Contemporary Mathematics
Volume **692**, 2017
http://dx.doi.org/10.1090/conm/692/13918

Around Anosov-Weil Theory

V. Grines and E. Zhuzhoma

Abstract. The survey is devoted to the exposition of main results of Anosov-Weil Theory that studies nonlocal asymptotic properties of simple curves on a surface with a non-positive constant curvature. This study consists of the lifting these curves to an universal covering and making a "comparison" in a sense with lines of constant geodesic curvature. We review some applications conserning constructions of topological invariants for surface dynamical systems and foliations.

Contents

Introduction

In 1966 our friend and teacher S.Kh. Aranson met Dmitrii Viktorovich Anosov at Tiraspol (Moldova former part of Soviet Union) at the Symposium on General Topology. It became clear that dynamical systems (even, structurally stable) can have complex dynamics with nontrivially recurrent orbits. This is related to the problem of the topological classification of dynamical systems with complex dynamics beginning with the simplest, in a sense, systems such as surface flows.

A classical example of an effective topological invariant is given by the Poincaré rotation number for fixed-point-free flows on the two-dimensional torus $\mathbb{T}^2$ [**62**]. This number determines the "asymptotical rotation" of trajectories along the meridians and parallels of the torus. It is well known that when all trajectories are nontrivially recurrent,[1] the rotation number is a complete topological invariant up to the recalculation by an integer unimodular matrix.

2010 *Mathematics Subject Classification.* Primary 37E30, 37E35, 37D05.
Key words and phrases. Surface, geodesic, dynamical system.

[1]Earlier, following Poincaré, such trajectories were called nonclosed Poisson stable trajectories [**62**].

In Tiraspol D.V. Anosov formulated the idea that a clue to the construction of effective topological invariants for dynamical systems with nontrivially recurrent motions (including foliations with nontrivially recurrent leaves) on surfaces consists of studying nonclosed curves without self-intersections that possess certain recurrent properties and in investigating the nonlocal asymptotic behavior of the lifts of these curves to the universal covering by means of the absolute (circle at infinity). Later, the development of this idea led to the topological classification of the basic classes of flows, foliations, 2-webs, nontrivial one-dimensional basic sets, and homeomorphisms with invariant foliations on closed surfaces of constant nonpositive curvature. In 1973, while developing the above-mentioned geometric interpretation of the Poincaré rotation number, S. Aranson and V. Grines [**19**] constructed a complete topological invariant for irrational flows on orientable closed surfaces of constant negative curvature.

An explicit use of the universal covering in the study of the nonlocal asymptotic behavior of the trajectories of fixed-point-free flows on $\mathbb{T}^2$ was first proposed by A. Weil [**65**] in 1932. Before him, following Poincaré, mathematicians used a global section and the first-return map on this section. A. Weil proposed an alternative definition for the rotation number. This definition employs the trajectories of a covering flow on the Euclidean plane. Namely, Weil proved that the rotation number is equal to the angular coefficient of a straight line that has the same asymptotic direction as the trajectories of the covering flow [**65**]. His arguments were based on the fact that the lifts of the trajectories are pairwise disjoint and that each lift divides the Euclidean plane. This fact prompted Weil to suppose that curves without self-intersections, not necessarily defined by differential equations, should possess similar properties. In his lecture delivered at the Moscow International Topological Conference in 1935, Weil formulated two conjectures on the behavior of covering curves for curves without self-intersections [**66**]. The first conjecture (formulated as a theorem and referred to as the Weil theorem below) stated that the lift of a curve without self-intersections on $\mathbb{T}^2$ to the universal covering has an asymptotic direction if this lift is an unbounded curve and goes to infinity. The second conjecture was similar to the first one but referred to closed surfaces of negative Euler characteristic (exact statements of the conjecture and theorem are given below).

Unfortunately, Weil's approach was not developed and was soon forgotten. However, in the early 1960s, interest in this subject was renewed by Anosov within the context of a general upsurge in the theory of dynamical systems. The problem from which Anosov started his studies consisted of determining the common features in the asymptotic behavior of trajectories and geodesics. This problem naturally led Anosov to the investigation of trajectories on the universal covering and to the study of their nonlocal asymptotic behavior. In Tiraspol in 1966, Anosov communicated the theorem stating that the coverings for the trajectories of a smooth flow with a finite number of fixed points on a compact surface of nonpositive Euler characteristic have asymptotic directions. He also formulated several conjectures (one of which generalized the Weil conjecture) on the behavior of coverings for curves without self-intersections. These conjectures, the Anosov theorem, and a number of his subsequent works [**2**]–[**10**] catalyzed the development of the whole theory. In view of these circumstances, the field of inquiry in question was called the "Anosov–Weil Problem" or "Anosov–Weil Theory" in the studies of mathematicians from

Nizhni Novgorod (and later in the studies of other mathematicians). Now, roughly speaking, Anosov-Weil Theory includes the following two parts:

- ***a study of nonlocal asymptotic properties of simple curves on a surface by lifting these curves to an universal covering, and making a "comparison" in a sense with lines of constant geodesic curvature;***
- ***an application of nonlocal asymptotic properties for constructions of topological invariants for surface dynamical systems and foliations.***

Simultaneously with Anosov's works, Weil's approach was also considered by N. Markley [**50**]–[**52**], who paid more attention to the flows. However, these studies were not so widely recognized in the USA as Anosov's works in the USSR.

Acknowledgments. We thank A. Katok and Ya. Pesin for the invitation to submit this survey. We also thank M. Brin for very careful reading of the manuscript and helpful comments. This work was supported by the Russian Foundation for Basic Research (project nos. 15-01-03687-a, 13-01-12452-ofi-m,16-51-10005 KO-a), Russian Science Foundation (project no 14-41-00044) and was done in the frames of Basic Research Programs at the HSE (project 98 in 2016) and (project T-90 in 2017).

1. Mathematical background

We give here the main definitions of Anosov-Weil Theory. We consider surfaces being complete Riemannian manifolds M^2 of constant nonpositive curvature. For simplicity, we restrict ourself to closed surfaces.

1.1. Universal covering and the circle at infinity. The universal covering space $\overline{M}^2$ for M^2 is isometric either to the Euclidean plane $\mathbb{R}^2$ (in the case of zero curvature and Euler characteristic $\chi(M^2) = 0$) or to the hyperbolic plane Δ (in the case of negative curvature and Euler characteristic $\chi(M^2) < 0$). Accordingly, M^2 is isometric either to $\mathbb{R}^2/\Gamma$ or to Δ/Γ (hyperbolic surface), where Γ is a properly discontinuous group of isometries. Denote by $\pi : \overline{M}^2 \to M^2$ the universal covering map, which is a local isometry. Given a curve $C \subset M^2$, a *lift* of C is an arcwise connected component of the pull back $\pi^{-1}(C)$. Often the choice of this component is clear from the context.

The Euclidean plane $\mathbb{R}^2$ endowed with the standard quadratic form $ds^2 = dx^2 + dy^2$ is the simplest *flat* surface. Sometimes it is convenient to use the unit disk D^2 with coordinates ξ, η as a universal covering space: $D^2 = \{(\xi,\eta) : \xi^2 + \eta^2 < 1\}$. One can check that the map (1) is a homeomorphism denoted by $\tau : D^2 \to \mathbb{R}^2$. Then $\pi \circ \tau$ is also a universal covering map, see Fig. 1, (a). The boundary $S_\infty = \partial D^2$ is called the *circle at infinity* or *absolute*.

$$x = \frac{\xi}{\sqrt{1-\xi^2-\eta^2}}, \qquad y = \frac{\eta}{\sqrt{1-\xi^2-\eta^2}}. \tag{1}$$

For the hyperbolic plane Δ, we use the Poincaré model $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, Fig. 1, (b). Sometimes we consider Δ as the unit disk on $\mathbb{R}^2$ with the topology and metric induced by $\mathbb{R}^2$. Denote by $\overline{d}_{\mathrm{E}}(\cdot,\cdot)$ (respectively, $\overline{d}_{\mathrm{NE}}(\cdot,\cdot)$) the Euclidean (respectively, non-Euclidean) metric on Δ.

