}_p\big(b_{d/kp^{i+1}}\big)^{kp^i}}{kp^{i+1}} \bmod \mathcal{O}_p \\
&= b_d + \frac{\operatorname{Frob}_p a_{d/p}}{d} \bmod \mathcal{O}_p
\end{aligned} \tag{2.19}$$

Therefore, $\frac{1}{d}(\operatorname{Frob}_p a_{d/p} - a_d) \in \mathcal{O}_p$ implies $b_d \in \mathcal{O}_p$ for all $p \nmid D$.

Given this, (i) immediately implies

$$Y = \prod_{d=1}^{\infty} (1 - b_d z^d)^{-1} \in 1 + z\mathcal{O}_D[[z]] \tag{2.20}$$

i.e., (iii). Given $Y \in 1 + z\mathcal{O}_D[[z]]$, it can be factored as in (2.20), with $b_d \in \mathcal{O}_D$. Then

$$\frac{\operatorname{Frob}_p Y_p}{(Y_p)^p} = \prod_d \frac{(1 - b_d z^d)^p}{1 - \operatorname{Frob}_p b_d z^{dp}} = \prod_d \frac{1 - (b_d)^p z^{dp}}{1 - \operatorname{Frob}_p(b_d) z^{dp}} = 1 \bmod zp\mathcal{O}_p \tag{2.21}$$

implying (ii). Finally, given (ii), taking the logarithm on the two sides, and using $\log(1 + pz\mathcal{O}_p[[z]]) \subset pz\mathcal{O}_p[[z]]$ implies (i). □

Framing of 2-functions In [8], we motivated framing as an ambiguity in the choice of variables in which to write our formal power series. Given a 1-function $V \in zK[[z]]$, we can write $Y = \exp(V) \in 1 + z\mathcal{O}_D[[z]]$ as a series in z, or as a series in $z_f = z(-Y)^f \in (-1)^f z + z\mathcal{O}_D[[z]]$, for any integer f. The resulting series, $Y_f \in 1 + z_f\mathcal{O}_D[[z_f]]$, will also have integral coefficients, and define a "framed" 1-function $V_f = \log Y_f$. Clearly, these "framing transformations" preserving integrality are generated by $f = 1$, and can be extended to include $Y \mapsto Y^{-1}$. Our main theorem says that if V "comes from" (in the sense of being the logarithmic derivative of) a 2-function, then so do all its framed versions. We first state a somewhat more special result, and return to the general case in section 4.

Theorem 8. *Let $W \in zK[[z]]$ be a 2-function. Define $Y = \exp(-\delta W)$ and $\tilde{Y}(\tilde{z})$ via the inverse series $\tilde{z} = -zY(z)$, $z = -\tilde{z}\tilde{Y}(\tilde{z})$. Then $\tilde{W} = -\tilde{\int} \log \tilde{Y}(\tilde{z}) \in \tilde{z}K[[\tilde{z}]]$ is also a 2-function (where $\tilde{\int}$ is the logarithmic integral w.r.t. $\tilde{z}$).*

We prove this theorem in section 3. It will then become clear that the minus sign in the relations between z, $\tilde{z}$, Y, $\tilde{Y}$ is important for preserving integrality at $p = 2$. (Whereas the sign in the relation between Y and W is conventional.) In the rest of this section, we discuss some examples and ask questions about possible further theoretical developments.

2.2 Bases of s-functions

Let us denote by $\mathcal{S}_K \subset zK[[z]]$ the set of s-functions with coefficients in a fixed number field K. One sees immediately that $\mathcal{S}_K$ is a free module over $\mathbb{Z}[\frac{1}{D}]$, where D is the discriminant of $K/\mathbb{Q}$. We view it as an important challenge, and especially for $s = 2$, to characterize a submodule of s-functions by suitable algebraic or analytic properties, and a class of distinguished generators for this submodule. For the following considerations, we endow $\mathcal{S}_K$ with the topology of formal power series in one variable, with neighborhood basis $\mathcal{S}_{K,l} := (z^l)K[[z]] \cap \mathcal{S}_K$ for $l = 1, 2, \ldots$.

Lemma 9. *(i) If $V(z) = \sum \frac{a_k}{k^s} z^k \in \mathcal{S}_K$ is an s-function, then for $l > 1$ $\mathrm{Sh}_l(V)(z) := V(z^l) = \sum \frac{a_k}{k^s} z^{lk} \in \mathcal{S}_{K,l}$ is also an s-function.*
(ii) $\mathcal{S}_K = \mathcal{S}_{K,1}$ and $\mathcal{S}_{K,1}/\mathcal{S}_{K,2} \cong \mathcal{O}_D$ is a free module over $\mathbb{Z}[\frac{1}{D}]$ of rank $d = [K : \mathbb{Q}]$. In fact, $\mathcal{S}_{K,l}/\mathcal{S}_{K,l+1} \cong \mathcal{O}_D$ for all l.
(iii) If $\{V_1, V_2, \ldots, V_d\} \subset \mathcal{S}_K$ is a set of s-functions whose image in $\mathcal{S}_{K,1}/\mathcal{S}_{K,2}$ generates

$\mathcal{O}_D$, then (the image of) $\mathrm{Sh}_l\{V_1,\ldots,V_d\}$ generates $\mathcal{S}_{K,l}/\mathcal{S}_{K,l+1}$, and

$$\cup_{l=1}^{\infty}\mathrm{Sh}_l\{V_1,\ldots,V_d\} \tag{2.22}$$

is a (Schauder) basis of $\mathcal{S}_K$ in the z-adic topology of formal power series.

Proof. (i) is obvious. (ii) follows from the fact that the leading coefficient of any s-function is in $\mathcal{O}_D$ (this was noted, e.g., in Lemma 4). To verify (iii) one may show recursively that for any $L=1,2,\ldots$

$$\mathcal{S}_K - \langle \cup_{l=1}^{L}\mathrm{Sh}_d\{V_1,\ldots,V_d\}\rangle_{\mathbb{Z}[\frac{1}{D}]} \subset (z^L)K[[z]] \tag{2.23}$$

□

In is natural to call such a set $\{V_1,\ldots,V_d\}$ that generates $\mathcal{S}_K$ over $\mathbb{Z}[\frac{1}{D}]$ and under Sh_l a "basis of s-functions with coefficients in K". To construct such a basis, in view of the Lemma, it is enough to show that for every algebraic integer $x\in\mathcal{O}_{\overline{\mathbb{Q}}}$, there exists an s-function with coefficients in $K=\mathbb{Q}(x)$ and leading coefficient $a_1=x$, as in the Introduction. Indeed, the congruences (2.13) relate all coefficients with the Galois orbit of a_1 modulo $\mathcal{O}_D$, so that (if a solution to the congruences exists, which we will show momentarily) the coefficients will all be in K. Note that this is true even if K is not Galois over $\mathbb{Q}$ since all local extensions are. As preliminary restrictions on the class of allowed functions, we will call such an s-function $V\in zK[[z]]$ *algebraic* if $Y:=\exp\bigl(-\delta^{s-1}V\bigr)$ is the series expansion of an algebraic function of z around 0. We call an s-function *locally analytic* if (for some embedding $K\hookrightarrow\mathbb{C}$) it converges (in the complex topology) in a finite neighborhood of the origin, and we say that V is *analytic* if it can be analytically continued to a dense subset of the complex plane. Clearly, *algebraic* $\Rightarrow$ *analytic*. Moreover,

Lemma 10. *For every algebraic integer $x\in\mathbb{C}$ there exists an s-function $V\in xz+z^2K[[z]]$ that is locally analytic.*

Proof. By Lemma 4, we need to find a convergent power series $V=\sum\frac{a_k}{k^s}z^k$ with coefficients a_k that satisfy for all unramified $p|k$ the condition

$$\mathrm{Frob}_p(a_{k/p})-a_k=0 \bmod p^{s\alpha}\mathcal{O}_p \tag{2.24}$$

To this end, given $a_1:=x$, we determine $a_k\in K=\mathbb{Q}(x)$ for $k>1$ (outside the discriminant) recursively by (i) fixing for each $p|k$ a lift $\mathrm{Frob}_p^{(k,s)}:\mathcal{O}\to\mathcal{O}$ of Frobenius at p mod$p^{s\alpha}$ to $\mathcal{O}$, and (ii) solving the congruences

$$a_k = \mathrm{Frob}_p^{(k,s)}(a_{k/p}) \bmod p^{s\alpha}\mathcal{O} \tag{2.25}$$

jointly for all $p|k$. (This is possible by the CRT.) Since for every embedding $K \hookrightarrow \mathbb{C}$, there exists a $B > 0$ such that any disk of radius B contain an element of $\mathcal{O}$, we can choose a_k such that $|\frac{a_k}{k^s}| < B$. We put $a_k = 0$ when $(D,k) \neq 1$. Then $V = \sum \frac{a_k}{k^s} z^k$ has radius of convergence at least B. □

Remark 11. This algorithm of course is far from specifying a unique solution to the problem, and the condition of local analyticity is clearly too weak to select a finitely generated submodule of s-functions, motivating us to seek s-functions with stronger analytic properties. We will next show that when $K = \mathbb{Q}(x)$ is an abelian extension, there exists a basis of algebraic s-functions in the above sense. An important consequence of Theorem 22 is that algebraic cycles on Calabi-Yau three-folds provide a source of 2-functions that are analytic, and even satisfy a differential equation with algebraic coefficients, albeit in a different variable $q(z)$, that is related to z by a transcendental "mirror" transformation which however does not preserve 2-integrality. This class includes examples with non-abelian Galois group, but does not teach us how to specify a basis in general.

2.3 Abelian field extensions

We have already remarked in the introduction that if ζ is a root of unity, then $\mathrm{Frob}_p(\zeta) = \zeta^p$ for all p, and as a consequence

$$\mathrm{Li}_s(\zeta z) = \sum_{k=1}^{\infty} \frac{\zeta^k}{k^s} z^k \tag{2.26}$$

is an (analytic) s-function for any s.

Let us now assume that K is a number field that is Galois over $\mathbb{Q}$ with Galois group $\mathrm{Gal}(K/\mathbb{Q})$ that is *abelian*. Then, by the Kronecker-Weber theorem, there exists a root of unity ζ, say primitive of degree N, such that K is a subextension of $\mathbb{Q}(\zeta)$. By elementary Galois theory, there is an (abelian) subgroup $\Gamma \subset \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ such that $K = \mathbb{Q}(\zeta)^\Gamma$, and $\mathrm{Gal}(K/\mathbb{Q}) = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})/\Gamma$.

Then for any algebraic integer $x \in \mathcal{O}_K$, there are rational numbers $c_i \in \mathbb{Q}$ such that

$$x = \sum_{i=0}^{N-1} c_i \zeta^i \tag{2.27}$$

and for every $(p, N) = 1$

$$\mathrm{Frob}_p(x) = \sum_{i=0}^{N-1} c_i \zeta^{pi} \in \mathcal{O} \tag{2.28}$$

In particular, $c_i = c_{p^{-1}i \bmod N}$ whenever $\mathrm{Frob}_p \in \Gamma$. Clearly then,

$$L_D(x; z) := \sum_{i=0}^{N-1} c_i \mathrm{Li}_s(\zeta^i z) \in xz + z^2 K[[z]] \tag{2.29}$$

is an s-function with coefficients in K for every s. Since

$$\delta^{s-1} L_D(x; z) = -\sum_i c_i \log(1 - \zeta^i z) \tag{2.30}$$

we see that $L_D(x; z)$ is *algebraic*. As a consequence

Theorem 12. *If $K = \mathbb{Q}(x)$ is an abelian extension of $\mathbb{Q}$, there exists an algebraic basis of s-functions with coefficients in K.*

As a physical example from the introduction 1.1, consider x a root of x^3+x^2-2x-1. We have $K = \mathbb{Q}(x) = \mathbb{Q}(\zeta)^\Gamma$, where ζ is a primitive 7-th root of unity, and $\Gamma = \mathbb{Z}/2$ whose non-trivial element acts by $\zeta \mapsto \zeta^{-1}$. Namely, $x = \zeta + \zeta^{-1}$, and

$$\delta^{s-1} L_D(x; z) = -\log(1 - xz + z^2) \tag{2.31}$$

2.4 When do 2-functions come from 3-functions?

We reiterate here a few comments from [8] concerning the special status of 2-functions. First of all, given our results on framing of 2-functions (Theorem 8 and its generalization, Theorem 14), it seems natural to ask whether starting from an s-function with $s > 2$ and taking $s-1$ logarithmic derivatives, framing à la Thm. 8 might produce a 1-function that can be integrated back to an s'-function with $s' > 2$. In general, this is not the case (the proof that we give below makes it plain why one should not expect it, and one easily produces counterexamples). However, there can be special cases in which it is, and so one comes to ask which pairs of s-, s'-functions are related by framing in this fashion.

The simplest example for this phenomenon (with $s' = 3$) comes from the ordinary polylogarithms, Li_s, which are of course s-functions with rational coefficients for any s. For $f \in \mathbb{Z}$, cmp. Theorem 14, we solve

$$z_f = z(-\exp(\mathrm{Li}_1(z)))^f = \frac{z}{(z-1)^f} \tag{2.32}$$

f	2	3	4	5
1	-2	3	-4	5
2	1	$\frac{3}{2}$	4	5
3	$-\frac{2}{3}$	3	-8	$\frac{50}{3}$
4	1	$\frac{15}{2}$	28	75
5	-2	24	-124	425
6	$\frac{13}{3}$	$\frac{171}{2}$	624	$\frac{8240}{3}$
7	-10	339	-3452	19605

Table 1: Framed sequence $N_d^{(f)}$ from (2.35) for various d, f.

for z,

$$z = (-1)^f z_f Y_f \tag{2.33}$$

with $Y_f \in 1 + z_f \mathbb{Z}[[z_f]]$, and claim that

$$F_f = \iint \log Y_f \tag{2.34}$$

is a 3-function for all f, except perhaps at $p = 2$ and 3. Namely, writing

$$F_f = \sum_{d=1}^{\infty} N_d^{(f)} \mathrm{Li}_3(z_f^d) \tag{2.35}$$

we claim that the $N_d^{(f)}$ are integers (after multiplication by a power of 6) for all d and f. (See Table 1 for some examples; it seems that in fact $6N_d^{(f)}/f \in \mathbb{Z}$.)

To prove our claim, we note that the explicit solution of (2.32) is given by

$$V_f = -\log Y_f = (-1)^f \sum_{k=1}^{\infty} \frac{1}{k} \binom{kf}{k} z_f^k \tag{2.36}$$

so that in view of Lemma 4, the statement is equivalent to

$$\binom{pkf}{pk} \equiv \binom{kf}{k} \bmod p^{3(\alpha+1)} \tag{2.37}$$

for all k, f and primes $p > 3$ (and as before $\alpha = \mathrm{ord}_p(k)$). The congruence (2.37) now follows from the generalization of the classical Wolstenholme theorem that is known to experts [15] as the Jacobsthal-Kazandzidis congruence [16].[8]

[8]This congruence is usually stated as

$$\binom{pn}{pm} \equiv \binom{n}{m} \bmod p^q \tag{2.38}$$

Meanwhile, further explicit examples of algebraic 3-functions with rational coefficients have appeared in [17] as solutions of so-called extremal A-polynomials of certain knots, and all framings of these 3-functions are also (algebraic) 3-functions. In this case, the integrality can be seen to follow from the relation with quiver representation theory [18]. (Alternatively, it has been suggested that the integrality can be proved by clarifying the relation between the K-theoretic "quantizability condition" of the A-polynomial [19] and the K-theoretic interpretation of 2-functions that we have given in [8].)

Given that the polylogarithms are arguably the simplest s-functions, it seems unlikely that framings of s-functions can be s'-functions with $\min(s, s') > 3$ (outside a finite number of primes). We would also be interested to learn about any other examples of algebraic 3-functions with rational or algebraic coefficients.[9]

Another reason for the distinguished status of 2-functions is that the multi-variable generalization of framing that we discuss in section 4 only makes sense for 2-functions, although we find it conceivable that $s = 3$ could again harbour some exception.

3 Proof of Integrality of Framing

We will prove theorem 8 separately for each rational prime p. In fact, our main calculation goes through with the following slightly more general set of coefficients (which we will have occasion to exploit in multi-dimensional framing in section 4). Abusing the notation of section 2, we let $\mathcal{O}_p$ be a (commutative, unital) ring in which p is not a zero divisor and all integers outside of $(p) = p\mathbb{Z}$ are invertible, and we let $K_p \supset \mathcal{O}_p$ be a ring extension in which also p is invertible (in other words, K_p contains $\mathbb{Q}$ as a field; we do not need K_p to be complete w.r.t. the p-adic norm).

We suppose $\mathrm{Frob}_p : K_p \to K_p$ to be a ring homomorphism fixing $\mathbb{Q} \subset K_p$, and such that for $a \in \mathcal{O}_p$, $\mathrm{Frob}_p(a) - a^p \in p\mathcal{O}_p$. We consider $K_p[[z]]$ ($\supset \mathcal{O}_p[[z]]$) the ring of formal power series with coefficients in K_p ($\supset \mathcal{O}_p$). We extend Frob_p to $K_p[[z]]$ by

where q is the power of p dividing $p^3 mn(n-m)$. Plugged into (2.37) it shows that F_f can be the derivative of a local s'-function with $s' > 3$ for special values of f and p. But even (2.38) is not in general optimal in q.

[9]Transcendental 3-functions with algebraic coefficients (in real quadratic number fields) have appeared as solutions of certain Calabi-Yau-type differential equations studied by Bogner, van Straten et al. (private communication).

$z \mapsto z^p$ as usual. A crucial property of this extension is that if $X \in \mathcal{O}_p[[z]]$, then

$$X^p - \operatorname{Frob}_p X \in p\mathcal{O}_p[[z]] \tag{3.1}$$

(This follows by a two line calculation from the definition $\mathcal{O}_p[[z]] = \varprojlim \mathcal{O}_p[z]/z^n$.)

Now let $W \in zK_p[[z]]$ be a formal power series satisfying the following "local 2-function property" (In this section, we work with a fixed prime p, so we drop the subscript from W etc.)

$$X := \frac{1}{p^2} \operatorname{Frob}_p W - W \in z\mathcal{O}_p[[z]] \tag{3.2}$$

As in lemma 4, this can be equivalently rewritten in terms of the coefficients of W and X. With

$$W = \sum_{k=1}^{\infty} \frac{a_k}{k^2} z^k \tag{3.3}$$

$$X = \sum_{k=1}^{\infty} x_k z^k \tag{3.4}$$

we have

$$x_k = \frac{a_k - \operatorname{Frob}_p(a_{k/p})}{k^2} \in \mathcal{O}_p \tag{3.5}$$

(with the understanding that $a_{k/p} = 0$ if $p \nmid k$).

Lemma 13. *With these relaxed assumptions on the coefficients, we may still define* $Y = \exp(-\delta W)$*, and solve the relation*

$$\tilde{z} = -zY \tag{3.6}$$

for z*,*

$$z = -\tilde{z}\tilde{Y} \tag{3.7}$$

Then $\tilde{Y} \in 1 + \tilde{z}\mathcal{O}_p[[\tilde{z}]]$ *and*

$$\tilde{W} := -\int \log \tilde{Y} \in \tilde{z}K_p[[\tilde{z}]] \tag{3.8}$$

is a 2-function at p*.*

Our proof relies only on the manipulation of formal power series, property (3.1), and elementary p-adic estimates.

Proof. We shall verify that the coefficients of $\tilde{W}$,

$$\tilde{W} = \sum \frac{\tilde{a}_k}{k^2} \tilde{z}^k \tag{3.9}$$

satisfy the congruence

$$\tilde{a}_{pk} = \mathrm{Frob}_p(\tilde{a}_k) \bmod p^{2(\alpha+1)}\,, \quad \text{where } \alpha = \mathrm{ord}_p(k). \tag{3.10}$$

To this end, we briefly digress to recall the Lagrange inversion formula: If $f(z)$ is a general kind of formal power series without constant term and coefficient of z invertible in its coefficient ring (this may sometimes be written as $f(0) = 0$, $f'(0) \neq 0$), and if $g(\tilde{z})$ is the compositional inverse of f (*i.e.*, $g(f(z)) = z$, $f(g(\tilde{z})) = \tilde{z}$; a unique such g exists by the assumptions on f), then for every k,

$$(\text{coefficient of } \tilde{z}^k \text{ in } g) = \frac{1}{k}(\text{coefficient of } z^{k-1} \text{ in } (z/f)^k) \tag{3.11}$$

This formula is most readily understood with complex coefficients as a consequence of Cauchy's theorem: If f converges and is suitably analytic in a neighborhood of 0, we have

$$g(\tilde{z}) = \oint \frac{f'(z)}{f(z) - \tilde{z}}\, z\, dz \tag{3.12}$$

for a suitably small contour (and $\frac{1}{2\pi i}$ included in $\oint$). Expanding

$$g(\tilde{z}) = \oint \sum_{k=0}^{\infty} \frac{f'(z)}{f(z)^{k+1}} \tilde{z}^k z\, dz \tag{3.13}$$

and noting that inside a small enough circle, f will only vanish at the origin, we learn that the $k = 0$ term vanishes, while for $k > 0$ we may integrate by parts to obtain

$$g(\tilde{z}) = \sum_{k=1}^{\infty} \frac{\tilde{z}^k}{k} \oint \frac{1}{f(z)^k}\, dz \tag{3.14}$$

This shows that (3.11) is valid for convergent power series with complex coefficients, but since the coefficients of g are a priori algebraic in those of f, the formula will be valid for formal power series as well.

Along similar lines, the coefficients of (integer and complex) *powers* g^l of g can be obtained from the expression

$$g^l(\tilde{z}) = \oint \frac{f'(z)}{f(z) - \tilde{z}}\, z^l\, dz \tag{3.15}$$

and we may also take a derivative at $l=0$ to obtain an expression for the coefficients of $\log g(\tilde{z})$. (In the analytic approach, one needs to be somewhat careful with the right choice of contours, but this is irrelevant at the formal algebraic level.)

Applied to our situation, with $\tilde{z} = -zY(z)$, $z/\tilde{z} = -\tilde{Y}(\tilde{z})$ the formula reads

$$\tilde{\delta}\tilde{W}(\tilde{z}) = -\log \tilde{Y}(\tilde{z}) = -\oint \frac{(zY(z))'}{zY(z)+\tilde{z}} \log\bigl(-\frac{z}{\tilde{z}}\bigr)\, dz \tag{3.16}$$

so that, for $k>0$,

$$\tilde{a}_k = (-1)^{k-1} \oint \frac{1}{z^k Y(z)^k}\, d\log z \tag{3.17}$$

From now on, we think of $\oint \cdots d\log z$ as a formal device for extracting the constant term of a power series. In particular, it is unchanged if we replace z by z^p in the integrand. On the other hand, from outside the $\oint$-sign, Frob_p would act only on the coefficients of Y, so that in combination, we obtain

$$\mathrm{Frob}_p \tilde{a}_k = (-1)^{k-1} \oint \frac{1}{z^{pk}(\mathrm{Frob}_p Y)^k}\, d\log z \tag{3.18}$$

By (3.2) and the other definitions, we have

$$\mathrm{Frob}_p Y = Y^p \exp(-p\delta X) \tag{3.19}$$

Therefore

$$\mathrm{Frob}_p \tilde{a}_k = (-1)^{k-1} \oint \frac{1}{z^{pk} Y^{pk}} \exp\bigl(pk\, \delta X(z)\bigr)\, d\log z \tag{3.20}$$

Now let's first assume that $(-1)^{kp} = (-1)^k$, which is the case if p is odd, or $p=2$ and k even. Then

$$\mathrm{Frob}_p \tilde{a}_k - \tilde{a}_{pk} = (-1)^{k-1} \oint \frac{1}{z^{pk} Y^{pk}} \Bigl(\exp\bigl(pk\, \delta X\bigr) - 1\Bigr)\, d\log z \tag{3.21}$$

Our goal now is to control the order at p of the contribution of each term in the expansion,

$$\exp\bigl(pk\, \delta X\bigr) - 1 = \sum_{r=1}^{\infty} \frac{(pk)^r}{r!} \bigl(\delta X\bigr)^r \tag{3.22}$$

exploiting the fact that all power series involved have coefficients in $\mathcal{O}_p$. To this end, we use the well-known (or easily checked) estimate,

$$\mathrm{ord}_p(r!) = \sum_{j=1}^{\infty} \Bigl\lfloor \frac{r}{p^j} \Bigr\rfloor \le r \sum_{j=1}^{\infty} \frac{1}{p^j} - \frac{1}{p-1} = \frac{r-1}{p-1} \tag{3.23}$$

in which equality holds if and only if $r = p^s$ is a prime power. Therefore,

$$\mathrm{ord}_p\Big(\frac{(pk)^r}{r!}\Big) \geq r(\alpha+1) - \frac{r-1}{p-1} \tag{3.24}$$

Now if $p > 2$,

$$\frac{r-1}{p-1} \leq \frac{r-1}{2} \leq r-2 \tag{3.25}$$

where the latter inequality holds if in addition $r \geq 3$. In that case then

$$r(\alpha+1) - \frac{r-1}{p-1} \geq 2(\alpha+1) + (r-2)\alpha \geq 2(\alpha+1) \tag{3.26}$$

If p is still odd, but $r = 2$, we have $\frac{r-1}{p-1} = \frac{1}{p-1} < 1$, and since the left-hand side of (3.24) has to be integral, it can be no less than $2(\alpha+1)$ (in fact, it is equal to that).

If $p = 2$, and $\alpha \geq 1$ (*i.e.*, k is even), and also $r \geq 3$, then

$$\begin{aligned} r(\alpha+1) - \frac{r-1}{p-1} &= 2(\alpha+1) + (r-2)(\alpha+1) - r + 1 \\ &\geq 2(\alpha+1) + 2(r-2) - r + 1 = 2(\alpha+1) + r - 3 \\ &\geq 2(\alpha+1) \end{aligned} \tag{3.27}$$

Summarizing, when $p > 2$, $\alpha \geq 0$, and $r \geq 2$, or when $p = 2$, $\alpha \geq 1$, and $r \geq 3$,

$$\mathrm{ord}_p\Big(\frac{(pk)^r}{r!}\Big) \geq 2(\alpha+1) \tag{3.28}$$

Therefore, $\mathrm{mod} p^{2(\alpha+1)}$, we can ignore the contribution of those terms to (3.21), since all power series involved have integral coefficients.

To begin dealing with the remaining terms, we observe that when $r = 1$, we may improve the manifest order at p by "integrating by parts" (in other words, using $\oint \delta(\cdots)\, d\log z = 0$, and that δ is a derivation)

$$\oint \frac{1}{z^{pk}Y^{pk}}(pk)\,\delta X\, d\log z = \oint \frac{\delta(zY)}{z^{pk+1}Y^{pk+1}}(pk)^2 X\, d\log z \tag{3.29}$$

which vanishes $\bmod p^{2(\alpha+1)}$ since X still has coefficients in $\mathcal{O}_p$. (This is the place where the original 2-function property enters in the crucial way. Note also that this step can only be taken exactly once, *i.e.*, it cannot be repeated for s-functions with $s > 2$.)

When $p = 2$, $r = 2$ (but still $\alpha \geq 1$), we find $\mathrm{ord}_2((2k)^2/2) = 2(\alpha+1) - 1$, so what we have to show is that

$$\oint \frac{1}{z^{2k}Y^{2k}}(\delta X)^2\, d\log z = 0 \bmod 2 \tag{3.30}$$

To see this, using def. (3.4), we first reduce mod2,

$$(\delta X)^2 = \sum_i i^2 x_i^2 z^{2i} = \sum_{i \text{ odd}} x_i^2 z^{2i} \tag{3.31}$$

and *then* integrate by parts each term in (3.30)

$$\oint \frac{1}{z^{2k}Y^{2k}} z^{2i}\, \mathrm{dlog}\, z = \oint \frac{1}{z^{2k}Y^{2k}} \frac{1}{2i} \delta z^{2i}\, \mathrm{dlog}\, z = -\frac{k}{i} \oint \frac{\delta(zY)}{z^{2k+1}Y^{2k+1}} z^{2i}\, \mathrm{dlog}\, z \tag{3.32}$$

which vanishes mod2 if $\alpha \geq 1$, and i is odd.

Finally, we consider the situation $(-1)^{pk} = -(-1)^k$, which happens when $p = 2$, and k is odd (*i.e.*, $\alpha = 0$). This leads to a sign change in (3.21), so we have to study

$$\exp\bigl(2k\, \delta X\bigr) + 1 = 2 + \sum_{r=1}^{\infty} \frac{(2k)^r}{r!} \bigl(\delta X\bigr)^r \tag{3.33}$$

When r is not a power of 2, (in particular, $r \geq 3$), we easily see that

$$\mathrm{ord}_2\bigl(\frac{(2k)^r}{r!}\bigr) \geq 2 \tag{3.34}$$

so we can ignore those terms. When $r = 2^s$ is a power of 2, we are confronted with the fact that $\mathrm{ord}_2\bigl((2k)^{2^s}/(2^s!)\bigr) = 1$. So to verify (3.10) in this case, we remain with showing that

$$\oint \frac{1}{z^{2k}Y^{2k}} \Bigl(1 + \sum_{s=0}^{\infty} \frac{(2k)^{2^s}}{2 \cdot (2^s!)} (\delta X)^{2^s}\Bigr)\,, \tag{3.35}$$

which we now know is integral, in fact vanishes mod2. Referring back to eq. (3.4), we find mod2,

$$(\delta X)^{2^s} = \sum_{i=1}^{\infty} i^{2^s} x_i^{2^s}\, z^{i\,2^s} \tag{3.36}$$

and we can again ignore the terms with i even. On the other hand, when i is odd, we have from eq. (3.5)

$$x_i = a_i \bmod 2 \tag{3.37}$$

In fact, since $\delta W = \sum_i \frac{a_i}{i} z^i$ is a 1-function, we have for all s, for i odd, and mod2,

$$\bigl(x_i\bigr)^{2^s} = \bigl(a_i\bigr)^{2^s} = (\mathrm{Frob}_2)^s(a_i) = a_{i\,2^s} \bmod 2 \tag{3.38}$$

So what remains of (3.35) becomes

$$\begin{aligned} \oint \frac{1}{z^{2k}Y^{2k}} \Bigl(1 + \sum_{s=0}^{\infty} \sum_{i \text{ odd}} a_{i\,2^s} z^{i\,2^s}\Bigr)\, \mathrm{dlog}\, z &= \oint \frac{1}{z^{2k}Y^{2k}} \bigl(1 + \sum_{j=1}^{\infty} a_j z^j\bigr)\, \mathrm{dlog}\, z \\ &= \oint \frac{1}{z^{2k}Y^{2k}} \bigl(1 - \delta^2 W\bigr)\, \mathrm{dlog}\, z \end{aligned} \tag{3.39}$$

These manipulations were valid mod2, but we now claim that the RHS of (3.39) in fact vanishes identically in K_p (we're at $p = 2$). Indeed,

$$\begin{aligned} \frac{1}{2k}\delta\Big(\frac{1}{z^{2k}Y^{2k}}\Big) &= -\frac{1}{z^{2k}Y^{2k}} - \frac{1}{z^{2k}Y^{2k+1}}\delta Y \\ &= -\frac{1}{z^{2k}Y^{2k}}(1-\delta^2 W) \end{aligned} \tag{3.40}$$

Thus we see that the integrand at the end of (3.39) is in fact a total derivative, and therefore its constant term vanishes. This completes the proof. □

4 Multi-dimensional Framing

Theorem 8 shows that replacing z with $\tilde{z} = -z\exp(-\delta W)$ transforms a 2-function W into another 2-function $\tilde{W}$ related to W via

$$\tilde{Y} := \exp\big(-\tilde{\delta}\tilde{W}\big) = \exp\big(\delta W\big) =: Y^{-1} \tag{4.1}$$

Since replacing $\tilde{W}$ with $-\tilde{W}$ clearly also preserves 2-integrality, we learn that δW itself integrates to a 2-function with respect to $\tilde{z} = -z\exp(-\delta W)$. This can be iterated to conclude

Theorem 14. *Let $W \in zK[[z]]$ be a 2-function, $Y := \exp(-\delta W)$. For integer "framing parameter" $f \in \mathbb{Z}$, let*

$$z_f := z(-Y)^f \tag{4.2}$$

Then δW, viewed as a formal power series in z_f, is the logarithmic derivative of a 2-function $W_f \in z_f K[[z_f]]$.

The point is that while the "elementary" framing operation studied so far is involutive, *i.e.*, $\tilde{\tilde{z}} = z$, $\tilde{\tilde{W}} = W$, framing in the sense of Theorem 14 defines an action of the group of integers on the set of 2-functions with coefficients in K. We have $\tilde{W} = -W_1$, *etc.*.

More explicitly, framing identifies

$$\delta_f W_f := z_f \frac{dW_f}{dz_f} = z\frac{dW}{dz} =: \delta W \tag{4.3}$$

Given (4.2), we have

$$\frac{z}{z_f}\frac{dz_f}{dz} = 1 - f\delta^2 W \tag{4.4}$$

so that

$$\delta W_f = \frac{z}{z_f}\frac{dz_f}{dz}\,\delta W = \delta W(1 - f\delta^2 W) \tag{4.5}$$

Thus, we can write the relation between W and W_f more succinctly as

$$W_f = W - \frac{f}{2}\big(\delta W\big)^2 \tag{4.6}$$

and Theorem 14 says that if W is a 2-function of z, then W_f is a 2-function of z_f for any $f \in \mathbb{Z}$.