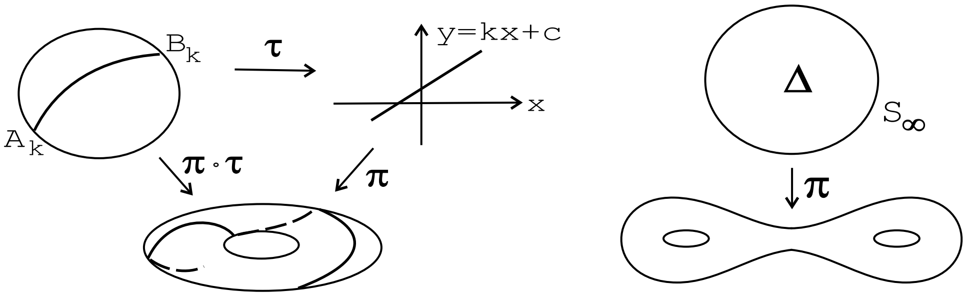

FIGURE 1. (a) Coverings of the torus; (b) Covering of a hyperbolic surface

One introduces the *absolute*, or the *circle at infinity* (sometimes, one say *circle composed of infinitely remote points*) $S_\infty = \{z \in \mathbb{C} : |z| = 1\}$. These points do not belong to the hyperbolic plane; however, they play a very important role in hyperbolic geometry. The geodesics are arcs of Euclidean circles and straight lines orthogonal to S_∞. We will suppose that endpoints of geodesics, *ideal endpoints*, belong to S_∞.

Let $l^+ \subset M^2$ be a semi-infinite continuous curve endowed with an injective parametrization $t \to l^+(t)$, $t \in \mathbb{R}^+$, and $\overline{l}^+$ a lift of l^+ under a universal covering map $\pi : \overline{M}^2 \to M^2$. The universal covering surface $\overline{M}^2$ can be thought of as the open unit disk $D^2 \subset \mathbb{R}^2$ provided M^2 is a flat or hyperbolic surface. Let $\overline{d}_{\mathrm{E}}(\cdot,\cdot)$ be the metric on $S_\infty \cup D^2$ induced by the standard metric of $\mathbb{R}^2$. The parametrization of l^+ induces the parametrization of $t \to \overline{l}^+(t)$ such that $\pi(\overline{l}^+(t)) = l^+(t)$. A point $\sigma \in S_\infty$ is called the *remote limit point* of $\overline{l}^+$ if there is a sequence t_k, $\lim_{k\to\infty} t_k = \infty$, such that $\overline{d}_{\mathrm{E}}(\sigma, \overline{l}^+(t_k)) \to 0$ as $k \to \infty$. The ***limit set at infinity*** $lim_\infty(\overline{l}^+)$ of $\overline{l}^+$ is the union of all remote limit points of $\overline{l}^+$. Denote by $\lim(\overline{l}^+)$ the set of (ordinary) limit points that belong to $\overline{M}$. The union of the limit set that belongs to $\overline{M}$ and the limit set at infinity is called the *complete limit set*, $Lim(\overline{l}^+) = \lim(\overline{l}^+) \cup lim_\infty(\overline{l}^+)$.

Now, we present some ways for specifying points of S_∞. For the flat surfaces (torus and Klein bottle), the universal covering $\overline{M}$ is $\mathbb{R}^2$. Every point $\sigma \in S_\infty$ corresponds to oriented parallel rays $y = kx + c$, $c \in \mathbb{R}$, with the same angular coefficient (including ∞) k. Rationality (irrationality) of k corresponds to rationality (irrationality) of σ. For the sake of generality, assume that ∞ is a rational "number." We see that pairs of diametrically opposite points are parameterized by angular coefficients $k \in \mathbb{R} \cup \{\infty\}$. This specification of points of S_∞ is often quite sufficient for the case of flat surfaces. Any ray with $k \in \mathbb{Q}$ (respectively, $k \notin \mathbb{Q}$) projects to a closed (respectively, unclosed) geodesic, and vise versa, rational points of S_∞ are exactly ideal points of lifts of closed geodesics.

Now, we consider the description of points of S_∞ for the hyperbolic plane Δ, which is the universal covering for hyperbolic surfaces. Let Γ be the group of deck transformations that acts on Δ. Each deck transformation is extended to S_∞. By definition, a fixed point of a deck transformation that belongs to S_∞ is called a *rational point*. Denote by $\mathcal{R}$ the union of all rational points. Rational points are exactly ideal endpoints of lifts of all closed geodesics. The remaining points $\mathcal{I} = S_\infty - \mathcal{R}$ are called *irrational points*. Every point of S_∞ corresponds to a family of oriented collinear parallel geodesics. However, there does not exist a convenient generally accepted method for assigning a certain number to such a family of geodesics. There are different types of coding that depend on the choice of the generators of the fundamental group of a surface (see [**47**], [**55**], [**56**]). Below, to specify points of S_∞, we'll consider ideal endpoints of the geodesics that belong to lifts of special geodesic laminations. We'll see that the description of irrational points is much more rich than in the case of flat surfaces.

One may introduce the circle at infinity through families of parallel directed geodesics or geodesic rays, see for example [**34, 35**]. Any geodesic from this family is called a *representative* of the point at infinity. Our definition of S_∞ gives the same object.

1.2. Asymptotic directions and co-asymptotic geodesics. We describe some possible types of asymptotic behavior of semi-infinite curves. Let $l^+ = \{m(t) : t \geq 0\} \subset M^2$ be a semi-infinite simple (i.e. without self-intersections) curve and $\overline{l}^+ = \{\overline{m}(t) : t \geq 0\}$ its lift to the universal covering $\overline{M}^2$ endowed with the metric $\overline{d}$ ($\overline{d} = \overline{d}_{\mathrm{E}}$ if $\overline{M}^2 = \mathbb{R}^2$, and $\overline{d} = \overline{d}_{\mathrm{N}E}$ if $\overline{M}^2 = \Delta$).

One says that $\overline{l}^+$ *leaves any compact subset* of $\overline{M}^2$, or is *unbounded*, if

$$\limsup_{t\to+\infty} \overline{d}(\overline{a}_0, \overline{m}(t)) = +\infty, \tag{2}$$

where $\overline{a}_0 \in \overline{M}^2$ is an arbitrary point, Fig. 2 (a). It is clear that this definition does not depend on the choice of $\overline{a}_0$. A curve that belongs to some compact subset of $\overline{M}^2$ is called *bounded.*

We say that $\overline{l}^+$ *goes to infinity* if

$$\lim_{t\to+\infty} \overline{d}(a_0, \overline{m}(t)) = +\infty. \tag{3}$$

In general, (2) does not imply (3). Obviously, the converse is true: a curve that goes to infinity sooner or later leaves any compact subset of $\overline{M}^2$ and never returns to it, Fig. 2 (b). A basic definition of the Anosov-Weil Theory is the following one.

Let $l^+ = \{m(t) : t \geq 0\} \subset M^2$ be a semi-infinite simple curve and $\overline{l}^+ = \{\overline{m}(t) : t \geq 0\} \subset \overline{M}^2$ its lift, where $\overline{M}^2$ is either D^2 or Δ. If $\overline{l}^+$ tends exactly to one point $\sigma \in S_\infty$, $Lim(\overline{l}^+) = lim_\infty(\overline{l}^+) = \sigma$, then we say that $\overline{l}^+$ ***has an asymptotic direction*** σ.

Roughly speaking, for an observer situated on $\overline{M}^2$, the curve $\overline{l}^+$ goes exactly to one point of the horizon, Fig. 2 (c). The point σ is called a *point accessible (or reached) by the curve* $\overline{l}^+$. One also says that $\sigma = \omega(\overline{l}^+)$ is *attained* by $\overline{l}^+$. An asymptotic direction is called *rational* (*irrational*) if the point $\sigma \in S_\infty$ is rational (respectively, irrational).

Clearly, if some lift has an asymptotic direction, then any lift also has an asymptotic direction. For a curve $\overline{l}^-$ that is semi-infinite in the negative direction, the asymptotic direction and its accessible point $\alpha(\overline{l}^-)$ are defined similarly.

Let $l = \{m(t) \in M^2 : -\infty < t < +\infty\} \subset M^2$ be a simple infinite continuous curve and $\overline{l} = \{\overline{m}(t) : -\infty < t < +\infty\} \subset \overline{M}^2$ a lift of l. The point $\overline{m}(0)$ divides $\overline{l}$ into two semi-infinite curves: the *positive* $\overline{l}^+ = \{\overline{m}(t) : t \geq 0\}$ and *negative* $\overline{l}^- = \{\overline{m}(t) : t \leq 0\}$. Suppose that $\overline{l}^+$ and $\overline{l}^-$ have asymptotic directions $\omega(\overline{l}) \in S_\infty$ and $\alpha(\overline{l}) \in S_\infty$, respectively, and $\alpha(\overline{l}) \neq \omega(\overline{l})$. Then there exists a geodesic $\overline{g}(\overline{l})$ with

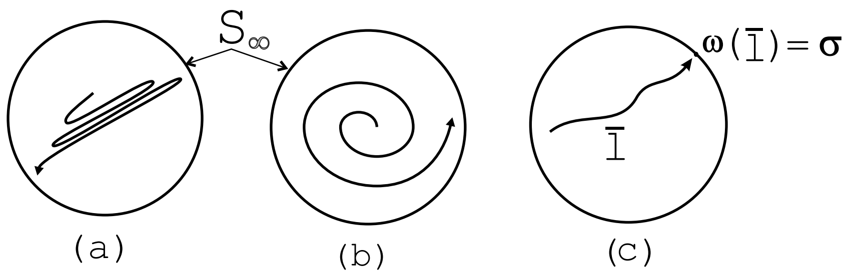

FIGURE 2. Unbounded curve (a); a curve goes to infinity (b); the point $\omega(\bar{l}^+)$ is reached by $\bar{l}^+$.

the same ideal endpoints $\alpha(\bar{l})$ and $\omega(\bar{l})$ oriented from $\alpha(\bar{l})$ to $\omega(\bar{l})$. The geodesic $\overline{g}(\bar{l})$ is called *co-asymptotic* or *corresponding* to $\bar{l}$, Fig. 3. It is easy to see that the geodesic $\pi(\overline{g}(\bar{l})) \overset{\text{def}}{=} g(l)$ on M^2 does not depend on the choice of $\bar{l} \in \pi^{-1}(l)$ and is called *co-asymptotic* or *corresponding* to l.