The generalization to the multi-variable case is now clear: If $z^1, z^2, \ldots, z^n$ are n independent formal variables, we continue the Frobenius endomorphism at prime p to the ring of formal power series $K[[z^1, \ldots, z^n]]$ via $\mathrm{Frob}_p(z^i) = (z^i)^p$ for each i. We denote by $(z)K[[z^1, \ldots, z^n]]$ the maximal ideal generated by the z^i (and $(z)\mathcal{O}[[z^1, \ldots, z^n]]$ that with integral coefficient *etc.*). We say that $V \in (z)K[[z^1, \ldots, z^n]]$ is an s-function if

$$\frac{1}{p^s}\,\mathrm{Frob}_p\, V_p - V_p \in (z)\mathcal{O}_p[[z^1, \ldots, z^n]] \tag{4.7}$$

for all p as before, see (2.11).

Now let $W \in (z)K[[z^1, \ldots, z^n]]$ be a 2-function with coefficients in K, and let $\kappa = (\kappa^{ij}) \in \mathbb{Z}^{n^2}$, $\kappa^{ij} = \kappa^{ji}$ be a *symmetric matrix with rational integer coefficients.*[10] We then define *framing of W with respect to* κ by the pair of formulas

$$z^i_\kappa = \sigma_i\, z^i \exp\big(-\textstyle\sum_k \kappa^{ik}\delta_k W\big) \tag{4.8}$$

$$\delta^{(\kappa)}_i W_\kappa = \delta_i W \tag{4.9}$$

where $\delta_j := z^j \frac{d}{dz^j}$, $\delta^{(\kappa)}_j := z^j_\kappa \frac{d}{dz^j_\kappa}$, and $\sigma_i \in \{\pm 1\}$ is a sign inserted to guarantee integrality at $p = 2$, and determined by the diagonal elements of κ,

$$\sigma_i := (-1)^{\kappa^{ii}} \tag{4.10}$$

To see that this multi-dimensional framing is well defined, we first note that $\sum_k \kappa^{ik}\delta_k W$ is a 1-function, and hence by Lemma 7, $z^i_\kappa \in z^i \mathcal{O}_D[[z^1, \ldots, z^n]]$. Moreover,

$$\begin{aligned} \frac{z^j}{z^i_\kappa}\frac{dz^i_\kappa}{dz^j} &= \Delta^i_j - \sum_k \kappa^{ik}\delta_j\delta_k W \\ &= \Delta^i_j \bmod (z)\mathcal{O}_D[[z^1, \ldots, z^n]] \end{aligned} \tag{4.11}$$

[10]It appears possible that with some extra care, this can be generalized to algebraic integer κ^{ij}, but we have not studied this question in any detail.

where $\Delta^i_j = 1$ if $i = j$, and 0 otherwise is the unit matrix. As a consequence, the relation (4.8) can be inverted to find $z^i \in z^i_\kappa \mathcal{O}_D[[z^1_\kappa, \dots z^n_\kappa]]$. To see that (4.9) is integrable with respect to the z^j_κ, we observe that as a consequence of (4.11), the (formal) one-form

$$\begin{aligned}\sum_i \delta_i^{(\kappa)} W_\kappa \frac{dz^i_\kappa}{z^i_\kappa} &= \sum_i \delta_i W \Big(\frac{dz^i}{z^i} - \sum_{j,k} \kappa^{ik} \delta_j \delta_k W \frac{dz^j}{z^j} \Big) \\ &= \sum_i \Big(\delta_i W - \sum_{j,k} \kappa^{jk} \delta_j W \delta_i \delta_k W \Big) \frac{dz^i}{z^i} \\ &= \delta_i \big(W - \frac{1}{2} \sum_{j,k} \kappa^{jk} \delta_j W \delta_k W \big) \frac{dz^i}{z^i}\end{aligned} \tag{4.12}$$

is exact in virtue of the symmetry of κ^{ij}. In other words, we have,

$$W_\kappa = W - \frac{1}{2} \sum_{j,k} \kappa^{jk} \delta_j W \delta_k W \tag{4.13}$$

viewed as a formal power series in the variables (z^i_κ), obtained by inverting (4.8).

The following is obvious from (4.8), (4.9):

Proposition 15. *Let κ and κ' be two symmetric integral matrices. Then $\big((z_\kappa)_{\kappa'}\big)^i = z^i_{\kappa+\kappa'}$ and*

$$(W_\kappa)_{\kappa'} = W_{\kappa+\kappa'} \tag{4.14}$$

In other words, the group of framing transformation in n variables is the additive group of symmetric integral $n \times n$ matrices.

□

It appears to be true that whenever $W \in (z)K[[z^1, \dots, z^n]]$ is a 2-function, then for every symmetric integral matrix κ, $W_\kappa \in (z_\kappa)K[[z^1_\kappa, \dots, z^n_\kappa]]$ is also a 2-function. In view of Proposition 15, it suffices to establish this for the generators of the group of framing transformations, in other words for
(i) "single variable framing", $\kappa^{ii} = 1$ for some i, all other $\kappa^{jk} = 0$, and
(ii) "exchange framing", $\kappa^{ij} = 1 = \kappa^{ji}$ for some fixed $i \neq j$, all other $\kappa^{kl} = 0$.
We are able to prove (i) for all primes p, and case (ii) for all primes except $p = 2$. The last remaining case appears to depend on an improvement of the estimates of Lemma 13 that is as yet missing.

Proposition 16. *For every 2-function W, and κ of type (i), W_κ is also a 2-function.*

Proof. Clearly, it is enough to treat the case $i=1$. Writing

$$K[[z^1,\dots,z^n]] = K[[z^2,\dots,z^n]][[z^1]] \tag{4.15}$$

etc., we view W as a 2-function with coefficients in $K[[z^2,\dots,z^n]]$, albeit with *in general non-zero constant coefficient*, let us call it $a_0 \in K[[z^2,\dots,z^n]]$.

Indeed, for every prime p, the pair $K_p[[z^2,\dots,z^n]] \supset \mathcal{O}_p[[z^2,\dots,z^n]]$ satisfies the hypotheses of section 3 and we have

$$\frac{1}{p^2}\operatorname{Frob}_p W_p - W_p \in \mathcal{O}_p[[z^2,\dots,z^n]][[z^1]] \tag{4.16}$$

Namely $(W-a_0)_p \in z^1 K_p[[z^2,\dots,z^n]]$ is a 2-function without constant coefficient, and

$$\frac{1}{p^2}\operatorname{Frob}_p (a_0)_p - (a_0)_p \in \mathcal{O}_p[[z^2,\dots,z^n]] \tag{4.17}$$

Moreover, for $\kappa^{ii}=1$, all other $\kappa^{jk}=0$, we see that $z^j_\kappa = z^j$ for $j=2,\dots,n$, while $z^1_\kappa = \tilde{z}^1$ in the notation of Lemma 13 (notice that $\sigma_1=-1$). As a consequence,

$$(W-a_0)_\kappa = -(\widetilde{W-a_0}) \tag{4.18}$$

is a 2-function with coefficients in $K[[z^2,\dots,z^n]]$. The claim follows by adding back the constant coefficient, which is unchanged and therefore still satisfies (4.17). ☐

Proposition 17. *For every 2-function W, and κ of type (ii), W_κ is a 2-function at all odd primes.*

Proof. We can assume $i=1$ and $j=2$, and by relaxing the coefficients analogous to the proof of the previous proposition, we might as well pretend that $n=2$.

Under the substitution

$$z^1 = w^1 w^2\,, \qquad z^2 = \frac{w^1}{w^2} \tag{4.19}$$

the ring of formal power series $K[[z_1,z_2]]$ is identified isomorphically with the ring $\bigl(K[w^2,(w^2)^{-1}][[w^1]]\bigr)_+$ of formal powers series in w^1 with coefficients that are Laurent polynomials in w^2, and the following restrictions on the $w^{1,2}$-degrees $m_{1,2}$, indicated by the subscript $+$:

$$m_1 \geq |m_2| \quad \text{ and } m_1 \equiv m_2 \bmod 2\,. \tag{4.20}$$

These conditions ensure that substituting back,

$$w^1 = \bigl(z^1 z^2\bigr)^{1/2}, \qquad w^2 = \Bigl(\frac{z^1}{z^2}\Bigr)^{1/2} \tag{4.21}$$

returns a power series in z^1, z^2. The conditions (4.20) also guarantee that formal manipulations in $w^{1,2}$ are equivalent to those in $z^{1,2}$. For all p, the Frobenius endomorphism lifts to w^1, w^2 as

$$\mathrm{Frob}_p(w^{1,2}) = (w^{1,2})^p \tag{4.22}$$

(This lift is not unique at $p = 2$, but this is not the origin of our problems there.) The logarithmic derivatives with respect to the $w^{1,2}$ are related to those w.r.t. $z^{1,2}$ as

$$\begin{aligned}\gamma_1 &:= w^1 \frac{d}{dw^1} = z^1 \frac{d}{dz^1} + z^2 \frac{d}{dz^2} = \delta_1 + \delta_2 \\ \gamma_2 &:= w^2 \frac{d}{dw^2} = z^1 \frac{d}{dz^1} - z^2 \frac{d}{dz^2} = \delta_1 - \delta_2\end{aligned} \tag{4.23}$$

so that framing w.r.t. κ is diagonal in the $w^{1,2}$: With

$$\begin{aligned}w^1_\kappa &= -w^1 \exp\bigl(-\tfrac{1}{2}\gamma_1 W\bigr) \\ w^2_\kappa &= -w^2 \exp\bigl(\tfrac{1}{2}\gamma_2 W\bigr)\end{aligned} \tag{4.24}$$

we recover

$$\begin{aligned}z^1_\kappa &= w^1_\kappa w^2_\kappa = w^1 w^2 \exp\bigl(-(\tfrac{1}{2}\gamma_1 - \tfrac{1}{2}\gamma_2)W\bigr) = z^1 \exp(-\delta_2 W) \\ z^2_\kappa &= \frac{w^1_\kappa}{w^2_\kappa} = \frac{w^1}{w^2} \exp\bigl(-(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2)W\bigr) = z^2 \exp(-\delta_1 W)\end{aligned} \tag{4.25}$$

If now W is a 2-function in $K[[z^1, z^2]]$, eq. (4.22) allows us to view it as a 2-function in $\bigl(K[w^2, (w^2)^{-1}][[w^1]]\bigr)_+$. Thanks to (4.20), we can continue to use the Lagrange formula to invert the transformation (4.24) so that the proof of Lemma 13 (used as in the previous proposition, for each variable separately) still goes through, *for all primes* $p \neq 2$. This shows that W_κ is a 2-function in $\bigl(K[w^2_\kappa, (w^2_\kappa)^{-1}][[w^1_\kappa]]\bigr)_+$. Substituting $w^1_\kappa = (z^1_\kappa z^2_\kappa)^{1/2}$, $w^2_\kappa = (z^1_\kappa / z^2_\kappa)^{1/2}$, we conclude that W_κ is a 2-function in $K[[z^1_\kappa, z^2_\kappa]]$ at all primes $p \neq 2$. $\square$

Remark 18. Because of the $\frac{1}{2}$ in the framing (4.24) of the $w^{1,2}$-variables, the proof of Lemma 13 does not directly apply at $p = 2$ for the two variables separately. A promising line of attack is to use multivariate Lagrange inversion in the $z^{1,2}$ variables, but we have not been able to carry this to the end so far.

5 2-functions from algebraic cycles on Calabi-Yau threefolds

In the first part of this section, we recall the standard setup of the B-model of mirror symmetry. Namely, following [20, 21], we describe the variation of Hodge structure attached to a family of complex Calabi-Yau threefolds, and the special properties of that variation around a point of maximal degeneration. In particular, we review the interpretation of the canonical coordinate (a.k.a. the mirror map), as well as the Yukawa coupling, as extension classes in the category of mixed Hodge structures. The comparison with the p-adic analogue of this interpretation is the first ingredient in the integrability proofs of [1, 2, 3]. (The second ingredient is the identification of the limiting behaviour of these extension classes in the complex and p-adic setup, see [3].)

In the second part of this section, we describe, following [11, 5], the extension of the B-model by a family of algebraic cycles varying inside the family of threefolds. This includes the extension of the local system and the relation between the superpotential and the Griffiths infinitesimal invariant characterizing the extension of Hodge structure.

Finally, we add the assumption that the maximal degeneration of our family is defined over the integers. This assumption implies that the local period ring is the ring of power series with rational coefficients. The limit of the algebraic cycle then is defined over an algebraic number field, which leads to an extension of the residue field of the period ring. Our main integrality statement is that the superpotential is a 2-function (with coefficients in the extended residue field). The statement will be proven in the two subsequent sections.

5.1 Variation of Hodge structure

Let B be a smooth quasi-projective complex curve and let $\pi : Y \to B$ be a smooth family of projective Calabi-Yau threefolds parametrized by B. We assume (for convenience) that the generic member of the family $Y_b = \pi^{-1}(b)$ (b a point in B) is simply connected, has middle-dimensional Betti number $b_3(Y_b) = 4$, and that there is no torsion in cohomology (ever).

To such a family is associated a polarized, integral variation of pure Hodge structure (VHS) $\mathcal{H}$ of weight 3 over B. The data for the VHS arises as follows.
(1) The local system is the higher direct image $\mathcal{H}_{\mathbb{Z}} = R^3\pi_*\mathbb{Z}$ of the constant sheaf $\mathbb{Z}$ on Y. The fibers $(\mathcal{H}_{\mathbb{Z}})_b$ of this local system are the middle-dimensional integral cohomology groups $H^3(Y_b, \mathbb{Z})$. Under our assumptions, $\mathcal{H}_{\mathbb{Z}}$ is torsion free of rank 4.
(2) The decreasing Hodge filtration $F^0 \supset F^1 \supset F^2 \supset F^3$ on $\mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_B = \mathcal{H} = F^0$ originates in the natural filtration on the relative de Rham complex $(\Omega^*(Y/B), d)$. The fibers of F^s are $(F^s)_b = \oplus_{s' \geq s} H^{3-s'}(\Omega^{s'}(Y_b))$. The assumption that Y_b is Calabi-Yau implies that F^3 has rank one.

(3) The anti-symmetric polarization form $\langle\cdot,\cdot\rangle : \mathcal{H}_\mathbb{Z}\otimes\mathcal{H}_\mathbb{Z} \to \mathbb{Z}(-3)$ is induced from the cup-product on cohomology, and extended linearly to $\mathcal{H}$. Here, $\mathbb{Z}(-3) = (2\pi i)^{-3}\mathbb{Z} \hookrightarrow \mathbb{C}$ denotes the trivial, constant VHS of weight 6 on $R^6\pi_*\mathbb{Z}\otimes\mathcal{O}_B$.

We will denote as usual by ∇ the Gauss-Manin connection on $\mathcal{H}$ as a vector bundle, characterized by the property that its horizontal sections are precisely the sections of $R^3\pi_*\mathbb{C} = \mathcal{H}_\mathbb{C} = \mathcal{H}_\mathbb{Z}\otimes\mathbb{C}$. The connection preserves the polarization and together with the Hodge filtration enjoys Griffiths transversality, that is $\nabla F^s \subset F^{s-1}\otimes\Omega^1(B)$.

Now let us assume that our curve B is embedded into a larger, smooth and projective curve $\bar{B}$, and that our family can be continued to a semi-stable map $\bar{\pi} : \bar{Y} \to \bar{B}$. Fix a boundary point $a \in \bar{B}\setminus B$, and restrict to a simply connected neighborhood $\bar{U}$ of a in $\bar{B}$ such that $U = \bar{U}\setminus\{a\} \subset B$. We denote the restricted data by the same letters as above. Let $M : (\mathcal{H}_\mathbb{Z})_b \to (\mathcal{H}_\mathbb{Z})_b$ be the local monodromy operator of the local system around a. By the monodromy theorem, M is quasi-unipotent. We assume that M is in fact unipotent, and define its logarithm $N = \log M : (\mathcal{H}_\mathbb{Q})_b \to (\mathcal{H}_\mathbb{Q})_b$, where $\mathcal{H}_\mathbb{Q} = \mathcal{H}_\mathbb{Z}\otimes\mathbb{Q}$.

In this situation, the monodromy weight filtration, W_*, is the unique increasing filtration on $\mathcal{H}_\mathbb{Q}$ such that $W_{-1} = 0$, $W_6 = \mathcal{H}_\mathbb{Q}$, $NW_k \subset W_{k-2}$, and that for $k = 0,1,2,3$, N^k induces an isomorphism $\mathrm{Gr}^W_{3+k} \xrightarrow{\cong} \mathrm{Gr}^W_{3-k}$ between the graded pieces, $\mathrm{Gr}^W_k = W_k/W_{k-1}$.

Because of the monodromy, the local system $\mathcal{H}_\mathbb{Z}$ can not be continued from U across a to $\bar{U}$. However, because the monodromy is unipotent, the vector bundle $\mathcal{H} = \mathcal{H}_\mathbb{Z}\otimes\mathcal{O}_U$ has a (Deligne) canonical continuation$\bar{\mathcal{H}} \to \bar{U}$. The characteristic property of the continuation is that the flat connection acquires a first order pole at a with residue (conjugate to) $-N/(2\pi i)$. It can be explicitly constructed as follows. One picks a local coordinate z on $\bar{U}$ vanishing at the boundary point a, and then introduces on $\mathcal{H}\to U$ the "un-twisted" connection

$$\nabla^c = \nabla + \frac{N}{2\pi i}\frac{dz}{z} \tag{5.1}$$

This connection has no monodromy (the ∇^c-horizontal sections are of the form $\exp\bigl(-\frac{\log z}{2\pi i}N\bigr)g$ for ∇-horizontal section g), which allows continuation of $\mathcal{H}$ to $\bar{U}$ as a "constant" bundle. The continued connection is $\bar{\nabla} = \nabla^c - \frac{N}{2\pi i}\frac{dz}{z}$. This explicit construction will enter later in the definition of the limiting mixed Hodge structure, and the comparison with the p-adic setup.

A result of central importance for the present description of the VHS in the neighborhood of a is the nilpotent orbit theorem, which guarantees that not only $\mathcal{H}$, but in fact the entire Hodge filtration can be continued across the boundary point. We will denote it by $\bar{F}^*$.

Now, the key assumption that makes such a family $Y \to B$ interesting for us is that the distinguished boundary point a be a *point of maximal degeneration*. By definition, this means that the local monodromy operator M is unipotent of maximal rank 3. In other words, $(M - \mathrm{id})^4 = 0$, but $(M - \mathrm{id})^3 \neq 0$. The logarithm N of M is then nilpotent of rank 3.

Under the assumption that the monodromy is maximally unipotent, the monodromy weight filtration W_* on $\mathcal{H}_{\mathbb{Q}}$ pairs up with the Hodge filtration F^* on $\mathcal{H}$ to define a variation of mixed Hodge structure in a punctured neighborhood U of a in B as above [21], to which we restrict the following discussion.[11] This mixed Hodge structure is Hodge-Tate, meaning that the pure Hodge structures induced on the even graded pieces Gr^W_{2s} are constant of Hodge type (s, s), while the odd pieces Gr^W_{2s-1} all vanish. In our case, the Gr^W_{2s} are all constant of rank 1, and have Hodge structure isomorphic to $\mathbb{Z}(-s)$, see [20].

The full variation of mixed Hodge structure over U then has a composition series $\mathcal{L}_0 \to \mathcal{L}_2 \to \mathcal{L}_4 \to \mathcal{L}_6$ with successive quotients of Tate type. For example, $\mathcal{L}_2 = W_2 \otimes \mathcal{O}_U$ as a mixed Hodge structure fits into the exact sequence

$$\mathbb{Z}(0) \xrightarrow{\alpha} \mathcal{L}_2 \xrightarrow{\beta} \mathbb{Z}(-1) \tag{5.2}$$

that can be described explicitly as follows [21]. Let $g_0 = \alpha(1)$ be an integral generator of $W_0 \subset W_2$ (*i.e.*, a primitive monodromy invariant section of $\mathcal{H}_{\mathbb{Z}}$), and g_1 be a complementary integral generator of W_2. Thus, g_1 is a multi-valued ∇-horizontal section on which the monodromy acts as $N(g_1) = mg_0$ for some non-zero integer m. Specifying the two-step Hodge filtration on $\mathcal{L}_2$ is equivalent to giving a generator e^1 of $F^1 \subset \mathcal{L}_2 = (\mathbb{Z}g_0 + \mathbb{Z}g_1) \otimes \mathcal{O}_U$. The image of this section under β must generate $\mathbb{Z}(-1)$, whose Hodge filtration consists only of an F^1. So the image cannot vanish, and we can normalize e^1 such that $\beta(e^1) = 1$. By the nilpotent orbit theorem mentioned above, F^1 can be continued across a. This implies that e^1 can be chosen such that it is single-valued on U and has a limit at a. Since $\beta(g_1) = (2\pi i)^{-1}$, and $\beta(g_0) = 0$, this means that we can write

$$e^1 = (2\pi i) g_1 - m \log q \; g_0 \tag{5.3}$$

[11] We emphasize that we are not here talking about the "nilpotent orbit", which is a different VMHS obtained by extending back the limiting Hodge filtration $\bar{F}^*_a$ as a ∇^c-constant filtration F^*_{nilp} on $\mathcal{H}$. The theorem is important, but the nilpotent orbit itself will not play a role in our discussion.

for some holomorphic function q on $\bar{U}$ with a simple zero at a. A change of basis $g_1 \to g_1 + g_0$ can be compensated by a change of q by an m-th root of unity, so that the invariant characterizing the extension (5.2) is the class

$$q^m \in \mathrm{Ext}^1_{\mathrm{VMHS}}(\mathbb{Z}(-1), \mathbb{Z}(0)) = \mathcal{O}^*_U \tag{5.4}$$

The subsequent extensions can be discussed along similar lines [21], but we will only present, illustrated with explicit formulas, the final result under the two additional assumptions that (i) the monodromy is small, *i.e.*, $m = 1$, and (ii) the polarization form is unimodular. We can then complete (g_0, g_1) to a "good integral basis" $(g_s)_{s=0,1,2,3}$ of $\mathcal{H}_\mathbb{Z}$ such that $g_s \in (W_{2s} \cap \mathcal{H}_\mathbb{Z})$ (locally around some base point $b \in U$, or as multi-valued sections) and that is primitive in the sense that the matrix $I \in \mathrm{Mat}_{4\times 4}(\mathbb{Z}(-3))$ representing the polarization in this basis has the form

$$I = \frac{1}{(2\pi i)^3} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{5.5}$$

i.e., $\langle g_0, g_3 \rangle = -\langle g_3, g_0 \rangle = \langle g_1, g_2 \rangle = -\langle g_2, g_1 \rangle = (2\pi i)^{-3}$, while all other pairings vanish.

The assumption that the monodromy is small implies that we can pick g_2 such that $M(g_2) = g_2 + \kappa g_1$ with $\kappa \in \mathbb{Z}$ (*i.e.*, the coefficient of g_0 in $M(g_2)$ can be eliminated by a suitable choice of g_2). The condition that $\langle M(g_3), M(g_i) \rangle = \langle g_3, g_i \rangle$ for $i = 0, 1, 2, 3$ is then seen to imply that there exists an integer λ such that the matrix representing M in this basis takes the form

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & \kappa & 1 & 0 \\ \lambda & -\kappa & -1 & 1 \end{pmatrix} \tag{5.6}$$

Thus the matrix representing N takes the form

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{\kappa}{2} & \kappa & 0 & 0 \\ -\frac{\gamma}{12} & -\frac{\kappa}{2} & -1 & 0 \end{pmatrix} \tag{5.7}$$

where $\gamma = -12\lambda - 2\kappa$.

The extension

$$\mathbb{Z}(-2) \to \mathcal{L}_6/\mathcal{L}_2 \to \mathbb{Z}(-3) \tag{5.8}$$

dual to (5.2) w.r.t. the polarization, can then be described as follows. Following common practice, we let e^3 be a (single-valued) generator of F^3 normalized such that $\langle g_0, e^3 \rangle = 1$ (the non-vanishing of $\langle g_0, e^3 \rangle$ follows from the non-degeneracy of the polarization), and e^2 be a complementary generator of F^2 such that $e^2 = (2\pi i)^2 g_2 \bmod \mathcal{L}_2$. Under these conditions, and with the monodromy (5.6) (*i.e.*, $g_3 \to g_3 - g_2 \bmod W_2$), we must have

$$e^3 = (2\pi i)^3 g_3 + (2\pi i)^2 \log q^\vee \, g_2 \bmod \mathcal{L}_2 \tag{5.9}$$

where $q^\vee$ is a holomorphic function on $\bar{U}$ with a simple zero at the puncture. Since F^3 must be orthogonal to F^1 w.r.t. $\langle \cdot, \cdot \rangle$, we find by pairing e^3 with e^1 from (5.3) that $q^\vee = q$. Then, pairing (5.9) with g_1, we find that

$$q = \exp 2\pi i \frac{\langle g_1, e^3 \rangle}{\langle g_0, e^3 \rangle} \tag{5.10}$$

This is the standard formula for the so-called canonical coordinate on the neighborhood $\bar{U}$ of the maximal degeneracy point a. (Given that q has a simple zero at a, it is indeed a good local coordinate to use.)

The canonical coordinate is useful to describe the remainder of the mixed Hodge-Tate structure, as follows. We introduce the logarithmic vector field $\delta = \frac{d}{d \log q} = q \frac{d}{dq}$ and denote its contraction with the Gauss-Manin connection by $\nabla_t = \nabla(\delta)$. Following (5.9), we write

$$e^3 = (2\pi i)^3 g_3 + (2\pi i)^2 \log q \; g_2 - 2\pi i \mathcal{A} g_1 - \mathcal{B} g_0 \tag{5.11}$$

for some locally holomorphic functions (periods) $\mathcal{A} = (2\pi i)^2 \langle g_2, e^3 \rangle$ and $\mathcal{B} = (2\pi i)^3 \langle g_3, e^3 \rangle$. We then define $e^2 = \nabla_t e^3$ and note that $e^2 \in F^2$ by Griffiths transversality. Since $\nabla_t (g_i) = 0$, we find

$$e^2 = (2\pi i)^2 g_2 - (2\pi i) \delta\mathcal{A} g_1 - \delta\mathcal{B} g_0 \tag{5.12}$$

which shows consistency with our previous definition. Then $F^3 \perp F^2$, *i.e.*$\langle e^2, e^3 \rangle = 0$ implies

$$\mathcal{A} - \delta\mathcal{A} \, \log q - \delta\mathcal{B} = 0 \tag{5.13}$$

This relation allows us to express $\mathcal{A}$ and $\mathcal{B}$ in terms of the single function (the prepotential)

$$\mathcal{F} = \frac{1}{2}\Big(\mathcal{B} + \log q\ \mathcal{A}\Big) \tag{5.14}$$

Namely

$$\begin{aligned} \mathcal{A} &= \delta\mathcal{F} \\ \mathcal{B} &= 2\mathcal{F} - \log q\ \delta\mathcal{F} \end{aligned} \tag{5.15}$$

A short calculation then shows that

$$\nabla_t e^2 = -\delta^3\mathcal{F}\Big((2\pi i)g_1 - \log q\ g_0\Big) = -\mathcal{C}e^1 \tag{5.16}$$

where $\mathcal{C} = \delta^3\mathcal{F}$, and e^1 is from (5.3). Finally,

$$\nabla_t e^1 = -g_0 = -e^0 \tag{5.17}$$

Thus, the connection matrix in the basis (e^0, e^1, e^2, e^3) takes the form

$$\nabla_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -\mathcal{C} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5.18}$$

The most non-trivial entry of this matrix is the "normalized Yukawa coupling in canonical coordinates", $\mathcal{C} = \delta^3\mathcal{F}$. From (5.12), we recognize

$$\exp \delta\mathcal{A} \in \mathrm{Ext}_{\mathrm{VMHS}}(\mathbb{Z}(-2), \mathbb{Z}(-1)) \tag{5.19}$$

as the class of the extension $\mathbb{Z}(-1) \to \mathcal{L}_4/\mathcal{L}_0 \to \mathbb{Z}(-2)$, and $\mathcal{C}$ as the logarithmic derivative of this class [21]. The nilpotent orbit theorem guarantees that $\exp(\delta\mathcal{A})$ is holomorphic and has a zero of order κ at a. Alternatively, we can write $\mathcal{C}$ as the contraction of the third iterate of the infinitesimal period mapping

$$\langle \nabla^3 \cdot, \cdot \rangle \in \mathrm{Sym}^3 T^*B \otimes (F^3 \otimes F^3)^* \tag{5.20}$$

with $\delta^3 \otimes e^3 \otimes e^3$, which is perhaps the more frequent interpretation [20]. Namely,

$$\mathcal{C} = \langle \nabla_t^3 e^3, e^3 \rangle \tag{5.21}$$

All this data can be conveniently summarized in terms of the expansion of the prepotential (viewed as a locally holomorphic function on U) in the canonical coordinate q. The monodromy (5.6) dictates that the prepotential be of the form

$$\mathcal{F} = \frac{\kappa}{6}(\log q)^3 - \frac{\kappa}{4}(2\pi i)(\log q)^2 - \frac{\gamma}{24}(2\pi i)^2 \log q + \bar{\varphi} \tag{5.22}$$

where by the nilpotent orbit theorem $\bar{\varphi}$ is holomorphic on $\bar{U}$. The periods are

$$\begin{aligned}
\langle g_0, e^3 \rangle &= 1 \\
(2\pi i)\langle g_1, e^3 \rangle &= \log q \\
(2\pi i)^2 \langle g_2, e^3 \rangle = \mathcal{A} = \delta\mathcal{F} &= \frac{\kappa}{2}(\log q)^2 - \frac{\kappa}{2}(2\pi i)\log q - \frac{\gamma}{24}(2\pi i)^2 + \delta\bar{\varphi} \\
(2\pi i)^3 \langle g_3, e^3 \rangle = \mathcal{B} = 2\mathcal{F} - \log q\, \delta\mathcal{F} &= -\frac{\kappa}{6}(\log q)^3 - \frac{\gamma}{24}(2\pi i)^2 \log q + 2\bar{\varphi} - \log q\, \delta\bar{\varphi}
\end{aligned} \tag{5.23}$$

and the Yukawa coupling

$$\mathcal{C} = \kappa + \delta^3 \bar{\varphi} \tag{5.24}$$

5.2 The limiting mixed Hodge structure

To describe in greater detail the relation between the canonical coordinate q and a general local coordinate z, we return to the continuation of the Hodge bundle discussed around eq. (5.1).

We repeat that the local system $\mathcal{H}_{\mathbb{Z}}$ can not be continued to $\bar{U}$ because of the monodromy $N : (\mathcal{H}_{\mathbb{Q}})_b \to (\mathcal{H}_{\mathbb{Q}})_b$ ($b \in U$). However, in conjunction with the choice of the local coordinate z, the local system can be used to induce an integral structure on the fiber $V = \bar{\mathcal{H}}_a$ of the continued Hodge bundle at a. If g is a local section of $\mathcal{H}_{\mathbb{Z}} \subset \mathcal{H}$ away from the puncture, the combination $\bar{g} = \exp\bigl(-\frac{\log z}{2\pi i} N\bigr) g$ is horizontal with respect to the untwisted connection $\nabla^c = \nabla + \frac{N}{2\pi i}\frac{dz}{z}$. It thus becomes a section of $\bar{\mathcal{H}}$ in a neighborhood of a, and we put

$$\Psi_z(g) = \exp\Bigl(-\frac{\log z}{2\pi i} N\Bigr) g \Big|_a \in V \tag{5.25}$$

Putting $V_{\mathbb{Z}} = \mathrm{Im}(\Psi_z)$, the isomorphism $V \cong V_{\mathbb{Z}} \otimes \mathbb{C}$ defines an integral structure on V. (Following common practice, we partially suppress the dependence on z in the notation.)

Using Ψ_z, we can also carry the monodromy N and associated filtration W_* over to $V_{\mathbb{Q}} = V_{\mathbb{Z}} \otimes \mathbb{Q}$. Together with the Hodge filtration $\bar{F}^*_a$ (which we recall does not depend on the choice of z), this defines a mixed Hodge structure on $V = \bar{\mathcal{H}}_a$, known as the limiting mixed Hodge structure (LMHS). It will play a central role in the following, so we illustrate it with a few formulae.