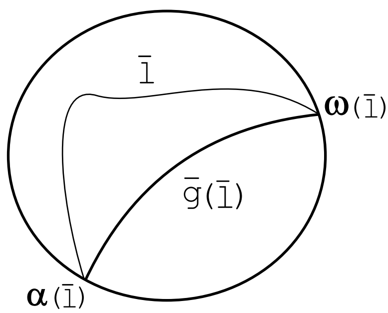

FIGURE 3. Co-asymptotic geodesic.

Clearly, a *co-asymptotic* or *corresponding geodesic to a simple non-null-homotopic closed curve* is the closed geodesic that is freely homotopic to the closed curve. For the hyperbolic surface, the co-asymptotic geodesic is unique, due to properties of hyperbolic geometry. For the torus, the co-asymptotic geodesic is not unique and must be specified.

2. Historical background

André Weil [**65**] applied a covering flow to get a geometrical interpretation of the Poincaré rotation number for a fixed-points-free flow on $\mathbb{T}^2$. On account of the above terminology A. Weil proved that the covering trajectories of such flows must have an asymptotic direction. Weil's method of the study of asymptotic directions is more geometric than the Poincaré's method which consists of the study of the first return maps on global cross-sections to the flows. What is more important Weil apparently inferred that his method works not exclusively for the torus flows but also for the higher genus surface flows and is applicable as well to arbitrary families of curves not necessarily given by the differential equations. This led him to the two conjectures (quoted below) on a nonlocal asymptotic behavior of a lift for any curve without topologically transversal self-intersections. We quote the original of his talk at the First international topological conference in Moscow [**66**] held in 1935:

"Dans la présente communication, l'auteur discute deux méthodes pouvant servir à l'etude de la question et d'autres analogues. La première, qui a déjà été développée dans un article du [**65**]*, consiste à considérer dans le plan (x, y) en même temps que la courbe C de la famille, toutes les courbes $C_{p,q}$ qui s'en déduisent par une translation (p, q), p et q étant des entiers: la position relative de ces courbes par rapport à C permet, non seulement de déterminer le nombre de rotation, mais encore la transformation qui ramène la famille étudiée à une forme canonique. La méthode s'applique dans le cas de Poincaré, et plus généralement chaque fois que la famille ne présante pas de 'col à l'infini' (au sens de Niemytzky). D'ailleurs cette dernière circonstance ne peut vraisemblablement pas se présenter si la famille ne contient pas de courbe fermée. À cette méthode se relie encore le théorème suivant, d'ailleurs obtenu par une voie quelque peu différente:*

Soit, sur le tore, une courbe de Jordan, image continue de la demi-droite $0 \leq t < +\infty$; on suppose que cette courbe soit sans point double; alors, si l'image de la courbe dans le plan (x, y), surface de recouvrement universelle du tore, tend vers l'infini avec t, elle y tend avec une direction asymptotique bien déterminée, c'est-à-dire que la rapport $\frac{x(t)}{y(t)}$ tend vers une limite quand t tend vers $+\infty$.

Une généralisation très intéressante du probllème étudié, qui paraît susceptible d'être abordée par la même méthode, est l'étude, sur une surface close de genre p, des solutions d'une équation différentielle du premier ordre n'ayant d'autres points singuliers que de cols, ou en termes topologiques, d'une famille de courbes dont tous les points singuliers sont d'indice négatif. Un premier résultat est suivant:

Sur le cercle hyperbolique, surface de recouvrement universelle de la surface étudiée, toute courbe de la famille tend, dans chaque direction, vers un point à l'infinie bien détetminé. ...

Actually, A.Weil singled out two conjectures on the behavior of the covering of curves without self-intersections. The first conjecture says that the covering of a curve without self-intersections on the torus has an asymptotic direction, provided this covering goes to infinity. Since A.Weil informed that this statement was proved by Magnier, one formulates this conjecture as a theorem.

THEOREM 2.1 (Theorem of Weil). *Let $l = \{m(t) : t \geq 0\}$ be a semi-infinite (continuous) curve without self-intersections on the torus T^2, and let $\overline{l} = \{\overline{m}(t) : t \geq 0\}$ be its lift to D^2. If the curve $\overline{l}$ goes to infinity, it has an asymptotic direction.*

The second conjecture is similar to the first conjecture and is applied to the higher genus surfaces.

CONJECTURE 2.1 (Conjecture of Weil). *Let $l = \{m(t) : t \geq 0\}$ be a semi-infinite (continuous) curve without self-intersections on a closed hyperbolic surface M^2, and let $\overline{l} = \{\overline{m}(t) : t \geq 0\}$ be its lift to Δ. If the curve $\overline{l}$ goes to infinity, it has an asymptotic direction.*

Proof of Conjecture 2.1. Suppose that $\overline{l}$ does not have an asymptotic direction. Since the curve $\overline{l}$ goes to infinity, its limit set at infinity contains at least two points and coincides with the complete limit set, $Lim(\overline{l}) = \lim_{\infty}(\overline{l})$. The complete limit set of the lift of a semi-infinite curve is closed and connected. Therefore, $Lim(\bar{l})$ contains a nontrivial interval, which we denote by $I \subset S_\infty$.

Since the group Γ of deck transformations is a Fuchsian group of the first kind, there exists a hyperbolic isometry $\gamma \in \Gamma$ such that the ideal endpoints of its axis $O(\gamma)$ belong to the interval I. Note that $O(\gamma)$ is projected to a closed geodesic on the surface.

Take a sufficiently long interval $\overline{A} \subset O(\gamma)$ such that one of its endpoints is mapped by γ into $\overline{A}$. The interval $\overline{A}$ divides the axis $O(\gamma)$ into two subintervals $\overline{A}_1$ and $\overline{A}_2$. Each of these subintervals has one ideal endpoint in I. Since the curve $\bar{l}$ goes to infinity, it does not intersect $\overline{A}$ starting from a certain moment. The fact that I belongs to the limit set of the curve $\bar{l}$ implies that there exists an arc $\overline{S}$ of the curve $\bar{l}$ that intersects $O(\gamma)$ only at the endpoints, such that one of the endpoints is in $\overline{A}_1$ and the other in $\overline{A}_2$, Fig. 4. But then the curve l has self-intersections

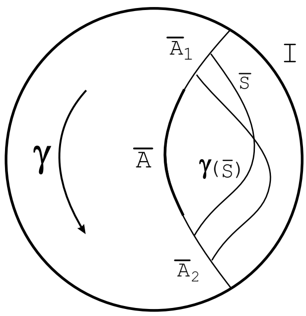

FIGURE 4. The arcs $\overline{S}$ and $\gamma(\overline{S})$ intersect.

because $\overline{S}$ and $\gamma(\overline{S})$ intersect. This contradicts the assumption. $\square$

Due to unclear reasons, neither Weil nor Magnier have ever published the proof of their statements. Unfortunately Weil's idea was ignored until the 1960s. In the 60s within the framework of the general progress in dynamical systems Anosov revived the interest to this problem. Anosov's study was motivated by the common asymptotic behavior which the trajectories of a surface flow and the geodesics curves can exhibit.

In the 1960s, on the the American continent apparently under M. Morse's influence, G. Hedlund brought to the attention of N.G. Markley (who was then his student) all the bunch of problems connected with the area [**50**]. Unfortunately only a minor part of Markley's results has been published [**51**, **52**]. N.G. Markley proved independently Weil's conjecture as well as several related results for the flows on surfaces of constant negative curvature.

In 1966 at Tiraspol's Symposium on General Topology Anosov communicated the theorem stating that unbounded coverings for semitrajectories of smooth flows with a finite number of fixed points on a closed surface of nonpositive Euler characteristic have an asymptotic direction. Besides, Anosov formulated a number of conjectures on the behavior of coverings to the curves without self-intersections. Anosov's theorem and Anosov's conjectures sparked the interest to the above area.

One of the conjectures of Anosov concerned a deviation of a curve from the co-asymptotic geodesic. In 1967, V. Pupko [**63**] stated the restricted deviation property for the curve without self-intersections but her proof was unclear. About 1972, Aranson and Grines came to Moscow to present their results on the classification of transitive flows on hyperbolic surfaces. The essential part of this presentation was the proof of the existence of an asymptotical direction for a nontrivially recurrent trajectory. Anosov asked Grines about the deviation property, and it looked like he doubted Pupko's statement. Soon, Grines realized that in the example by C. Robinson and F. Williams [**64**] there are curves with an unbounded deviation. Aranson and Grines constructed a counter-example (described by Anosov in [**2**]) to Pupko's statement even if a curve is a semi-trajectory of flow on closed orientable surface of genus $g = 2$. Later Anosov constructed counter-examples to Pupko's statement on other surfaces including $\mathbb{T}^2$ and the Klein bottle [**2**, **3**, **6**].

3. Nonlocal behavior of curves on universal coverings

Weil's theorem and conjecture say that a lift of a simple (i.e., without self-intersections) curve has an asymptotic direction provided the lift goes to the circle at infinity S_∞. However, this does not imply the existence of an asymptotic direction for an unbounded lift of a semitrajectory because *apriori* the covering semitrajectory can oscillate. Here we represent Anosov's results on the existence of asymptotic directions for semitrajectories of surface flows.

3.1. Anosov's theorems on asymptotic directions. The first theorem is on continuous (or topological) flows. To formulate this theorem, we need the following definition. A subset $F \subset M^2$ is called *contractible to a point* if there exists a continuous mapping $\varphi : F \times [0;1] \to M$ such that $\varphi(m, 0) = m$ and $\varphi(m, 1) = m_0$ for any $m \in F$, where $m_0 \in F$ is a certain point.

THEOREM 3.1. *If the set of fixed points of a topological flow f^t on a closed surface M^2 of nonpositive Euler characteristic is contractible to a point, then any semitrajectory of the covering flow $\overline{f}^t$ on $\overline{M}^2$ is either bounded or has an asymptotic direction.*

COROLLARY 3.1. *If the set of fixed points of a topological flow f^t on a closed surface M^2 of nonpositive Euler characteristic is finite, then any semitrajectory of the covering flow $\overline{f}^t$ on $\overline{M}^2$ is either bounded or has an asymptotic direction.*

Presently, Theorem 3.1 gives the most general sufficient conditions for an unbounded semitrajectory of a topological flow to have an asymptotic direction.

THEOREM 3.2. *If a flow f^t on a closed surface M^2 of nonpositive Euler characteristic is analytic, then any semitrajectory of the covering flow $\overline{f}^t$ on $\overline{M}^2$ is either bounded or has an asymptotic direction.*

Theorem 3.2 does not follow from Theorem 3.1 because the set of fixed points of an analytic flow may contain, for instance, homotopically nontrivial closed curves and, hence, may not be contractible to a point.

Note that the problem of whether a closed curve has an asymptotic direction is solved without difficulty.

THEOREM 3.3. *Let C be a closed curve on a surface M^2. Then,*

(1) *if C is null homotopic, then any of its lifts $\overline{C}$ to $\overline{M}^2$ is a closed (and, hence, bounded) curve;*

(2) *if C is non-null-homotopic, then any of its lifts $\overline{C}$ to $\overline{M}^2$ is a nonclosed infinite curve both of whose semi-infinite curves have a rational asymptotic direction.*

Now, we quote some sufficient conditions for the existence of asymptotic directions for so-called widely disposed simple semi-infinite continuous curve. These conditions are then applied to special curves. We'll consider the cases of flat and hyperbolic surfaces separately.

Let $\mathcal{T}$ be an arc or a simple closed curve that intersects a semi-infinite curve l^+ transversally. The curve l^+ is said to be *widely disposed with respect to* $\mathcal{T}$ if there do not exist any $\mathcal{T}$-loops that bound a disk. Recall that $\mathcal{T}$-loop is defined as follows. Suppose that l^+ intersect $\mathcal{T}$ at two points a and b. The arc $\widehat{ab}$ of l^+ with endpoints a, b is called a $\mathcal{T}$-*arc* if $\mathcal{T} \cap \widehat{ab} = a \cup b$. The $\mathcal{T}$-arc together with a subinterval $\overline{ab} \subset \mathcal{T}$ between the points a, b forms a simple closed curve $\widehat{ab} \cup \overline{ab}$ called a $\mathcal{T}$-*loop*.

On $\mathbb{T}^2$, the concept of wide disposition with respect to a non-null-homotopic simple closed curve coincides with the concept of orientability of the intersection with this curve (orientability means that the index of the intersection is the same at every points of intersection). It can easily be shown that the orientability of the intersection implies the wide disposition on any surface.

THEOREM 3.4. *Let C be a simple closed curve on $\mathbb{T}^2$, and suppose that a simple infinite curve l orientably intersects C infinitely many times in such a way that the positive and negative semi-infinite curves l^+ and l^- of l also intersect C infinitely many times. Then, any lift $\overline{l}$ of l to the universal covering $\mathbb{R}^2$ is an infinite curve whose positive and negative semi-infinite curves $\overline{l}^+$ and $\overline{l}^-$ have diametrically opposite asymptotic directions.*

On a hyperbolic surface, one can easily construct an example of a semi-infinite curve that is widely disposed with respect to C and intersects the curve C non-orientably. Let us formulate a sufficient condition for the existence of an asymptotic direction of a widely disposed semi-infinite curve on a hyperbolic surface.

THEOREM 3.5. *Let C be a simple closed curve on a hyperbolic surface M^2, and suppose that a simple semi-infinite curve l^+ is widely disposed with respect to C and transversally intersects C infinitely many times. Then any lift $\overline{l}^+ \subset \Delta$ of l^+ has an asymptotic direction. Moreover, the point of S_∞ that is accessible by $\overline{l}^+$ is the topological limit of the lifts $\overline{C}_i$ of C that are successively intersected by $\overline{l}^+$ as the parameter increases.*

This theorem also holds true for noncompact and non-orientable surfaces [**12**]. The theorem can be conveniently applied to study the existence of asymptotic directions for semi-infinite curves belonging to a simple curve that is infinite in both directions.

THEOREM 3.6. *Let C be a simple closed curve on a hyperbolic surface M^2, and suppose that a simple infinite curve l is widely disposed with respect to C; moreover, the positive and negative semi-infinite curves l^+ and l^- of l transversally intersect C infinitely many times. Then any lift $\overline{l}$ of l on the universal covering is an infinite curve whose positive and negative semi-infinite curves $\overline{l}^+$ and $\overline{l}^-$ have asymptotic directions. Moreover, $\omega(\overline{l}^+) \neq \alpha(\overline{l}^-)$.*

3.2. Anosov's theorems on the approximation of curves. In this section, we represent one of the fundamental results in this field, Anosov's theorem [**3**] on the approximation, from the viewpoint of the Frechet distance ρ_F, of a semi-infinite continuous curve by a semitrajectory of a smooth flow.

THEOREM 3.7. *Let $l = \{m(t) : t \geq 0\}$ be a semi-infinite continuous curve without self-intersections on a surface M^2. Then for any $r > 0$ there exists a C^∞ flow f^t on M^2 such that one of its semitrajectories $T = \{f^t(m_0) : m_0 \in M^2,\, t \geq 0\}$ lies at the Frechet distance $\leq r$ from l; i.e.,*

$$\rho_F([l], [T]) \leq r.$$

Recall that the inequality $\rho_F([l], [T]) \leq r$ means the following: there exists a homeomorphism $s : [0; +\infty) \to [0; +\infty)$ such that $\sup_{t \geq 0} d(m \circ s(t), f^t(m_0)) \leq r$, where $d(\cdot, \cdot)$ is the metric on M^2.

The main idea of the proof of Theorem 3.7 is to approximate the curve l by a C^∞-embedded curve l_∞ that is r-close to l in the sense of the Frechet metric and is obtained by a successive construction of arcs of increasing length. Since l_∞ is smoothly embedded, it is embedded into M^2 together with a certain strip. Next, we declare all boundary points of this strip fixed points and construct a C^∞ flow with a semitrajectory l_∞.

Note that the initial curve l may contain points of its own limit set or may even completely belong to its own limit set. Therefore, the construction of l_∞ must be accompanied by "extruding the tails" of intermediate semi-infinite curves from a certain neighborhood of their initial arcs.

In 1995, Anosov [**8**] generalized Theorem 3.7 and obtained its metric (in the sense of measure theory) version.