Starting from (5.23), we can express the (multi-valued) basis of integral sections (g_s) $(s = 0, 1, 2, 3)$ in terms of the sections $(e^s)_{s=0,1,2,3}$ of $\mathcal{H}$, the prepotential, and $\log q$:

$$\begin{aligned} g_0 &= e^0 \\ (2\pi i) g_1 &= e^1 + \log q\, e^0 \\ (2\pi i)^2 g_2 &= e^2 + \delta^2 \mathcal{F}\, e^1 + \delta \mathcal{F}\, e^0 \\ &= e^2 + \Big(\kappa \log q - \frac{\kappa}{2}(2\pi i) + \delta^2 \bar{\varphi}\Big) e^1 \\ &\qquad\qquad + \Big(\frac{\kappa}{2}(\log q)^2 - \frac{\kappa}{2}(2\pi i)\log q - \frac{\gamma}{24}(2\pi i)^2 + \delta\bar{\varphi}\Big) e^0 \\ (2\pi i)^3 g_3 &= e^3 - \log q\, e^2 + (\delta \mathcal{F} - \log q\, \delta^2 \mathcal{F}) e^1 + (2\mathcal{F} - \log q\, \delta \mathcal{F}) e^0 \\ &= e^3 - \log q\, e^2 + \Big(-\frac{\kappa}{2}(\log q)^2 - \frac{\gamma}{24}(2\pi i)^2 + \delta\bar{\varphi} - \log q\, \delta^2 \bar{\varphi}\Big) e^1 \\ &\qquad\qquad + \Big(-\frac{\kappa}{6}(\log q)^3 - \frac{\gamma}{24}(2\pi i)^2 \log q + 2\bar{\varphi} - \log q\, \delta\bar{\varphi}\Big) e^0 \end{aligned} \tag{5.26}$$

(Using (5.18), it is easy to check that the g_s are horizontal.) Then, in correspondence with our (multi-valued) basis (g_s), we introduce

$$\bar{g}_s = \exp\Big(-\frac{\log z}{2\pi i} N\Big) g_s \tag{5.27}$$

which by construction can be continued as sections of $\bar{\mathcal{H}} \to \bar{U}$. This untwisting amounts simply to the replacement of $\log q$ with $\log q/z$ in (5.26),[12] so that we have

$$\begin{aligned} \bar{g}_0 &= e^0 \\ (2\pi i) \bar{g}_1 &= e^1 + \log \frac{q}{z}\, e^0 \\ (2\pi i)^2 \bar{g}_2 &= e^2 + \Big(\kappa \log \frac{q}{z} - \frac{\kappa}{2}(2\pi i) + \delta^2 \bar{\varphi}\Big) e^1 \\ &\qquad + \Big(\frac{\kappa}{2}(\log \frac{q}{z})^2 - \frac{\kappa}{2}(2\pi i)\log \frac{q}{z} - \frac{\gamma}{24}(2\pi i)^2 + \delta\bar{\varphi}\Big) e^0 \\ (2\pi i)^3 \bar{g}_3 &= e^3 - \log \frac{q}{z}\, e^2 + \Big(-\frac{\kappa}{2}(\log \frac{q}{z})^2 - \frac{\gamma}{24}(2\pi i)^2 + \delta\bar{\varphi} - \log \frac{q}{z}\, \delta^2 \bar{\varphi}\Big) e^1 \\ &\qquad + \Big(-\frac{\kappa}{6}(\log \frac{q}{z})^3 - \frac{\gamma}{24}(2\pi i)^2 \log \frac{q}{z} + 2\bar{\varphi} - \log \frac{q}{z}\, \delta\bar{\varphi}\Big) e^0 \end{aligned} \tag{5.28}$$

[12]In contrast, the formation of the nilpotent orbit (see footnote 11 on page 38) amounts to keeping *only* the logarithmic terms.

And indeed, since q and z both vanish to first order at a, $\lim q/z = c \neq 0$ exists, which leads to the limiting period matrix

$$\Pi = (\Pi_{st}) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\log c}{2\pi i} & \frac{1}{2\pi i} & 0 & 0 \\ \frac{\kappa}{2}\frac{(\log c)^2}{(2\pi i)^2} - \frac{\kappa}{2}\frac{\log c}{2\pi i} - \frac{\gamma}{24} & \kappa\,\frac{\log c}{(2\pi i)^2} - \frac{\kappa}{2}\frac{1}{2\pi i} & \frac{1}{(2\pi i)^2} & 0 \\ -\frac{\kappa}{6}\frac{(\log c)^3}{(2\pi i)^3} - \frac{\gamma}{24}\frac{\log c}{2\pi i} + \frac{2\zeta}{(2\pi i)^3} & \frac{\kappa}{2}\frac{(\log c)^2}{(2\pi i)^3} - \frac{\gamma}{24}\frac{1}{2\pi i} & -\frac{\log c}{(2\pi i)^3} & \frac{1}{(2\pi i)^3} \end{pmatrix} \tag{5.29}$$

with $\lim_{z\to 0}(\bar{g}_s - \Pi_{st}e^t) = 0$. Here $\zeta = \lim_{z\to 0}\bar{\varphi}$ is a complex number and the only entry that is not determined by considerations of local monodromy.

5.3 Algebraic cycles and extensions

The starting point of this section was a smooth family $\pi : Y \to B$ of Calabi-Yau threefolds over a quasi-projective complex curve B, admitting a semi-stable compactification $\bar{\pi} : \bar{Y} \to \bar{B}$, with a distinguished boundary point $a \in \bar{B}\setminus B$ of maximal degeneration. A natural extension of this situation, considered in [5], is by a complex algebraic surface $i : C \to Y$, with the following properties:

(i) The composition $\pi \circ i : C \to B$ is a semi-stable flat family of curves, and the situation admits a semi-stable compactification over $\bar{B}$.

(ii) On a dense open subset, $i : \overset{\circ}{C} \to Y$ is a smooth immersion, and $\pi \circ i : \overset{\circ}{C} \to \overset{\circ}{B}$ is a smooth family. In other words, in the generic member $Y_b := \pi^{-1}(b)$ of the family, $i_b : C_b = (\pi \circ i)^{-1}(b) \to Y_b$ is an immersed curve. It is important that we allow the fibers C_b to be reducible. We assume that the irreducible components of C_b are homologically equivalent to each other in Y_b. In other words, writing $C_b = \cup_k C_{b,k}$, we assume that $[i_b(C_{b,k})] - [i_b(C_{b,k'})] = 0 \in H_2(Y_b, \mathbb{Z})$.

(iii) There exists an embedded surface $\bar{C}_0 \hookrightarrow \bar{Y}$ such that the composition $\bar{C}_0 \to \bar{B}$ is a smooth family, with irreducible fibers. We denote these fibers by $C_{b,0}$ and assume that for generic b, some fixed positive multiple of $C_{b,0}$ is homologically equivalent to the components of C_b in Y_b.

In the following, we'll pretend that this multiple is 1. Moreover, we shall restrict to a simply connected neighborhood $\bar{U}$ of a such that $U = \bar{U} \setminus \{a\} \subset \overset{\circ}{B}$ and such that for $b \in U$, the components $C_{b,k}$ are all smooth. We allow ourselves to drop a subset of the components of $C \to U$ (wihtout new notation), in order to satisfy conditions further specified below. We also assume that $C_{a,0}$ is smooth.

To this configuration $(Y, C) \to U$ is now attached a variation of mixed Hodge structure $\hat{\mathcal{H}}$, which can be thought of as an extension of a pure Hodge structure $\mathcal{I}$ of weight 4 by the pure Hodge structure $\mathcal{H}$ of weight 3 attached to $Y \to U$. It looks as follows.

(1) The local system $\mathcal{I}_{\mathbb{Z}} = (\pi \circ i)_* \mathbb{Z}$ is free of rank equal to the number of components of C_b. Its fibers are $H^0(C_b, \mathbb{Z}) \cong H_2(C_b, \mathbb{Z})$ by Poincaré duality on C_b. The extension of local systems

$$\mathcal{H}_{\mathbb{Z}} \to \hat{\mathcal{H}}_{\mathbb{Z}} \to \mathcal{I}_{\mathbb{Z}} \tag{5.30}$$

can be identified at each fiber as the exact sequence in relative homology

$$0 \to H_3(Y_b, \mathbb{Z}) \to \check{H}_3(Y_b, C_b, \mathbb{Z}) \to H_2(C_b, \mathbb{Z}) \to 0 \tag{5.31}$$

Here, we have "based" the relative homology group by letting

$$\check{H}_3(Y_b, C_b, \mathbb{Z}) := H_3(Y_b, C_b \cup C_{b,0}, \mathbb{Z}) \tag{5.32}$$

and we have identified $H_2(C_b, \mathbb{Z}) \cong \mathrm{Ker}(H_2(C_b \cup C_{b,0}, \mathbb{Z}) \to H_2(Y_b, \mathbb{Z}))$ in an obvious way (*i.e.*, by using that $C_{b,0}$ is homologous to all the irreducible components of C_b). The identification of the extension of local systems with (5.31) is induced by Poincaré duality from the exact sequence in cohomology

$$0 \to H^3(Y_b, \mathbb{Z}) \to \check{H}^3(Y_b \setminus C_b, \mathbb{Z}) \to H^0(C_b, \mathbb{Z}) \to 0 \tag{5.33}$$

(2) Although born as an H^0, $\mathcal{I} = \mathcal{I}_{\mathbb{Z}} \otimes \mathcal{O}_U$ in fact has weight 4 as a consequence of the embedding in Y_b, and is purely of Hodge type $(2,2)$. (By Poincaré duality on Y_b, the components of C_b are represented by 4-cochains or "currents" with delta-function support on the $C_{b,k}$). As a consequence, the Hodge filtration $\hat{F}^*$ on $\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\mathbb{Z}} \otimes \mathcal{O}_U$ satisfies $\hat{F}^s / \hat{F}^{s+1} = F^s / F^{s+1}$ except for $s = 2$, and we have $\hat{F}^2 / F^2 = \mathcal{I}$.

(3) The polarization does not extend in a canonical way to all of $\hat{\mathcal{H}}$. However, given that $F^2 \perp F^2$, and the above properties of the Hodge filtration, it makes sense to extend the pairing with F^2 as a bilinear form from $\mathcal{H} \times F^2$ to $\hat{\mathcal{H}} \times F^2$. (Namely, we define the pairing to be 0 on $(\hat{\mathcal{H}}/\mathcal{H}) \times F^2 = (\hat{F}^2/F^2) \times F^2$.) Doing so[13] allows us to identify extensions of Hodge structure in $\mathrm{Ext}^1_{\mathrm{VMHS}}(\mathbb{Z}(-2), \mathcal{H})$ with $J \cong (F^2)^* / \mathcal{H}_{\mathbb{Z}}$ (Abel-Jacobi map).

Needless to say, the Gauss-Manin connection extends to $\hat{\mathcal{H}}$ with horizontal sections $\hat{\mathcal{H}}_{\mathbb{C}}$, and Griffiths transversality $\nabla \hat{F}^s \subset \hat{F}^{s-1} \otimes \Omega_U$.

[13] Geometrically, this is accomplished by integrating three-forms representing elements of F^2 against three-chains of boundary $C_{b,k} - C_{b,0}$, see [11, 5].

As before, the monodromy theorem guarantees that the extended monodromy operator $\hat{M} : \hat{\mathcal{H}}_{\mathbb{Z}} \to \hat{\mathcal{H}}_{\mathbb{Z}}$ is quasi-unipotent. It preserves $\mathcal{H}_{\mathbb{Z}}$ and agrees with M there. On the quotient, $\mathcal{I}_{\mathbb{Z}}$, the monodromy is of finite order[14], which we denote by r. In other words, r is the smallest positive integer such that $\hat{M}^r$ is unipotent, and we define $\hat{N} = \log \hat{M}^r$. Note that $\hat{N}|_{\mathcal{H}_{\mathbb{Q}}} = rN$. We assume (possibly after dropping some of the fibers) that all the orbits of $\hat{M}$ on $\mathcal{I}_{\mathbb{Z}}$ are of the same order, and denote the number of orbits by $\hat{d}$.

We may trivialize this finite monodromy of $\mathcal{I}_{\mathbb{Z}}$ by passing to an r-fold cover $(\hat{U} \to U) \subset (\bar{\hat{U}} \to \bar{U})$ branched at a. We won't introduce new notation for those parts of the data that pull back trivially to $\hat{U}$, but for the local system $\hat{\mathcal{I}}_{\mathbb{Z}}$ whose rank drops from $r\hat{d}$ to $\hat{d}$. The extension of variation of Hodge structures is now of the form

$$\mathcal{H} \xrightarrow{\text{id}} \hat{\mathcal{H}} \xrightarrow{\beta} \mathbb{Z}(-2)^{\hat{d}} \tag{5.34}$$

To describe it explicitly, we first extend our basis $(g_s)_{s=0,\dots,3}$ with $g_s \in W_{2s} \subset \hat{W}_{2s}$ and monodromy (5.6) by a collection of complementary generators $(h_k)_{k=1,\dots,\hat{d}}$. We pause to explain certain ("torsion") subtleties that arise in the choice of the h_k.

Because $\hat{N}$ projects to 0 on $\hat{\mathcal{I}}_{\mathbb{Q}}$, the extension of the monodromy filtration is "concentrated in the middle". Namely, $\hat{W}_*$ satisfies $\hat{W}_k = W_k$ for $k < 3$, $\hat{W}_k/\hat{W}_3 = W_k/W_3$ for $k > 3$, and $\hat{W}_3/\hat{W}_2 \cong \hat{\mathcal{I}}_{\mathbb{Q}}$. However, we can not necessarily assume that the extending generators h_k are both integral generators of $\hat{\mathcal{H}}_{\mathbb{Z}}/\mathcal{H}_{\mathbb{Z}}$ *and* contained in $\hat{W}_3 \subset \hat{\mathcal{H}}_{\mathbb{Q}}$: The image of integral extending generators under monodromy might be contained in W_2, and obtaining generators of $\hat{W}_3$ might require a change of basis that is rational but not in general integral. This point was emphasized in [22], and explicit examples illustrating the phenomenon can be found in [23]. For simplicity, we will here assume that the h_k are both in $\hat{W}_3$ and that their images under β generate $(2\pi i)^{-2}\mathbb{Z}^{\hat{d}} \subset \hat{\mathcal{I}} = \hat{\mathcal{I}}_{\mathbb{Z}} \otimes \mathcal{O}_U$.

This assumption does however not remove the subtleties completely. Monodromy acts by $\hat{N}(h_k) = a_k g_0 \in \hat{W}_1 = W_0$, with $a_k \in \mathbb{Z}$. The h_k are canonical up to the addition of integral multiples of g_1 and g_0, which changes the integers a_k by integral multiples of $r \cdot m$. (Recall that $N(g_1) = mg_0$, and so $\hat{N}(g_1) = rm\, g_0$.) Thus, even assuming that the monodromy is small ($m = 1$), we cannot take $a_k = 0$ in general. We do not wish to assume $r = 1$ because it would exclude most of the examples of [5]. (Although the proofs are not significantly more complicated without the assumption.)

In opposition, we extend our basis $(e^s)_{s=0,\dots,3}$ with $e^s \in F^s \subset \hat{F}^s$ by a collection

[14] This follows from the nilpotent orbit theorem because $\mathcal{I}$ is Hodge-Tate.

of complementary generators $(f_k)_{k=1,\ldots,\hat{d}}$ of $\hat{F}^2$ that agree with $((2\pi i)^2 h_k) \bmod \mathcal{H}$, in other words, that $(2\pi i)^2\beta(h_k) = \beta(f_k)$. The Hodge structure is specified by lifting this relation between the h_k and f_k to $\hat{\mathcal{H}}$. In this process, the F^2-part of f_k remains arbitrary, so that we can choose the f_k such that (*cf.*, (5.26)),

$$(2\pi i)^2 h_k = f_k + \mathcal{V}_k e^1 + \mathcal{W}_k e^0 \tag{5.35}$$

where the $\mathcal{W}_k$ and $\mathcal{V}_k$ are locally holomorphic functions on U, and the nilpotent orbit theorem guarantees that the f_k continue across a.

To analyze the behaviour of (5.35) at the point a of maximal degeneration more precisely, we use the canonical coordinate q on $\bar{U}$ (see eq. (5.10)), or rather, its lift $q = \hat{q}^r$ to $\bar{\hat{U}}$. Since $\nabla_t h_k = 0 = \nabla_t e^0$, $\nabla_t e^1 = -e^0$, and $\nabla_t f_k \in \hat{F}^1$, we must have

$$\mathcal{V}_k = \delta \mathcal{W}_k \tag{5.36}$$

and

$$\nabla_t f_k = -\delta \mathcal{V}_k \, e^1 = -\delta^2 \mathcal{W}_k \, e^1 \tag{5.37}$$

where $\delta = q\frac{d}{dq}$ as before. The monodromy $\hat{N}(h_k) = a_k g_0 = a_k e^0$ implies that $\mathcal{W}_k$ must be of the form (*cf.*, (5.22))

$$\mathcal{W}_k = a_k (2\pi i) \log q^{1/r} + \bar{w}_k \tag{5.38}$$

where $\bar{w}_k$ is single valued on $\hat{U}$, and the nilpotent orbit theorem implies that it continues holomorphically to $\bar{\hat{U}}$.

In eq. (5.35), the combination

$$\hat{\nu}_k = \mathcal{V}_k e^1 + \mathcal{W}_k e^0 \in \mathcal{H} \cap W_2 \tag{5.39}$$

can be viewed as the normal function that generally classifies extensions of Hodge structures by algebraic cycles of this type, see [11]. More precisely, the normal function is the image ν_k of $\hat{\nu}_k$ in the intermediate Jacobian $J = \mathcal{H}_{\mathbb{Z}} \backslash \mathcal{H}/F^2$. As explained in [22], the maximal degeneration of $\mathcal{H}$ at a makes the lift (5.39) well-defined modulo $\mathcal{H}_{\mathbb{Z}} \cap W_2$ instead of $\mathcal{H}_{\mathbb{Z}}$. We also note that in terms of the pairing on $\hat{\mathcal{H}} \times F^2$ discussed above, we have

$$\mathcal{W}_k = (2\pi i)^2 \langle h_k, e^3 \rangle \tag{5.40}$$

which is the definition of the "superpotential" (the truncated normal function) used in [11].