THEOREM 3.8. *Let $l = \{m(t) : t \geq 0\}$ be a semi-infinite continuous curve without self-intersections on a surface M and μ be a smooth measure on M with everywhere positive C^∞ density. Then, for any $r > 0$, there exists a C^∞ flow f^t that preserves the measure μ and is such that one of its semitrajectories $T = \{f^t(m_0) : m_0 \in M^2,\, t \geq 0\}$ lies at the Frechet distance $\leq r$ from l; i.e.,*

$$\rho_F([l], [T]) \leq r.$$

3.3. Limit sets at infinity. Here, we consider the question concerning possible limit sets at infinity for arbitrary unbounded curves that cover simple semi-infinite curves on a surface. For an observer standing on the universal covering, this question can be reformulated as follows: What regions of the horizon can be covered by an unbounded curve that is a lift of a simple curve? In particular, do there exist "wild" covering curves $\overline{l}^+$ whose complete limit set $Lim(\overline{l}^+)$ contains the whole absolute? A positive answer to the latter question was obtained by Anosov in [**3**].

Anosov's wild curve. We provide a schematic example (in the form of an existence theorem) of a "wild" covering curve whose limit set contains the whole absolute and that is projected to a simple curve on a surface.

THEOREM 3.9. *Let M^2 be a closed surface of nonpositive Euler characteristic. There exists a continuous semi-infinite curve $l \subset M^2$ without self-intersections such that its lift $\overline{l}$ to the universal covering $\overline{M}^2$ contains the whole absolute in its limit set.*

Sketch of the proof. Take a countable family of neighborhoods $V_n \subset \overline{M}^2 \cup S_\infty$ such that 1) $\bigcup_n V_n \supset S_\infty$; 2) for any point $\sigma \in S_\infty$ and any of its neighborhoods $U(\sigma)$, there exists V_i such that $V_i \subset U(\sigma)$. Take a smooth semi-infinite curve $\bar{l}_0 = \{\overline{m}_0(t) : t \geq 0\}$ on the universal covering that intersects all V_n. Then, $\omega(\bar{l}_0) \supset S_\infty$. The curve $l_0 = \pi(\bar{l}_0) \subset M$, generally speaking, has self-intersections; we will deform it to obtain the required curve without self-intersections. After reparameterizing and slightly jiggling the curve l_0, we can divide it into arcs $l_{n,n+1} = \{m_0(t) : n \leq t \leq n+1\}$, $n \geq 0$, that satisfy the following conditions:

1) each arc $l_{n,n+1}$ has no self-intersections; 2) $m_0(n+1) \notin \{m_0(t) : 0 \leq t < n+1\}$; 3) there exists a sequence $t_n \geq 1$ such that $\overline{m}_0(t_n) \in V_n$ for any $n \geq 1$.

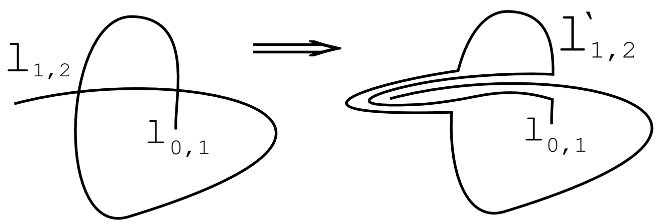

FIGURE 5. Deleting of self-intersections.

Let us fix the endpoints of the arc $l_{1,2}$ and deform it into an arc $l'_{1,2}$ so that the arc $l_{0,1} \cup l'_{1,2}$ has no self-intersections, see Fig. 5. Let us subject the curve $l_{2,3}$ (where the role of $l_{0,1}$ is now played by the arc $l_{0,1} \cup l'_{1,2}$) to a similar deformation. Continuing this process, we obtain a curve l without self-intersections on the surface M^2. By property (3), its lift $\bar{l}$ contains the whole absolute in its limit set. □

The following theorem on the existence of a wild semitrajectory follows immediately from Theorems 3.7 and 3.9.

THEOREM 3.10. *On any closed surface M^2 of nonpositive Euler characteristic, there exists a C^∞ flow f^t that has a positive semitrajectory l^+ such that its lift $\bar{l}^+$ to the universal covering contains the whole absolute in its limit set at infinnity, $S_\infty = lim_\infty(\bar{l}^+)$.*

Limit sets at infinity of the lifts of curves on the torus $\mathbb{T}^2$. All possible limit sets at infinity for unbounded curves that are the lifts of curves without self-intersections have been described only for $\mathbb{T}^2$. It is obvious that any point of S_∞ may serve as the limit set at infinity for a lift of a ray, which is projected to a simple curve on $\mathbb{T}^2$. Moreover, any pair of diametrically opposite points on S_∞ can serve as the limit set at infinity. It follows from Theorem 3.9 that the whole absolute may serve as the limit set at infinity. The following theorem, which was proved by Glutsyuk [**36**] after Anosov's questions, shows that any (either open or closed) arc of the absolute that covers more than half of S_∞ cannot be a limit set. Recall that the absolute S_∞ is a unit circle and, hence, has the length 2π.

THEOREM 3.11. *Let $\Omega \subset S_\infty$ be a closed set such that there exists an arc of length strictly less than π among the connected components of the set $S_\infty \setminus \Omega$. Then Ω cannot serve as the limit set at infinity for any curve without self-intersections on $\mathbb{T}^2$. However, any closed arc of S_∞ of length at most π is realized as the limit set at infinity for a certain curve without self-intersections on $\mathbb{T}^2$.*

D. Panov [**57**] constructed a pseudo-Anosov homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ such that a lift for any unstable leaf is dense in the part of the universal covering $\mathbb{R}^2$. For closed hyperbolic surfaces, the question concerning possible limit sets at infinity has not yet been solved in the general case.

4. Asymptotic properties of special curves

Now, we are mainly considering curves that have the dynamical origin e.g. trajectories of flows, and leaves of foliations, and one-dimensional invariant manifolds of diffeomorphisms with hyperbolic structure on non-wandering sets. Such curves often form so-called local laminations. The motivation for the definition of local lamination is the statement from theory of differential equations that says that trajectories locally looks like parallel straight lines away from the singularities.

Local laminations. Let $\mathcal{M} \subset M^2$ be a subset of M^2 (which may coincide with M^2) that contains some closed subset $S \subset \mathcal{M}$. Let $\mathcal{M}$ be the union $S \bigcup_\alpha L_\alpha$, where L_α are pairwise disjoint C^r-smooth simple curves. We say that the family $\{L_\alpha\}$ forms a $C^{r,l}$ ***local lamination*** if, for any point $P \in \mathcal{M} - S$, there exist a neighborhood $U(P)$ of P, and a C^l diffeomorphism $\psi : U(P) \to \mathbb{R}^2$, $\psi(P) = (0,0)$, such that any connected component of the intersection $U(P) \cap L_\alpha$ (provided that this intersection is nonempty) is mapped by ψ onto the line $y = const$ and the restriction $\psi|_{U(P)\cap L_\alpha}$ is a C^r diffeomorphism onto its image. The curves L_α are called *leaves*. Each point of the set S is called a *singularity*. A point that is not a singularity is called *regular*.

The concept of a local lamination generalizes the classical concepts of lamination and foliation. If $\cup_\alpha L_\alpha$ is closed and $S = \emptyset$, then $\mathcal{M}$ is called a $C^{r,l}$ *lamination*. An important example of a lamination is a geodesic lamination. Note that a local $C^{r,l}$ lamination without singularities is not always a lamination. If $\mathcal{M} = M^2$, then $\mathcal{M}$ is called a $C^{r,l}$ *foliation*. One may say that a local lamination with singularities is a "foliation" (with singularities) on a subset. If this subset is closed and there are no singularities, then we obtain a lamination. If this subset coincides with the manifold (and there may be some singularities), then the local lamination is a foliation. It follows from the above that the concept of a local lamination is a quite general concept, which includes, as particular cases, the concepts of lamination and foliation.

A foliation on a surface is called *transitive* if it has at least one everywhere dense leaf. A foliation is called *highly transitive* if every (one-dimensional) leaf is dense on a surface. Obviously, any highly transitive foliation is a transitive one. One can prove that if a transitive foliation has only isolated singularities, then each singularity is of saddle type (see Fig. 6). In general, a transitive foliation can have separatrix connections, while a highly transitive foliation has no separatrix connections (obviously, a separatrix connection can't be dense). A highly transitive foliation can have fake saddles whose number could be arbitrary with no connection with the topology of supporting surface. In this sense, fake saddles are artificial. Therefore, it is natural to distinguish transitive foliations without separatrix connections and fake saddles. A highly transitive foliation with no fake saddles is called *irrational* if it has only isolated singularities. An irrational foliation is called *strongly irrational* if it is without thorns.

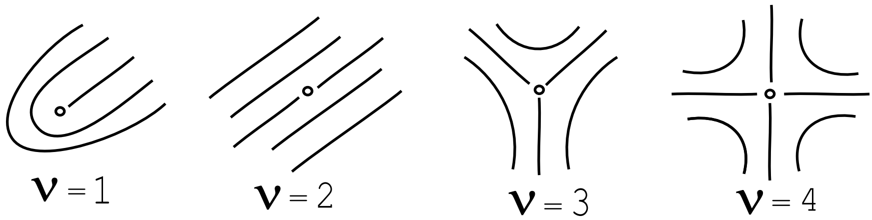

FIGURE 6. Singularities of saddle type: the thorn ($\nu = 1$), the fake saddle ($\nu = 2$), the tripod ($\nu = 3$), a saddle with four separatrices ($\nu = 4$), where ν is the number of separatrices.