Lastly, the (Griffiths) infinitesimal invariant, which is the analogue of the Yukawa coupling (5.21) characterizing the variation of Hodge structure locally is the combination[15]

$$\mathcal{D}_k = \langle \nabla_t{}^2 \hat{\nu}_k, e^3 \rangle = \delta^2 \mathcal{W}_k \tag{5.41}$$

Finally, if we choose as in subsection 5.2 a general local coordinate z vanishing to first order at a, we can define the limiting Hodge structure by untwisting the local system à la (5.25), (5.28),

$$(2\pi i)^2 \bar{h}_k = f_k + \Big(\frac{a_k}{r}(2\pi i) + \delta\bar{w}_k\Big)e^1 + \Big(a_k(2\pi i)\log\frac{q^{1/r}}{z^{1/r}} + \bar{w}_k\Big)e^0 \tag{5.42}$$

5.4 Integrality statements

We started the discussion with a family of Calabi-Yau varieties $Y \to B$ over a complex curve B, admitting a semi-stable compactification $\bar{Y} \to \bar{B}$. Assuming that the boundary point $a \in \bar{B} \setminus B$ is a point of maximal degeneration, we reviewed how the variation of Hodge structure is encoded locally in a set of holomorphic functions of a local coordinate z. Among these functions are the canonical coordinate q and the normalized Yukawa coupling $\mathcal{C}$. We then extended this family of varieties by a family of algebraic cycles $C \subset Y$ varying continuously with Y over B, and reviewed how the associated extension of Hodge structure can be encoded in another holomorphic function, the infinitesimal invariant $\mathcal{D}_k$. (Here, k is an index labelling components of the generic fiber of C, see previous subsection for precise definitions.)

We now add the assumption that the maximal degeneration is defined over the integers, and that z is an integral coordinate on B (we give the precise definitions momentarily). This implies that the functions q and $\mathcal{C}$, when expanded around $z = 0$, are power series with rational coefficients. It was proven in [1, 3] that for all primes $p > 3$ for which the reduction mod$p$ is smooth, the canonical coordinate $q(z)$ has p-integral coefficients, and that when the normalized Yukawa coupling is re-written as a power series in q, the coefficients satisfy congruence relations equivalent to the statement that the non-constant part of $\mathcal{C}$ is the third logarithmic derivative of a 3-function at p in the sense of section 2.

[15]Formally, the Griffiths infinitesimal invariant is the class of $\nabla\hat{\nu} \in F^1$ in $\mathrm{Ker}(\nabla\wedge)/(\mathrm{Im}(\nabla)$, where the equivalence accounts for the a priori ambiguity of f_k in F^2. Maximal degeneration provides a canonical lift, and the expression (5.41) is this "normalized canonical representative of the Griffiths infinitesimal invariant in canonical coordinates".

In [4], it was shown that under the assumption that the degeneration of the cycle C is also defined over the integers, the infinitesimal invariant is the second derivative of a 2-function at p. The main result of the remainder of this paper is the generalization to the situation in which the cycle is not defined over $\mathbb{Q}$.

To give precise definitions and state the results, we liberate the notation from the previous subsection, and rephrase the assumptions in scheme-theoretic language. The semi-stable map of complex algebraic varieties $\bar{\pi} : \bar{Y} \to \bar{B}$ can be viewed as a semi-stable morphism of schemes over $\operatorname{Spec}\mathbb{C}$. We assume that the field of definition of this morphism is $\mathbb{Q}$, which means that there exists a semi-stable morphism $\bar{\pi}_{\mathbb{Q}} : \bar{Y}_{\mathbb{Q}} \to \bar{B}_{\mathbb{Q}}$ over $\operatorname{Spec}\mathbb{Q}$ together with an isomorphism $\bar{\pi}_{\mathbb{Q}} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{C} \cong \bar{\pi}$. We mention that while it is in general not easy to identify the (smallest) field of definition of any given scheme, from the point of view of $\mathbb{Q}$, the important property of $\bar{Y}$ is that $\bar{Y}_{\mathbb{Q}} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\bar{\mathbb{Q}}$ remains irreducible as a scheme over $\mathbb{Q}$, where $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. We also assume that the boundary point a is rational, which means that it is the complexification of a section $a_{\mathbb{Q}} : \operatorname{Spec}\mathbb{Q} \to \bar{B}_{\mathbb{Q}}$.

An important consequence of these assumptions is that the singular fiber of the family, $\bar{Y}_a = \bar{Y} \times_{\bar{B}} \operatorname{Spec}\mathbb{C}$ (where $\operatorname{Spec}\mathbb{C} \xrightarrow{a} \bar{B}$) is also defined over $\mathbb{Q}$. Furthermore, letting z be a rational local coordinate on $\bar{B}$ vanishing at a (namely, given an identification of a neighborhood of $a_{\mathbb{Q}}$ in $\bar{B}_{\mathbb{Q}}$ with $\operatorname{Spec}\mathbb{Q}[[z]]$), the localization of $\bar{Y}$ at a, $\bar{Y} \times_{\bar{B}} \operatorname{Spec}\mathbb{C}[[z]]$ is also defined over $\mathbb{Q}$.[16]

We can not, in general, maintain these assumptions after extension by the algebraic cycle $(i : C \to Y) \subset (\bar{C} \to \bar{Y})$. Following the assumptions of subsection 5.3, and with similar notational conventions, we denote the localization of $(\bar{Y}, \bar{C})$ to the formal neighborhood $\bar{D} = \operatorname{Spec}\mathbb{C}[[z]]$ of a by the same letters. This formal neighborhood (or its underlying rational analogue) takes the place of the complex neighborhood $\bar{U}$ from subsection 5.3. We allow the finite part of the Stein factorization of the map $\bar{C} \to \bar{D}$ to be branched at a with ramification index r, and denote the r-fold cover $z = \hat{z}^r$ of $\bar{D}$ by $\hat{\bar{D}} = \operatorname{Spec}\mathbb{C}[[\hat{z}]]$. This corresponds to the monodromy of order r on $\mathcal{I}_{\mathbb{Z}}$ from subsection 5.3. In that subsection, we had allowed the fibers of $C \to \hat{U}$ to have $\hat{d} \geq 1$ irreducible components. We now specify that $\bar{C} \to \hat{\bar{D}}$ should be the complexification of a scheme that is irreducible over $\operatorname{Spec}\mathbb{Q}$, but whose field of definition K (the smallest

[16] It is an interesting question whether all maximal degenerations of complex algebraic families of Calabi-Yau 3-folds are necessarily defined over $\mathbb{Q}$. Mirror symmetry seems to strongly suggest that this is true, but we cannot imagine any reason for this statement from the point of view of the B-model.

intermediate field such that further extension leaves components of the fiber irreducible) can be a finite extension of $\mathbb{Q}$. We emphasize that K need not be Galois over $\mathbb{Q}$ and that its degree, $d = [K : \mathbb{Q}]$ need not equal $\hat{d}$. Rather, the complex cycles of subsection 5.3 each corresponds to a different embedding $K \hookrightarrow \mathbb{C}$, which could be permuted by the monodromy of order r around a (see [5, 23] for examples of this phenomenon).

In the algebraic setup, the irreducibility of the cycle implies that, after localization to a, the extension classes $\mathcal{V}_k$ from (5.35) for different values of k fit together to a single formal power series on the "extended disk" $\hat{\bar{D}}^K = \operatorname{Spec} K[[\hat{z}]]$. The same is true for their derivatives, $\mathcal{D}_k$, but *not* the superpotential $\mathcal{W}_k$ itself, which generically includes (apart from the log-term) a (conjecturally) transcendental constant, see [22, 23]. We shall denote these functions by $\mathcal{V}$, $\mathcal{D}$. We continue to denote by q, $\mathcal{C}$ the localization of the Hodge theoretic extension classes from subsection 5.1 to the formal neighborhood of a, possibly pulled back to $\hat{\bar{D}}$. We have

Lemma 19. *$q \in z\mathbb{Q}[[z]] \subset z\mathbb{C}[[z]]$, $\mathcal{C} \in \mathbb{Q}[[q]] \subset \mathbb{C}[[z]]$, and $\mathcal{D} \in \hat{q}K[[\hat{q}]] \Subset \hat{z}\mathbb{C}[[\hat{z}]]$.*

Remark 20. The first of these statements depends crucially on the assumption of smallness of monodromy over $\mathbb{Z}$, *i.e.*, $m = 1$ in (5.4). (In general, we can only prove that $q^m \in z\mathbb{Q}[[z]]$, see [3].) An equivalent statement is $q'(0) \in \mathbb{Q}$, which we will assume in the following.

It might happen that $\mathcal{D}$ has coefficients in a subfield of K, for instance if the algebraic cycle is rationally equivalent to a cycle defined over $\mathbb{Q}$.

In order to formulate the main integrality statements, we have to continue our families from the generic point $\operatorname{Spec}\mathbb{Q} \hookrightarrow \operatorname{Spec}\mathbb{Z}$ to some larger set of "good" primes. We exclude any primes at which $Y_{\mathbb{Q}} \to D_{\mathbb{Q}}$ or $C_{\mathbb{Q}} \to D_{\mathbb{Q}}$ are not smooth, or their compactifications are not semi-stable, all divisors of r, as well as those of the discriminant of the extension $K/\mathbb{Q}$. In order to apply the p-adic Hodge theory (section 6), we also need to exclude all prime $p \leq \dim(Y/B) + 1$ (in our case, this excludes 2 and 3). We now let $(N) \subset \operatorname{Spec}\mathbb{Z}$ be the union of all these excluded points and assume that there exist schemes over $S := \operatorname{Spec}\mathbb{Z}[N^{-1}] = \operatorname{Spec} Z \setminus (N)$ whose base change to $\mathbb{C}$ gives rise to the complex varieties from above. We also assume that the coordinate z has been chosen such that it is integral everywhere on S.

The following result was proven in [1, 3].[17]

[17] A comment for readers on return from one of the later sections: The theorems state that the power series q, $\mathcal{C}$, $\mathcal{D}$ as defined via *complex* Hodge theory have the indicated integrality property,

Theorem 21. *For all primes $(p) \in S$ (i.e., those with $(p, N) = 1$), we have $q \in z\mathbb{Z}_p[[z]]$, and there exists a formal power series $\psi_p(q) \in q\mathbb{Z}_p[[q]]$ such that*

$$\mathcal{C}(q^p) - \mathcal{C}(q) = \delta^3 \psi_p \tag{5.43}$$

In the terminology of section 2, the non-singular part of the prepotential (5.22) *(without the constant term) is a 3-function at p.*

In this paper, we extend the generalization of the integrality result [4] to the situation with a non-trivial residue field.

Theorem 22. *For all primes $(p, N) = 1$, there exists a formal power series $\omega_p(q) \in \hat{q}\mathcal{O}_p[[\hat{q}]]$ such that, in the notation of section 2, we have*

$$\mathrm{Frob}_p \, \mathcal{D} - \mathcal{D} = \delta^2 \omega_p \tag{5.44}$$

In other words, the non-singular part of the superpotential (without the constant term) is a 2-function at p with coefficients in $K \subset K_p$.

Remark 23. Theorem 22 provides an answer of sorts to the question of existence of analytic 2-functions that we have posed in subsection 2.2. To explain this, we recall that the Picard-Fuchs operator[18] annihilating the periods, which in the canonical coordinate takes the form

$$\mathcal{P} := \delta^2 \mathcal{C}^{-1} \delta^2 \tag{5.45}$$

is a differential operator with algebraic coefficients when written in a global algebraic coordinate z over B. Application to the superpotential (5.40), (5.41) does not return 0 in general. Let us define instead for each k,

$$j_k := \frac{d}{dz} \big(\delta \mathcal{C}^{-1} \delta^2 \mathcal{W}_k \big) \tag{5.46}$$

Then the j_k are local power series that determine $\mathcal{W}_k$ up to periods, *i.e.*, up to the constant and logarithmic terms in (5.38). On general grounds, the j_k are the local expansions of algebraic functions over the moduli space, explicitly calculated in [11, 5]. In other words, the $\mathcal{W}_k$ are at the same time (explicitly) 2-functions in the variable q, and (explicitly) analytic in z. We believe that this deserves further attention.

though the proofs depend on p-adic methods. The outcome of section 6 is that the complex power series essentially agree with the p-adic ones. Section 7 establishes integrality of the p-adic series. We felt that carrying the weight of subscripts $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}_p$ offered additional clarity only rather temporarily.

[18] As before, we only make statements for one-dimensional moduli spaces, but they all admit fairly obvious generalizations.

6 The (extended) p-adic B-model

The purpose of this section is to review the strategy of the integrality proofs of [1, 3, 4], and to explain some background on p-adic Hodge theory and its comparison with the more familiar complex Hodge theory.

Theorems 21 and 22 make integrality statements about formal power series that are attached to families of complex algebraic varieties $(Y, C) \to B$ by a Hodge theoretic construction described in subsections 5.1 and 5.3. The relevant assumption is that the complex algebraic varieties in fact come from a more abstract scheme that provides an underlying algebraic integral structure, locally around the point of maximal degeneration.

It is important to note that these power series cannot entirely be attached to the algebraic structure alone. One of the two main ingredients of the variation of Hodge structure is the topological integral structure (the local systems $\mathcal{H}_{\mathbb{Z}} \subset \hat{\mathcal{H}}_{\mathbb{Z}}$), and to define this topological integral structure, we require the fine topology of the complex numbers. In the way we have explained, the power series arise as "periods" during the pairing between the algebraic and topological cohomology groups (see eq. 5.23). Adding the algebraic cycle leads to an extension of the "local period ring" from $\mathbb{Q}[[z]]$ to $K[[\hat{z}]]$.

The clue for an explanation of the integrality is included in the formulation of the theorems, via their reference to a ("good") prime number (p) and an identification of the periods as power series with p-adic coefficients. The main idea of [1] is to relate these power series with the "periods" in the p-adic world, *i.e.*, with the coefficients involved in the comparison between the algebraic (deRham) cohomology and topological (étale or crystalline) cohomology. The p-adic integrality of the power series then follows from the properties of these p-adic theories. We will present the relevant calculations in the next section, and here attempt to convey an idea of the underlying concepts.

The crux is that a priori, the complex and p-adic definitions of the periods have little to do with each other: As we just mentioned, the topological integral structure in the complex setting comes from viewing the algebraic varieties as complex (in fact, real) topological manifolds. In contrast, in the p-adic setting, the role of the topological integral structure is played by the Frobenius symmetry acting on the algebraich cohomology groups. (We will explain this in more detail below.) Thus, a major step of the integrality proof is to show that the functions defined in the complex and p-adic setting in fact agree.

The identification between the complex and p-adically defined functions in turn divides in two parts. One first verifies that the functions satisfy the same differential relations in the neighborhood of the point of maximal degeneration. This is essentially a consequence of the fact that the differential equation satisfied by the periods (the homogeneous and inhomogeneous Picard-Fuchs equations) have rational and algebraic coefficients respectively. Then, one remains with checking that the initial conditions at the point of maximal degeneration also agree. This part (which is technically the hardest and will not be reviewed here) involves a comparison between the complex and p-adic Hodge structures in the strict degeneration limit.

6.1 Logarithmic de Rham cohomology over $\mathbb{Q}$

To begin with, we isolate those parts of the complex Hodge theory that can be defined purely algebraically, and which we can then complete to the p-adic setting (instead of to $\mathbb{C}$). For our purposes, it will be sufficient to work over the formal disk $\bar{D} = \operatorname{Spec}\mathbb{Q}[[z]]$, thought of as a rational neighborhood of $a = (z)$ in $\bar{B}$ as explained above. Thus, $\bar{\pi} : \bar{Y} \to \bar{D}$ is a semi-stable morphism such that $\pi : Y \to D$ (with $D = \operatorname{Spec}\mathbb{Q}[[z, z^{-1}]] = \bar{D}\backslash a$) is a smooth family of Calabi-Yau schemes of relative dimension 3, $a \cong \operatorname{Spec}\mathbb{Q} \hookrightarrow \bar{D}$ is the closed point and $\bar{Y}_a = \bar{Y} \times_{\bar{D}} \operatorname{Spec}\mathbb{Q}$ is the singular fiber.

In this setting, the rational analogue of the continued Hodge bundle $\bar{\mathcal{H}} \to \bar{D}$ can be defined, without reference to topology, via logarithmic de Rham cohomology[19]

$$\bar{\mathcal{H}} = H^3_{\log}(\bar{Y}/\bar{D}) = R^3\bar{\pi}_*\big((\Omega^*_{\bar{Y}/\bar{D}}(\log(\bar{Y}_a)), d)\big) \tag{6.1}$$

$\bar{\mathcal{H}}$ is a vector bundle over $\bar{D}$ and comes equipped with (see [24], and [3] for more complete information)

$*$ a decreasing filtration $\bar{\mathcal{H}} = \bar{F}^0 \supset \cdots \supset \bar{F}^3$ by subbundles $\bar{F}^s \to \bar{D}$ with $\operatorname{rank}\bar{F}^3 = 1$

$*$ a flat logarithmic connection $\nabla : \bar{\mathcal{H}} \to \bar{\mathcal{H}} \otimes \Omega^1_{\bar{D}}(\log a)$ satisfying Griffiths transversality $\nabla \bar{F}^s \subset \bar{F}^{s-1} \otimes \Omega^1_{\bar{D}}(\log a)$

$*$ a perfect pairing $\langle \cdot, \cdot \rangle : \bar{\mathcal{H}} \times \bar{\mathcal{H}} \to H^6_{\log}(\bar{Y}/\bar{D}) \cong \mathcal{O}_{\bar{D}}$ with $\langle \bar{F}^s, \bar{F}^{4-s} \rangle = 0$

We denote the fiber of $\bar{\mathcal{H}}$ at a by $V := \bar{\mathcal{H}}_a$ which at this point is a $\mathbb{Q}$-vector space of dimension 4. By the assumption of semi-stability, the residue $N_{\mathrm{dR}} = \operatorname{Res}_a(\nabla) : V \to V$ of the connection is nilpotent. Since, after complexification, N_{dR} is related to the

[19] In potential conflict with previous or later notation, all schemes are taken over $\mathbb{Q}$ in this subsection, unless stated or implied otherwise by context.

logarithm of the monodromy of the local system from section 5.1 by $N_{\mathrm{dR}} = -\frac{1}{2\pi i}N$ (see eq. (5.1)), and we have assumed that the complex degeneration has maximal unipotent monodromy, it follows that N_{dR} has maximal rank 3, and, just as in the complex case, induces a weight filtration W_* on V.