The question of whether closed leaves and non-closed leaves that tend to a closed leaf have an asymptotic direction is actually solved as follows:

- If a closed leaf is null homotopic (as a curve), then it has no asymptotic direction. If a closed leaf is non-null-homotopic, then its lift is an infinite curve both of whose semi-infinite curves have rational asymptotic directions.
- If a non-closed leaf tends to a null-homotopic closed leaf, then it has no asymptotic direction. If a non-closed leaf tends to a non-null-homotopic closed leaf, then it has a rational asymptotic direction.

It is convenient to consider flows as orientable foliations using the corresponding terminology. A similar statement holds true for a semitrajectory that tends to a loop composed of separatrix connections and saddles. Obviously, a semitrajectory that tends to a single fixed point has no asymptotic direction. It remains to consider the question of whether semitrajectories that tend to trajectories whose limit set contains regular points have an asymptotic direction. According to Maier theorem [**48**, **49**], such semitrajectories tend to nontrivially recurrent trajectories. Therefore, it is natural to consider first the question of whether nontrivially recurrent semitrajectories have an asymptotic direction.

Nontrivially recurrent semitrajectories and semileaves. Recall that a nontrivially recurrent semitrajectory is a nonclosed semitrajectory that belongs to its own limit set. Such semitrajectories may exist only on orientable surfaces of genus $g \geq 1$ and on non-orientable surfaces of genus $g \geq 3$ [**13**, **49**, **62**]. The Euler characteristic of these surfaces is nonpositive, and their universal covering is homeomorphic to a disk. The following theorem proved in [**19**] shows that a nontrivially recurrent semitrajectory of a flow with any set of fixed points has an asymptotic direction, and this asymptotic direction is irrational.

THEOREM 4.1. *Let l be a nontrivially recurrent semitrajectory of a flow f^t on a closed surface M^2 of nonpositive Euler characteristic, and let $\bar{l}$ be its lift to the universal covering $\overline{M}^2$. Then, $\bar{l}$ has an irrational asymptotic direction.*

COROLLARY 4.1. *Let l be a nontrivially recurrent trajectory of a flow f^t on a closed surface M^2 of nonpositive Euler characteristic, and let $\bar{l}$ be its lift to the universal covering $\overline{M}^2$. Then, $\bar{l}$ has irrational asymptotic directions $\omega(\bar{l}), \alpha(\bar{l}) \in S_\infty$; moreover, $\omega(\bar{l}) \neq \alpha(\bar{l})$.*

An analysis of the proof of Theorem 4.1 shows that similar assertions are valid for local laminations on a hyperbolic surface.

THEOREM 4.2. *Let C be a simple closed curve on a hyperbolic surface M^2, and suppose that all leaves of a local lamination $\mathcal{D}$ are widely disposed with respect to C. Suppose that a nontrivially recurrent leaf l of $\mathcal{D}$ transversally intersects C infinitely many times. Then, the positive and negative semileaves of the covering leaf $\bar{l}$ have different irrational asymptotic directions on the universal covering.*

4.1. Dynamical and asymptotical properties. Here we show how properties of remote limit points influence dynamical properties of flows and foliations. For example, the first theorem says that if a foliation (or a flow) with a finite set of singularities has a semi-leaf with an irrational asymptotic direction, then the foliation has a quasiminimal set. Recall that a quasiminimal set is the closure of a nontrivially recurrent semi-leaf. A quasiminimal set is called irreducible if any nontrivially homotopic closed curve on M intersects this quasiminimal set.

THEOREM 4.3. *If a foliation $\mathcal{F}$ with finitely many singularities on M^2 has a semi-leaf with an irrational direction, then $\mathcal{F}$ has a quasiminimal set (in particular, $\mathcal{F}$ has a nontrivially recurrent leaves).*

Let us introduce some notation. We consider only hyperbolic surfaces here. In this case, using geodesic laminations, we can get a good description for points of the circle at infinity.

A lamination whose leaves are geodesics is called a *geodesic lamination.* One can reformulate this definition in the traditional way: a geodesic lamination is a family of pairwise disjoint simple geodesics such that their union is a closed set. Here, a *simple geodesic* is either an infinite curve without self-intersections or a simple closed curve. Denote by $\mathcal{L}(M^2) = \mathcal{L}$ the set of geodesic laminations on M^2. A geodesic lamination is *trivial* if it consists of closed geodesics and isolated non-closed geodesics. Denote the set of trivial geodesic laminations by $\Lambda_{triv}(M^2) = \Lambda_{triv}$. So, it is natural to call a geodesic lamination *nontrivial* if it contains a non-closed geodesic that is non-isolated in the geodesic lamination. A nontrivial lamination is said to be *strongly nontrivial* if it consists of non-closed and non-isolated geodesics. Denote by Λ the set of strongly nontrivial geodesic laminations. A lamination is *minimal* if it contains no proper sub-laminations. A minimal strongly nontrivial geodesic lamination is called *weakly irrational.* It follows from [**32**, **35**] that if $\mathcal{L}$ is a strongly nontrivial geodesic lamination then 1) every geodesic of $\mathcal{L}$ is nontrivially recurrent; 2) $\mathcal{L}$ is a union of connected pairwise disjoint weakly irrational geodesic laminations; 3) every geodesic of a weakly irrational geodesic lamination is dense in this lamination.

So, Λ consists of weakly irrational geodesic laminations. Denote by Λ_{or} (respectively, Λ_{non}) the set of orientable (respectively, non-orientable) weakly irrational geodesic laminations on M, $\Lambda = \Lambda_{or} \cup \Lambda_{non}$. An important class of geodesic laminations is given by irreducible laminations. A geodesic lamination $G \in \Lambda$ is called *irreducible* if any closed geodesic on M^2 intersects G. On a closed orientable hyperbolic surface, this condition is equivalent to the fact that any component of the set $M - G$ is simply connected [**32**]. Denote by $\Lambda^{irr} \subset \Lambda$ the set of irreducible weakly irrational geodesic laminations. We'll call a geodesic lamination from Λ^{irr} *strongly irrational* (or simply, *irrational*). Set

$$\Lambda_{or} \cap \Lambda^{irr} \stackrel{\text{def}}{=} \Lambda^{irr}_{or}, \qquad \Lambda_{non} \cap \Lambda^{irr} \stackrel{\text{def}}{=} \Lambda^{irr}_{non}.$$

Let $G \in \mathcal{L}$ be a geodesic lamination on a hyperbolic surface M. It is clear that the preimage $\pi^{-1}(G) \stackrel{\text{def}}{=} \overline{G}$ is a geodesic lamination on the universal covering Δ. If $\overline{G}$ has a geodesic with an ideal endpoint $\sigma \in S_\infty$, we say that σ is *accessible (or, reached, or attained) by the lamination* $\overline{G}$. Taking a certain liberty, we will also say that σ is accessible by the lamination G, although this lamination lies on the surface. Denote by $\overline{G}_\infty \subset S_\infty$ the set of points on S_∞ that are accessible by the lamination $\overline{G}$. Again, taking a certain liberty, we will use the notation G_∞. Sometimes, when the subscript is in use, we will denote the set of accessible points by $\overline{G}(\infty)$ or $G(\infty)$. Thus, $\Lambda(\infty) \subset S_\infty$ is the set of points reached by all laminations from Λ, and $\Lambda^{irr}(\infty) \subset S_\infty$ is the set of points reached by the strongly irrational geodesic laminations.

THEOREM 4.4. *Let $\mathcal{F}$ be a foliation with finitely many singularities on M^2 and l^+ a positive semi-leaf of $\mathcal{F}$ such that its lifting $\overline{l}^+$ to Δ has the asymptotical direction $\sigma \in S_\infty$. If $\sigma \in \Lambda(\infty) - \Lambda^{irr}(\infty)$, then $\mathcal{F}$ is not highly transitive and there is a nontrivially homotopic closed curve that is not intersected by any nontrivially recurrent leaf. If $\sigma \in \Lambda^{irr}(\infty)$, then $\mathcal{F}$ has an irreducible quasiminimal set. Moreover, $\mathcal{F}$ is either highly transitive or can be obtained from a highly transitive foliation by a blow-up operation of at least countable set of leaves and by the Whitehead operation. When $\mathcal{F}$ is not highly transitive, $\mathcal{F}$ has a unique nowhere dense quasiminimal set.*

Take $G \in \Lambda^{irr}$. A point $\sigma \in \overline{G}(\infty)$ is a *point of the first kind* if there is only one geodesic of $\overline{G}$ with the endpoint σ. Otherwise, σ is called a *point of the second kind*. One can prove that this definition does not depend on the choosing of $G \in \Lambda^{irr}$. The following theorem shows that the type of asymptotic direction reflects certain "dynamical" properties of the foliation [**28**].

THEOREM 4.5. *Let $\mathcal{F}$ be an irrational foliation on M and l^+ a positive semi-leaf of $\mathcal{F}$ such that its lifting $\overline{l}^+$ to Δ has the asymptotical direction $\sigma \in S_\infty$. Then $\sigma \in \Lambda^{irr}(\infty)$. Moreover,*

(1) *If σ is a point of the first kind then l^+ belongs to a nontrivially recurrent leaf.*
(2) *If σ is a point of the second kind then l^+ belongs to a separatrix of a saddle singularity.*

We have the following sufficient condition for the existence of a continuum set of fixed points.

THEOREM 4.6. *Suppose that a flow f^t on M^2 reaches a point from $\Lambda^{irr}_{non}(\infty)$. Then f^t has a continuum of fixed points. Furthermore, f^t has neither nontrivially recurrent semitrajectories nor closed transversals nonhomotopic to zero.*

It turns that some points of S_∞ attained by C^∞ flows prevent these flows to be analytic. Recall that $\sigma \in S_\infty$ is called a *point achieved by f^t* if there is a positive (or negative) semitrajectory $l^\pm$ of f^t such that some lift $\overline{l}^\pm$ of $l^\pm$ has the asymptotic direction defined by σ.

Denote by A_{fl}, A_∞, $A_{an} \subset S_\infty$ the sets of points achieved by all topological, C^∞, and analytic flows respectively. Due to the remarkable result by Anosov [**4**], $A_{fl} = A_\infty$ (see Theorem 3.8). Obviously, $A_{an} \subset A_\infty$. It follows from the following theorem that $A_\infty - A_{an} \neq \emptyset$ [**29**].

THEOREM 4.7. *There exists a continual set $U(M^2) \subset A_\infty$ such that any C^∞ flow f^t that reaches a point from $U(M^2)$, is not analytic. The set $U(M^2)$ is dense and has zero Lebesgue measure on S_∞.*

One can prove that $\Lambda_{triv}(\infty) \subset A_{an} \subset \Lambda_{triv}(\infty) \cup \Lambda_{or}(\infty)$, and $\Lambda_{non}(\infty) \subset A_\infty - A_{an}$.

4.2. Geodesic frameworks of local laminations. Applying a medical terminology, one can say that the geodesic framework of local lamination is its geodesic skeleton around which the leaves that have asymptotic directions are grouped. The geodesic framework contains the full information on the asymptotic directions of leaves of a given local lamination. The geodesic framework of a local lamination is defined only if this lamination has a leaf or a generalized leaf (the union of separatrices and their singularities) that has a co-asymptotic geodesic. To be precise, let $\mathcal{D}$ be a local lamination on M^2. Denote by $\mathcal{A}^{\pm}(\mathcal{D})$ the union of all leaves and generalized leaves of $\mathcal{D}$ that have co-asymptotic geodesics. The topological closure

$$G(\mathcal{D}) \stackrel{\text{def}}{=} clos \bigcup_{l \in \mathcal{A}^{\pm}(\mathcal{D})} g(l)$$

is called the **geodesic framework of the local lamination** $\mathcal{D}$.

Since a lamination and a foliation are local laminations, we have defined the concepts of geodesic framework for foliations and laminations. It follows immediately from the definition that a geodesic framework is a geodesic lamination. The geodesic framework of an arbitrary invariant set of a local lamination is defined similarly.

On $\mathbb{T}^2$, a geodesic lamination either forms an irrational linear foliation (hence, this lamination fills the whole torus) or is a family of pairwise homotopic closed geodesics. Therefore, below in this section, we'll consider geodesic frameworks on closed orientable hyperbolic surfaces.

Geodesic frameworks of quasiminimal sets. Recall that by Theorem 3.6, a nontrivially recurrent leaf l has a co-asymptotic geodesic $g(l)$ provided l is widely disposed with respect to some simple closed curve C.

LEMMA 4.1. *Let l be a nontrivially recurrent leaf of a local lamination $\mathcal{D}$, and suppose that l is widely disposed with respect to a certain simple closedown curve C and transversally intersects C. Then the co-asymptotic geodesic $g(l)$ is nontrivially recurrent.*

It follows from a theorem of Cherry [**33**] (see generalizations in [**26**,**27**]) that any quasiminimal set with closed support contains a continuum of nontrivially recurrent leaves each of which is everywhere dense in the quasiminimal set. A quasiminimal set Q is called a *Maier quasiminimal set* if each semi-leaf from Q that does not tend exactly to one singularity is everywhere dense in Q. In particular, a leaf from Q that is different from a separatrix connection is everywhere dense in Q and is nontrivially recurrent in, at least, one direction. The following theorem describes a geodesic framework of the Maier quasiminimal set.

THEOREM 4.8. *Let Q be a Maier quasiminimal set containing a finitely many singularities and separatrices of a local lamination $\mathcal{D}$ with closed support supp $\mathcal{D}$. Suppose that every nontrivially recurrent leaf from Q is widely disposed with respect* *to some simple closed curve C. Then*

- *the geodesic framework $G(Q)$ is equal to clos $g(l)$ for any nontrivially recurrent leaf $l \in Q$;*
- *the geodesic framework $G(Q)$ is a weakly irrational geodesic lamination;*
- *any geodesic from $G(Q)$ is the co-asymptotic geodesic of a certain leaf or a generalized leaf that belongs to Q.*

Geodesic frameworks of special foliations. Consider a foliation with isolated singularities of negative index (in particular, with saddles of negative index). Then any semileaf of such a foliation that does not tend to a singularity has an asymptotic direction. In addition, any leaf or generalized leaf that is not a separatrix connection has a co-asymptotic geodesic.

THEOREM 4.9. *Let $\mathcal{F}$ be an irrational foliation on a closed orientable hyperbolic surface M^2 with singularities that are saddles of negative index. Then,*

(1) *the geodesic framework $G(\mathcal{F})$ of $\mathcal{F}$ is an irrational geodesic lamination;*
(2) *any geodesic from $G(\mathcal{F})$ is a co-asymptotic geodesic for a certain leaf or generalized leaf of $\mathcal{F}$. Moreover,*
   (a) *any point $\sigma \in \Lambda_{1,\infty}(M^2) \cap \overline{G}(\mathcal{F})_\infty$ is reached by a leaf projected to an internal nontrivially recurrent leaf on M^2 whose co-asymptotic geodesic is also internal;*
   (b) *any point $\sigma \in \Lambda_{2,\infty}(M^2) \cap \overline{G}(\mathcal{F})_\infty$ is reached by a leaf $\bar{l}$ that is an α-separatrix of a singularity, and the left and right Bendixson extensions of the leaf $\bar{l}$ in the negative direction*[2] *have different asymptotic directions α_1 and α_2. Two geodesics that connect σ with the points α_1 and α_2 are sides of a geodesic polygon with a finite number of sides that belong to $\overline{G}(\mathcal{F})$, and these geodesics are projected to boundary geodesics on M^2;*
(3) *each component of the set $M^2 - G(\mathcal{F})$ is a simply connected domain any of whose lifts to the universal covering is the interior of a geodesic polygon P with a finite number of sides and with vertices lying on S_∞. In this case, the sides of P belong to $\overline{G}(\mathcal{F})$, and each vertex is reached by exactly one separatrix of a certain saddle of the covering foliation $\overline{\mathcal{F}}$. Conversely, each saddle of $\overline{\mathcal{F}}$ corresponds to a unique geodesic polygon formed by geodesics from $\overline{G}(\mathcal{F})$, such that the separatrices of the saddle reach all vertices of the polygon and the number of separatrices is equal to the number of vertices.*

4.3. Deviations of curves from co-asymptotic geodesics. Here we focus our attention on the deviation of curves that have asymptotic directions from co-asymptotic geodesics on the universal covering. First, we consider examples of curves with unbounded deviation. Historically, the first example with an unbounded deviation was constructed by Aranson and Grines. They constructed a foliation that has a nontrivially recurrent leaf with unbounded deviation from the co-asymptotic geodesic. We now describe this example.

On a closed orientable surface $M^2_{g_1}$ of genus $g_1 \geq 1$, consider an irrational foliation $\mathcal{F}_1$ that has a topological saddle s_1 with $k \geq 3$ separatrices (hence, the index of the saddle is equal to $ind\ s_1 = 1 - \frac{k}{2}$). Since all saddles of the foliation $\mathcal{F}_1$ have a negative index, $\mathcal{F}_1$ is a widely disposed foliation with respect to any

[2]Without loss of generality, we may assume that the leaf $\bar{l}$ is oriented toward the point σ.

closed transversal and a transversal segment. On another surface $M^2_{g_2}$ of genus $g_2 \geq 1$, take a Denjoy-type foliation $\mathcal{F}_2$ with a minimal set $\Omega(\mathcal{F}_2)$ such that the set $M^2_{g_2} - \Omega(\mathcal{F}_2)$ has a component S_2 of index $ind\ s_1$, Fig. 7.