This weight filtration can be used to reconstruct a basis of ("algebraically rational") sections (e^s) of $\bar{\mathcal{H}}$ over $\bar{D}$: We begin by letting e^0 be a parallel section of $\bar{\mathcal{H}} = \bar{F}^0$ whose restriction to a generates the one-dimensional subspace $W_0 := \mathrm{Ker}(N_{\mathrm{dR}}) = \mathrm{Im}(N_{\mathrm{dR}}^3)$ of V. Note that e^0 is unique up to a rational number and generates a one-dimensional subbundle of $\bar{\mathcal{H}}$. We then let e^1 be a section of $\bar{F}^1$ such that

$$N_{\mathrm{dR}}(e^1(a)) = -e^0(a) \tag{6.2}$$

and $\nabla e^1 \in e^0 \otimes \Omega^1(\log a)$. In other words, the image of e^1 in the quotient $\bar{\mathcal{H}}/e^0$ is parallel w.r.t. the induced connection. Writing

$$\nabla e^1 = -e^0 \otimes d\log q_{\mathbb{Q}} \tag{6.3}$$

determines a "rational flat coordinate" $q_{\mathbb{Q}} \in \mathcal{O}_{\bar{D}}$ *up to a (multiplicative) integration constant*. More precisely, the condition (6.2) implies that $d\log q_{\mathbb{Q}} \in \big(1 + z\mathbb{Q}[[z]]\big)\frac{dz}{z}$, which we can integrate to a local coordinate $q_{\mathbb{Q}}$ on $\bar{D}$ that is well-defined up to overall normalization. We write $\nabla_{t_{\mathbb{Q}}}$ for the contraction with the corresponding logarithmic vector field $q_{\mathbb{Q}}\frac{d}{dq_{\mathbb{Q}}}$ which is independent of that normalization.

To obtain the other half of the basis, we first normalize the pairing $\langle\cdot,\cdot\rangle$ by choosing a constant (∇-parallel) section $\mathbf{1}_6$ of $H^6_{\log}(\bar{Y}/\bar{D})$. This trivialization allows us to identify e^3 as a section of the rank-one subbundle $\bar{F}^3$ such that $\langle e^0, e^3\rangle = \mathbf{1}_6$. Finally, we put $e^2 := \nabla_{t_{\mathbb{Q}}} e^3$, so that by compatibility of the pairing with the connection we obtain in the familiar fashion

$$\langle e^1, e^2\rangle = \langle e^1, \nabla_{t_{\mathbb{Q}}} e^3\rangle = -\langle \nabla_{t_{\mathbb{Q}}} e^1, e^3\rangle = \langle e^0, e^3\rangle = \mathbf{1}_6 \tag{6.4}$$

Thus, we learn that (e^s) is a symplectic basis of $\bar{\mathcal{H}}$ over $\bar{D}$. One also checks as usual that $\nabla_{t_{\mathbb{Q}}} e^2$ is proportional to e^1, and defines the "rational Yukawa coupling" $\mathcal{C}_{\mathbb{Q}} \in \mathcal{O}_{\bar{D}}$ by

$$\nabla_{t_{\mathbb{Q}}} e^2 = -e^1 \otimes \mathcal{C}_{\mathbb{Q}} \tag{6.5}$$

We emphasize that the seeming ease in obtaining the basis (e^s) is a consequence of solving the differential equation of parallelism over the field of rational numbers. Complexification of the basis will yield a basis of the complex Hodge bundle that agrees with the one used in subsection 5.1, *up to the normalization* of e^0 and $\mathbf{1}_6$. The normalization of e^0 (though not that of $\mathbf{1}_6$) drops out of (6.5), and both are fixed by the topological integrality that underlies the complex VHS, and which is expressed through relations of the type (5.26). With this out of the way, the single remaining difficulty in identifying $\mathcal{C}_{\mathbb{Q}}$, obtained from the local solutions of the differential equation, with the complex power series from section 5.1 is the proper normalization of $q_{\mathbb{Q}}$. Under our standing assumption that $q'_{\mathbb{C}}(0) \in \mathbb{Q}$, the results so far imply that $q_{\mathbb{Q}} = q_{\mathbb{C}}$ up to a rational factor, thus proving the first statement in Lemma 19.

The extension by the algebraic cycle is readily included. As explained in subsection 5.4, we first pass to an r-fold of $\bar{D}$, $\bar{\hat{D}} := \operatorname{Spec}\mathbb{Q}[[\hat{z}]] \to \bar{D}$ with $z = \hat{z}^r$, such that the composition $\bar{\pi} \circ i : \bar{C} \to \bar{\hat{D}}$ is unramified and irreducible over $\mathbb{Q}$. An important novelty is that we do not assume $\bar{C}$ to be defined over $\mathbb{Q}$. Namely, $\bar{C}$ could become reducible after base change to the algebraic closure $\bar{\mathbb{Q}}$. This affords $\bar{C}^{\bar{\mathbb{Q}}} = \bar{C} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\bar{\mathbb{Q}}$ with an action of the absolute Galois group $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and we can identify the field of definition, K, as the number field invariant under the subgroup of $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ that fixes the components of $\bar{C}^{\bar{\mathbb{Q}}}$.

We then define the "extended and continued rational Hodge bundle" via

$$\bar{\hat{\mathcal{H}}} = \check{H}^3_{\log}\big((\bar{Y} \setminus \bar{C})/\bar{\hat{D}}\big) \tag{6.6}$$

This fits into an exact sequence

$$\bar{\mathcal{H}} \to \bar{\hat{\mathcal{H}}} \to \bar{\hat{\mathcal{I}}} \tag{6.7}$$

with $\bar{\mathcal{H}}$ from (6.1) and

$$\bar{\hat{\mathcal{I}}} = (\bar{\pi} \circ i)_*(\mathcal{O}_{\bar{C}}) \tag{6.8}$$

$\bar{\hat{\mathcal{I}}}$ is a vector bundle over $\bar{\hat{D}}$ of rank equal to the degree $d = [K : \mathbb{Q}]$. In particular, defining $\hat{V} := \bar{\hat{\mathcal{H}}}_a$ we can write the extension of the fiber at a as

$$V \to \hat{V} \to K \tag{6.9}$$

where we view K either as a $\mathbb{Q}$-vector space of dimension d, or a K-vector space of dimension 1. The latter point of view is more convenient to study the differential equations, so we adopt it in what follows. In other words, we now study the differential equation over the "disk with scalar extension", $\bar{\hat{D}}^K = \bar{\hat{D}} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec} K = \operatorname{Spec} K[[\hat{z}]]$. We note that for reasons of degree, the extension of the filtration satisifies

$$\bar{\hat{F}}^2/\bar{F}^2 = \bar{\hat{\mathcal{I}}} \tag{6.10}$$

and that the residue $\hat{N}_{\mathrm{dR}}$ of the extended Gauss-Manin connection acts trivially on the quotient $K = \hat{V}/V$ in (6.9).

Lemma 24. *There exists a section f of $\bar{\hat{F}}^2$ that satisfies*

$$\nabla_{t_{\mathbb{Q}}} f \in e^1 \otimes \mathcal{O}_{\bar{\hat{D}}^K} \tag{6.11}$$

and whose restriction to the closed point $a \in \bar{\hat{D}}^K$ generates K.

Proof. Indeed, by Griffiths transversality and parellelism of the image in $\bar{\hat{\mathcal{I}}}$, we have $\nabla_{t_{\mathbb{Q}}} \bar{\hat{F}}^2 \subset \bar{F}^1$, which is spanned by (e^s) wih $s > 0$. Since $\nabla_{t_{\mathbb{Q}}}$ maps $\bar{F}^2$ surjectively onto $\bar{\mathcal{H}}/\bar{F}^2$, we can use the freedom (6.10) to fix f such that (6.11) is satsified. (See (5.35) for the complex analogue of this construction.) ☐

In close analogy to the absolute case, f is unique up to multiplication by a non-zero constant, and, defining the "K-rational Griffiths infinitesimal invariant" $\mathcal{D}_K \in \mathcal{O}_{\bar{\hat{D}}^K}$ by

$$\nabla_{t_{\mathbb{Q}}} f = -e^1 \otimes \mathcal{D}_K, \tag{6.12}$$

we can choose the normalization of f such that after complexification and choice of embedding $K \hookrightarrow \mathbb{C}$, $\mathcal{D}_K$ agrees with $\mathcal{D}_k$ from (5.37), (5.41).

At this point, the proof of Lemma 19 is complete. ☐

6.2 Fontaine-Lafaille modules

The de Rham cohomology over $\mathbb{Q}$ that we described in the previous subsection can be endowed with further structure in several different ways. One possibility is to complete our schemes with respect to the standard Archimedean norm, and after algebraic closure we obtain the standard topology of a family of complex manifolds $\bar{Y}_{\mathbb{C}} \to \bar{D}_{\mathbb{C}}$, later extended by the algebraic cycle $\bar{C}_{\mathbb{C}} \subset \bar{Y}_{\mathbb{C}}$. The cohomology (relative to $D_{\mathbb{C}}$) of the constant sheaf of integers over this topology is well-behaved outside of the boundary point $a_{\mathbb{C}} = \bar{D}_{\mathbb{C}} \setminus D_{\mathbb{C}}$. The resulting local system enriches the de Rham cohomology $\mathcal{H}_{\mathbb{C}} \to D_{\mathbb{C}}$ (resp. $\hat{\mathcal{H}}_{\mathbb{C}} \to \hat{D}_{\mathbb{C}}$) to the variation of Hodge structure (localized in the neighborhood of $a_{\mathbb{C}}$) that we described in section 5.1.[20] As we have alluded to before, as far as the calculation of invariants through the solution of differential equations is concerned, the main import of the topological integral structure is the proper normalization of the sections

[20]except that there were no $\mathbb{C}$-subscripts in that section to reduce cluttering.

e^0 and $\mathbf{1}_6$ (and later f), and the multiplicative normalization of the canonical coordinate $q_{\mathbb{Q}}$ (which is equivalent to an additive normalization of e^1). This normalization can be accomplished by interpreting the power series as representatives of extension classes in the category of mixed Hodge structures.

As an alternative to the complex topology, we can, for each "good" prime $p \in S$, complete our family with respect to the p-adic topology. This process is less familiar to some, but we are unable to overburden this paper with a full recollection. The ultimate idea is that the algebraic de Rham cohomology over $\mathbb{Q}_p$ can be provided with an additional "integral" substructure by identifying it as the cohomology of a constant sheaf with respect to a somewhat subtle "crystalline" topology on the geometry in question, tensored with a suitable "period ring". The main feature at the end of this process is the identification of the cohomology as a module over the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, which ends up taking the role played by the singular cohomology in the more familiar complex case. (See also [4] and [25] for some additional information.)

We actually do not need the full machinery of this theory, ref. [24], but only the action of the Frobenius element on the de Rham cohomology, which can be defined already from the scheme over $\mathbb{Z}_p$. In the situation relevant to us, the notion that is the p-adic analogue of the complex variation of Hodge structure is identified [3] as a Fontaine-Lafaille module over a p-adic scheme. For the continuation of our semi-stable morphism $\bar{Y} \to \bar{D}$ over $\mathbb{Q}$ to a p-adic family $\bar{Y}_p \to \bar{D}_p$, this amounts to the following.

Among the early steps, one needs to equip the p-adic disk $\bar{D}_p = \operatorname{Spec} \mathbb{Q}_p[[z]]$ with a continuous lift of the Frobenius endomorphism $z \mapsto z^p$ at the closed point $a_p \in \bar{D}_p$. (Recall that by assumption, z is integral at $p \in S$.) Very concretely, we have $\mathrm{Frob}_p(x) = x$ for $x \in \mathbb{Z}_p$ and $\mathrm{Frob}_p(z) = z^p(1+p\,\eta(z))$ for some, not necessarily zero, $\eta(z) \in \mathbb{Z}_p[[z]]$. It might seem that this step involves some abitrary choices beyond those present in the rational or complex algebraic setting. This is however not the case, as emphasized in [3], different lifts being related by canonical isomorphisms. In fact, the endpoint of the p-adic construction is precisely the identification of a canonical coordinate q_p in which Frobenius acts by $\mathrm{Frob}_p(q_p) = (q_p)^p$, as assumed in the statements of the Theorems 21 and 22. To account for the existence of this integral structure on the disk, it is convenient to substitute $\operatorname{Spec} \mathbb{Z}_p[[z]]$ (with distinguished $a_p = (z)$) for $\bar{D}_p$ in the following.

Let us now denote by $\bar{\mathcal{H}}_p \to \bar{D}_p$ the vector bundle (as before, of rank 4) of logarithmic de Rham cohomology over $\mathbb{Z}_p$. As over $\mathbb{Q}$, it continues to possess a filtration

$\bar{F}^*$, flat connection ∇ and pairing $\langle\cdot,\cdot\rangle$ with similar properties as before. The essential new ingredient is a canonical lift of the Frobenius morphism,

$$\Phi_p : (\mathrm{Frob}_p)^*\bar{\mathcal{H}}_p \to \bar{\mathcal{H}}_p \tag{6.13}$$

that is parallel in the sense that

$$\nabla \circ \Phi_p = \Phi_p \circ \nabla\,, \tag{6.14}$$

compatible with the filtration in the sense that

$$\Phi_p(\mathrm{Frob}_p^* \bar{F}^s) \subset p^s\bar{\mathcal{H}}_p \quad \text{ and } \sum_s p^{-s}\Phi_p(\mathrm{Frob}_p^* \bar{F}^s) = \bar{\mathcal{H}}\,, \tag{6.15}$$

and with the pairing in the sense that

$$\langle \Phi_p \circ \mathrm{Frob}_p^*(u), \Phi_p \circ \mathrm{Frob}_p^*(v)\rangle = p^3\, \mathrm{Frob}_p^*\langle u, v\rangle \tag{6.16}$$

This last equation being a transcription of the statement that

$$H^6_{\log}(\bar{Y}_p/\bar{D}_p) \cong \mathbb{Z}_p(-3) \tag{6.17}$$

is an instance of a Fontaine-Lafaille module of the type

$$\mathbb{Z}_p(-k) = \left(\bar{F}^k = \mathcal{O}_{\bar{D}_p}, \bar{F}^{k+1} = 0, \Phi_p = p^k\cdot \mathrm{id}\right), \tag{6.18}$$

which is the p-adic version of the Hodge-Tate structure $\mathbb{Z}(-k)$.

A central observation of [3] in this context is that, in the category of Fontaine-Lafaille modules over the *punctured* disk $D_p = \mathrm{Spec}\big(\mathbb{Z}_p((t))\big)$,

$$\mathrm{Ext}^1_{\mathrm{MF}(D_p)}(\mathbb{Z}_p(-k), \mathbb{Z}_p(-k+1)) \cong \hat{\mathcal{O}}^*(D_p) \tag{6.19}$$

where $\hat{\mathcal{O}}^*(D_p)$ is the p-adic completion of $\mathcal{O}^*(D_p)$, the invertible functions on D_p. This statement is the analogue of (5.4) in the complex case and implies that *the power series parameterizing extensions of Fontaine-Lafaille modules have integral coefficients*, provided of course that *they are calculated with respect to an integral basis and coordinate.*

As a final ingredient, we require the residue N_{dR} of the flat connection at a_p in order to induce a weight filtration on the limiting Fontaine-Lafaille module $V_p = V \otimes \mathbb{Q}_p$ at a_p. Notice that the F-L structure on V_p (especially the Frobenius) depends on the choice of coordinate (namely, through the choice of Frobenius, Frob_p) on $\bar{D}_p$.

Given all this, it is shown in [3] that the Fontaine-Lafaille module

$$\mathcal{L}_p = (\bar{\mathcal{H}}_p, \Phi_p, \bar{F}^*, \langle\cdot,\cdot\rangle) \tag{6.20}$$

coming from our Calabi-Yau threefold family has a composition series very much analogous to that discussed in section 5.1 in the complex case. We shall not retrace these steps here. The essential result is the identification of the p-adic canonical coordinate, q_p, and the p-adic Yukawa coupling, $\mathcal{C}_p$ as representatives of extensions classes in the category $\mathrm{MF}(D_p)$ of Fontaine-Lafaille modules over D_p, with respect to a distinguished basis of sections ($e_p^s \in \bar{F}^s$) of the Hodge filtration. This data satisfies the same differential equations as over $\mathbb{Q}$ (and $\mathbb{C}$).

To include the algebraic cycle, we turn to working over the extended disk at p, $\bar{\hat{D}}_p^K = \operatorname{Spec}\mathcal{O}_p[[\hat{z}]]$. Notice that in this case, Frobenius (still denoted Frob_p) acts non-trivially already on the residues at a_p, cmp. eq. (2.8). With the extended Hodge bundle $\bar{\hat{\mathcal{H}}}_p \to \bar{\hat{D}}_p^K$, the p-adic continuation of eq. (6.7) (whose complex version is eq. (5.34)) takes the form

$$\bar{\hat{\mathcal{I}}}_p := \bar{\hat{\mathcal{H}}}_p/\bar{\mathcal{H}}_p \cong \mathcal{O}_p(-2) \tag{6.21}$$

where

$$\mathcal{O}_p(-2) = \big(\bar{F}^2 = \mathcal{O}_{\bar{\hat{D}}_p^K}, \bar{F}^3 = 0, \Phi_p = p^2 \cdot \mathrm{Frob}_p\big) \tag{6.22}$$

is the Fontaine-Lafaille module of rank 1 over $\bar{\hat{D}}_p^K$ with Frobenius inherited from K_p. On the preimage of $\bar{\hat{\mathcal{I}}}_p$ in $\bar{\hat{\mathcal{H}}}_p$, this lifts to

$$\Phi_p = p^2 \cdot \mathrm{Frob}_p \bmod \bar{F}^2 \tag{6.23}$$

so that similarly to $\mathbb{Q}$ or $\mathbb{C}$, we can obtain a section f_p whose restriction to a_p generates $\hat{V}_p/V_p \cong K_p$ and such that

$$\Phi_p(f_p) - p^2 f_p \in \mathrm{Span}_{\mathcal{O}_{\bar{\hat{D}}_p^K}}(e_p^0, e_p^1) \tag{6.24}$$

This allows us to identify the p-adic infinitesimal invariant $\mathcal{D}_p$,

$$\nabla_{t_p} f_p = -\mathcal{D}_p e_p^1 \tag{6.25}$$

as the derivative of the extension class

$$\nu_p \in \mathrm{Ext}^1_{\mathrm{MF}(\hat{D}_p^K)}(\mathcal{O}_p(-2), \mathcal{L}_p) \tag{6.26}$$

in the category of Fontaine-Lafaille modules over the punctured extended p-adic disk (cmp. (5.41)). Specifically,

$$\mathcal{D}_p = \langle \nabla_{t_p}^2 \nu_p, e_p^3\rangle = -\langle \nabla_{t_p}\nu_p, e_p^2\rangle \tag{6.27}$$

6.3 Identification of extension classes

We have now defined the p-adic power series q_p, $\mathcal{C}_p$, and $\mathcal{D}_p$ parameterizing the composition of the (extended) Fontaine-Lafaille module associated to our Calabi-Yau scheme (with cycle) over the disk. For clarity, we let $q_{\mathbb{C}}$, $\mathcal{C}_{\mathbb{C}}$ and $\mathcal{D}_{\mathbb{C}}$ be the complex power series that were introduced in section 5 without the subscript. (We'll also add that subscript to the cohomology basis to write $(e^s_{\mathbb{C}}, f_{\mathbb{C}})$.)

Proposition 25. *In* $\mathbb{Q}[[z]]$*,* $\mathbb{Q}[[q]]$*, and* $K[[\hat{q}]]$ *respectively,*

$$\begin{aligned} q_p &= q_{\mathbb{C}} \\ \mathcal{C}_p &= \mathcal{C}_{\mathbb{C}} \\ \mathcal{D}_p &= \mathcal{D}_{\mathbb{C}} \end{aligned} \tag{6.28}$$

Proof. We notice that these power series satisfy the same differential equation over $\mathbb{C}$ and $\mathbb{Q}_p$ (or K_p in the extended case of $\mathcal{D}$) as they do over $\mathbb{Q}$ (or K). In particular, these differential equations imply that $d\log q_p = d\log q_{\mathbb{C}}$ and that the other two equations hold up to an overall factor (in $\mathbb{Q}^*$ or K^*, respectively). To show equality, we need to compare the normalization of the cohomology bases $(e^s_{\mathbb{C}}, f_{\mathbb{C}})$ of the VHS and (e^s_p, f_p) of the F-L structure. Again by virtue of the differential equations, and duality with respect to the pairing, it is in fact sufficient to establish equality for the subvariations spanned by (e^0, e^1) in the two cases. We refer to section 4 of [3] for the proof of this statement. □

7 Integrality Proofs

By using the results reviewed in the previous section, Theorems 21 and 22 follow from a couple lines of simple algebra. Thanks to Proposition 25, it is enough to verify the p-adic integrality of the p-adically defined functions. We shall drop the subscript p from most of the notation in what follows.

We recapitulate some notation:

K is an algebraic extension of $\mathbb{Q}$, $\mathcal{O}$ the ring of integers in K. p is a rational prime, and $\mathcal{O}_p$ the p-adic completion of $\mathcal{O}$. Frob_p is Frobenius as defined in subsection 2.1.

$\bar{D}$ is a formal disk over $\mathbb{Z}_p$, $\bar{\hat{D}}^K$ its r-fold cover, extended over $\mathcal{O}_p$. Given a local coordinate z (*i.e.*, an identification $\bar{D} \cong \operatorname{Spec} \mathbb{Z}_p[[z]]$), we obtain an endomoprhism of

$\bar{D}$ lifting Frobenius by putting

$$\mathrm{Frob}_p^{(z)}(z) = z^p \tag{7.1}$$

We have added the superscript to emphasize the dependence on the local coordinate. One is tempted to drop it when z is clear from the context. But we will do so only after replacing z with the canonical coordinate q. In the lift to $\bar{\bar{D}}^K$, $\mathrm{Frob}_p^{(z)}$ acts non-trivially also on the coefficients in the residue "product-of-fields" K_p.

$\bar{\bar{\mathcal{H}}}$ is a bundle of $\mathcal{O}_p$-modules over $\bar{\bar{D}}^K$, with a filtration $\bar{\bar{F}}^*$ by bundles of $\mathcal{O}_p$ submodules.

$\Phi_p^{(z)}$ is a bundle map $(\mathrm{Frob}_p^{(z)})^*\bar{\bar{\mathcal{H}}} \to \bar{\bar{\mathcal{H}}}$ lifting $\mathrm{Frob}_p^{(z)}$ to the extended Hodge bundle. We usually identify $\Phi_p^{(z)}$ with $\Phi_p^{(z)} \circ (\mathrm{Frob}_p^{(z)})^*$.

$\Phi_p^{(z)}$ preserves the weight filtration and is divisible by p^s on $\bar{\bar{F}}^s$. $\Phi_p^{(z)}$ is also compatible with the pairing $\langle\cdot,\cdot\rangle : \bar{\mathcal{H}} \times \bar{\mathcal{H}} \to \mathbb{Z}_p(-3)$ (namely, with $\Phi_p^{(z)} = p^3\mathrm{id}$ on $\mathbb{Z}_p(-3)$).

7.1 Integrality of cohomology basis

The first item on the list is to verify that the basis element e^0, which is defined in 6.1 as the parallel section of the rational bundle $\bar{\mathcal{H}}$ with $(e^0)_a \in W_0 = \mathrm{Ker}(N_{\mathrm{dR}})$, is p-adically integral, *i.e.*, continues to a section $e^0_p \in \bar{F}^0\bar{\mathcal{H}}_p$. To see this (cf. Lemma 7 of [3]), one first notices that $(e^0)_a$ is eigenvector of $(\Phi_p^{(z)})_a$ with eigenvalue of square 1 (this follows from the invariance of the pairing and $N_{\mathrm{dR}} \circ (\Phi^{(z)})_a = p \cdot (\Phi^{(z)})_a \circ N_{\mathrm{dR}}$). Then, letting $\tilde{e}^0$ be any section of $\bar{F}^0\bar{\mathcal{H}}_p$ with $(\tilde{e}^0)_a = (e^0)_a$, one observes that

$$\lim_{k\to\infty} \left(\Phi^{(z)}\right)^{2k}(\tilde{e}^0) \tag{7.2}$$

is a parallel and integral section that agrees with $(e^0)_a$ at a. By uniqueness of the solution of the differential equation, this is e^0.

The integrality of the remainder of the cohomology basis follows from its construction via duality and taking derivative with respect to the canonical coordinate, whose integrality is established next.

7.2 Integrality of mirror map

q_p is the class of the extension

$$\mathbb{Z}_p(0) \to \mathcal{L}_2 \to \mathbb{Z}_p(-1) \tag{7.3}$$

in the category of Fontaine-Lafaille modules over the formal disk $\mathrm{Spec}\,\mathbb{Z}_p[[z]]$ over $\mathbb{Z}_p$. Lemma 5 of [3] (a.k.a. the Dwork integrality Lemma) implies $q_p \in z\mathbb{Z}_p[[z]]$.

7.3 Integrality of Griffiths-Yukawa coupling

The rest of the Fontaine-Lafaille structure and the integrality of the Yukawa coupling is best analyzed in the canonical coordinate q. From now on, we drop the superscript from Frobenius. The following calculation first appeared in [1].

We let $(p^s m_t^s)$ be the matrix representing Frobenius with respect to the basis (e^s). Namely, we write

$$\Phi_p((\mathrm{Frob}_p)^*(e^s)) = p^s \sum_{t=0}^{s} m_t^s e^t \tag{7.4}$$

By the above, $m_t^s \in \mathbb{Z}_p[[z]]$.

In the same basis, the Gauss-Manin connection contracted with the canonical vector field $t = q\partial_q$ has the representation

$$\nabla_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -\mathcal{C} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{7.5}$$

We already know that $\mathcal{C} \in \mathbb{Z}_p[[z]]$. Since $\mathcal{C}$ is the logarithmic derivative of the extension class of $\mathcal{L}_4/\mathcal{L}_0$ in $\mathrm{Ext}_{\mathrm{MF}}(\mathbb{Z}_p(-2), \mathbb{Z}_p(-1))$ (*cf.*, (5.19) for the corresponding complex statement), Dwork's lemma implies that $\mathrm{Frob}_p(\mathcal{C}) - \mathcal{C} = \delta\varphi$ for some $\varphi \in \mathbb{Z}_p[[q]]$. We wish to improve this to the statement that

$$\mathrm{Frob}_p(\mathcal{C}) - \mathcal{C} = \delta^3\psi \tag{7.6}$$

for $\psi \in \mathbb{Z}_p[[q]]$. To evaluate

$$\nabla_t \Phi_p = p\Phi_p \nabla_t \tag{7.7}$$

we calculate

$$\begin{aligned}
\nabla_t \Phi_p &= \delta \begin{pmatrix} m_0^0 & 0 & 0 & 0 \\ pm_0^1 & pm_1^1 & 0 & 0 \\ p^2m_0^2 & p^2m_1^2 & p^2m_2^2 & 0 \\ p^3m_0^3 & p^3m_1^3 & p^3m_2^3 & p^3m_3^3 \end{pmatrix} + \begin{pmatrix} m_0^0 & 0 & 0 & 0 \\ pm_0^1 & pm_1^1 & 0 & 0 \\ p^2m_0^2 & p^2m_1^2 & p^2m_2^2 & 0 \\ p^3m_0^3 & p^3m_1^3 & p^3m_2^3 & p^3m_3^3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -\mathcal{C} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \delta \begin{pmatrix} m_0^0 & 0 & 0 & 0 \\ pm_0^1 & pm_1^1 & 0 & 0 \\ p^2m_0^2 & p^2m_1^2 & p^2m_2^2 & 0 \\ p^3m_0^3 & p^3m_1^3 & p^3m_2^3 & p^3m_3^3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ -pm_1^1 & 0 & 0 & 0 \\ -p^2m_1^2 & -p^2\mathcal{C}m_2^2 & 0 & 0 \\ -p^2m_1^3 & -p^3\mathcal{C}m_2^3 & p^3m_3^3 & 0 \end{pmatrix}
\end{aligned} \tag{7.8}$$

and

$$p\Phi_p\nabla_t = p\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -\operatorname{Frob}_p(\mathcal{C}) & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} m_0^0 & 0 & 0 & 0 \\ pm_0^1 & pm_1^1 & 0 & 0 \\ p^2m_0^2 & p^2m_1^2 & p^2m_2^2 & 0 \\ p^3m_0^3 & p^3m_1^3 & p^3m_2^3 & p^3m_3^3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -pm_0^0 & 0 & 0 & 0 \\ -p^2\operatorname{Frob}_p(\mathcal{C})m_0^1 & -p^2\operatorname{Frob}_p(\mathcal{C})m_1^1 & 0 & 0 \\ p^3m_0^2 & p^3m_1^2 & p^3m_2^2 & 0 \end{pmatrix} \tag{7.9}$$

The compatibility of Φ_p with the pairing in cohomology takes the form

$$m_s^t I^{ss'} m_{s'}^{t'} = p^3 I^{tt'} \tag{7.10}$$

where

$$I = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{7.11}$$

Now, by equating the diagonal terms in (7.8) and (7.9), we find that $\delta m_s^s = 0$ for $s = 0, 1, 2, 3$. At the top of the first lower diagonal, we learn that

$$\delta m_0^1 = m_1^1 - m_0^0 \tag{7.12}$$

Since $m_0^1 \in \mathbb{Z}_p[[q]]$ (in particular, it contains no negative powers of q), evaluation at $q = 0$ shows that the constant $m_1^1 - m_0^0$ in fact vanishes. Continuing down, we find that $m_0^0 = m_1^1 = m_2^2 = m_3^3$.

Putting $t = 0$, $t' = 3$ in (7.10), we find that $1 = m_0^0 m_3^3 = (m_0^0)^2$. Let us assume that $m_0^0 = 1$. (The case $m_0^0 = -1$ can be treated *mutatis mutandis*.) Then the remaining entries of (7.10) become $m_2^3 + m_0^1 = 0$ and $m_0^2 + m_1^2 m_2^3 - m_1^3 = 0$.

Returning to (7.12), we see that m_0^1 is a constant. In fact, by the results of section 6.3, this constant is 0.

Finally, the lower left 2×2 square of (7.8) and (7.9) becomes

$$\begin{aligned} \delta m_1^2 &= \mathcal{C} - \operatorname{Frob}_p(\mathcal{C}) \\ \delta m_0^2 &= m_1^2 \\ \delta m_1^3 &= m_1^2 \\ \delta m_0^3 &= m_1^3 + m_0^2 \end{aligned} \tag{7.13}$$

Put together, this gives

$$\mathrm{Frob}_p(\mathcal{C}) - \mathcal{C} = -\frac{1}{2}\delta^3 m_0^3 \tag{7.14}$$

The claim follows since $p \neq 2$. This ends the proof of Theorem 21. $\square$

7.4 Integrality of infinitesimal invariant

This calculation takes place over the extended disk in canonical coordinates, localized at p in the sense of (2.1), $\bar{\hat{D}}_p^K := \mathrm{Spec}\, \mathcal{O}_p[[\hat{q}]]$.

By eq. (6.24), we can write

$$\Phi_p(f) = p^2 f + p^2 n_0 e^0 + p^2 n_1 e^1 \tag{7.15}$$

with $n_0, n_1 \in \mathcal{O}_p[[\hat{q}]]$. On the other hand (cf. (5.37), (6.25)), we have

$$\nabla_t f = -\mathcal{D} e^1 \tag{7.16}$$

The equality (7.7) becomes

$$\nabla_t \Phi_p f - p\Phi_p \nabla_t f = p^2(\delta n_0 - n_1)e^0 + p^2(\delta n_1 - \mathcal{D} + \mathrm{Frob}_p(\mathcal{D}))e^1 = 0 \tag{7.17}$$

(where we used $m_1^1 = 1$ and $m_0^1 = 0$ from the previous subsection). As a result,

$$\begin{aligned} \delta n_0 &= n_1 \\ \delta n_1 &= \mathcal{D} - \mathrm{Frob}_p(\mathcal{D}) \end{aligned} \tag{7.18}$$

and by combining the two, we find

$$\mathrm{Frob}_p(\mathcal{D}) - \mathcal{D} = -\delta^2 n_0 \tag{7.19}$$

This concludes the proof of Theorem 22. $\square$

Acknowledgments The research of V.V. was supported in part by Laboratory of Mirror Symmetry NRUHSE, RF government grant, ag. number 14.641.31.0001. The research of J.W. was supported in part by an FRQNT New Researcher grant, an NSERC discovery grant, a Tier II Canada Research Chair at McGill University, and by the German Research Foundation in the framework of the Excellence Initiative through MATCH at Heidelberg University. The research of A.Sch. was supported in part by an NSF grant.

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