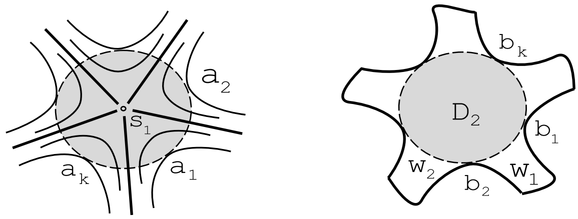

FIGURE 7. The saddle s_1 and the component S_2 of index $ind\ s_1$.

Let us place the saddle s_1 inside a disk D_1 whose boundary ∂D_1 is transversal to the foliation $\mathcal{F}_1$ everywhere except for points $a_1, \dots, a_k \in \partial D_1$ that are arranged in the order corresponding to the positive (counterclockwise) orientation of ∂D_1. Without loss of generality, we may assume that the leaves passing through the points $a_1, \dots, a_k$ are pairwise different and are not separatrices of any saddles. Between the points a_i and a_{i+1}, $i = 1, \dots, k$ (where $a_{k+1} = a_1$), on ∂D_1, there is a unique point of intersection of a separatrix of the saddle s_1 with ∂D_1, which we denote by c_i, such that the arc $(s_1; c_i)$ of the separatrix does not intersect ∂D_1 (we assume that $c_{k+1} = c_1$). Then the foliation $\mathcal{F}_1$ induces in D_1 the first-return map

$$\phi_1 : \partial D_1 - \bigcup_{i=1}^{k} c_i \to \partial D_1 - \bigcup_{i=1}^{k} c_i,$$
$$\phi_1|_{(c_i;a_i]} : (c_i; a_i] \to [a_i; c_{i+1}),\ \phi_1|_{[a_i;c_{i+1})} : [a_i; c_{i+1}) \to (c_i; a_i]$$

where $i = 1, \dots, k$. By the construction, $\phi_1^2 = \mathrm{id}$. In the component S_2, take an open disk $D_2 \subset S_2$ whose boundary ∂D_2 intersects $\Omega(\mathcal{F}_2)$ only at points $b_1, \dots, b_k \in \partial D_2$ that are arranged in the order corresponding to the negative (clockwise) orientation of ∂D_2. In addition, let us require that the disk D_2 divides S_2 into k domains W_i, $i = 1, \dots, k$, that are homeomorphic to an open strip, Fig. 7. Since the index of the component S_2 is $ind\ s_1 = 1 - \frac{k}{2}$, this can be done.

Let us declare that the points a_i and c_i, $i = 1, \dots, k$, are the singularities of the foliation $\mathcal{F}_1$, and denote the obtained foliation by $\mathcal{F}_1'$. Note that since the leaves passing through the points $a_1, \dots, a_k$ are pairwise different and are not separatrices, any one-dimensional leaf of the foliation $\mathcal{F}_1'$ different from the leaves of the form $(s_1; c_i)$ is everywhere dense on $M^2_{g_1}$.

Let us modify the foliation $\mathcal{F}_2$ by placing a Reeb foliation in each strip W_i, $i = 1, \dots, k$, and declaring each point of the set $\Omega(\mathcal{F}_2)$ a singularity. The points $d_i \in (b_i; b_{i+1}) \subset \partial D_2$, $i = 1, \dots, k$, are chosen arbitrarily, where $b_{k+1} = b_1$. The foliation is extended arbitrarily into the interior of D_2. Let us glue together the two surfaces $M^2_{g_1} - Int\ D_1$ and $M^2_{g_2} - Int\ D_2$ by $\Theta : \partial D_1 \to \partial D_2$. As a result, we obtain a closed surface $M^2_{g_1+g_2}$ of genus $g_1 + g_2 \geq 2$, which is the connected sum $M^2_{g_1} \sharp M^2_{g_2}$ of the surfaces $M^2_{g_1}$ and $M^2_{g_2}$. The foliations $\mathcal{F}_1'$ and $\mathcal{F}_2'$ form a foliation on $M^2_{g_1+g_2}$, which we denote by $\mathcal{F}$. It follows from the construction that there is a leaf l of $\mathcal{F}$ which is everywhere dense on the surface. Hence, $\bar{l}$ has an irrational asymptotic direction. One can prove that $\bar{l}$ possesses the property of unbounded deviation.

Examples of curves with rational and irrational asymptotic directions and with the property of unbounded deviation on $\mathbb{T}^2$ and the Klein bottle were first constructed by Anosov [**4**, **6**]. Note that the construction of a curve with a rational direction is a more complicated. An elegant example of the flow with a semi-trajectory that has a rational asymptotic direction and possesses the property of unbounded deviation was constructed in [**53**] (see also [**54**]).

It is natural to consider conditions under which the deviation from special curves is bounded. The following theorem was proved in [**8**].

THEOREM 4.10. *Let f^t be a topological flow on $\mathbb{T}^2$ and $\bar{l}$ be a lift to $\mathbb{R}^2$ of a semitrajectory $l = \pi(\bar{l})$ that has an asymptotic direction. Suppose that one of the following conditions is fulfilled: 1) the set of fixed points of the flow f^t is contractible to a point; 2) l is a nontrivially recurrent semitrajectory. Then $\bar{l}$ possesses the property of bounded deviation.*

For flows on hyperbolic surfaces, the following theorem was proved in [**24**].

THEOREM 4.11. *Let f^t be a topological flow with a finite set of fixed points on a closed hyperbolic surface M. Let $\overline{l}^+$ be a positive semitrajectory of a covering flow $\overline{f}^t$ on $\overline{M} = \Delta$ that has an asymptotic direction. Then $\overline{l}^+$ possesses the property of bounded deviation.*

As to analytic flows, Anosov [**8**] proved the following result.

THEOREM 4.12. *Let f^t be an analytic flow on a closed orientable surface of constant nonpositive curvature, and let $\bar{l}$ be a semitrajectory of the covering flow that has an asymptotic direction. Then $\bar{l}$ possesses the property of bounded deviation.*

Similar statements hold for surface foliations. Now, we pass on to the local laminations that play an important role in studying surface diffeomorphisms, namely, to one-dimensional stable or unstable manifolds of points that belong to hyperbolic nonwandering sets. The following theorem was proved in [**41**].

THEOREM 4.13. *Let $f : M \to M$ be an A-diffeomorphism of a closed surface M of nonpositive Euler characteristic. Let Ω be a one-dimensional widely disposed attractor (repeller) of f, and let $l_x^{u(s)}$ be the unstable (respectively, stable) manifold of a point $x \in \Omega$. Then both curves $l_x^{u(s)} - x$ has asymptotic direction and possess the property of bounded deviation.*

The analysis of the aforementioned example of Robinson and Williams [**64**] shows that for stable (respectively, unstable) manifolds of points of a one-dimensional attractor (respectively, repeller), Theorem 4.13 is generally incorrect. The above arguments do not work because the theorem on the product structure cannot be applied to all points of stable (respectively, unstable) manifolds of points of a one-dimensional attractor (respectively, repeller). However, if we require that $f : M^2 \to M^2$ is a structurally stable diffeomorphism, then we obtain the following result [**40**, **41**]:

THEOREM 4.14. *Let $f : M^2 \to M^2$ be a structurally stable A-diffeomorphism of a closed orientable hyperbolic surface M and let Ω be a one-dimensional widely disposed attractor (respectively, repeller) of f. Let $l_x^{s(u)}$ be the stable (respectively,* *unstable) manifold of a point $x \in \Omega$ and L^σ be one of the connected components of the set $l_x^{s(u)} - x$ that does not contain a periodic boundary point. Then L^σ has an asymptotic direction and possesses the property of bounded deviation.*

On the torus, Theorem 4.14 is valid without the requirement that the diffeomorphism should be structurally stable [**38**, **41**].

THEOREM 4.15. *Let $f : M^2 \to M^2$ be an A-diffeomorphism of $\mathbb{T}^2$, and let Ω be a one-dimensional widely disposed attractor (repeller) of the diffeomorphism f. Let $l_x^{s(u)}$ be the stable (respectively, unstable) manifold of a point $x \in \Omega$ and L^σ be one of the connected components of the set $l_x^{s(u)} - x$ that does not contain a periodic boundary point. Then L^σ has an asymptotic direction and possesses the property of bounded deviation.*

5. Applications to foliations and dynamical systems

Recall that two foliations $\mathcal{F}_1$, $\mathcal{F}_2$ on a surface M are *topologically equivalent* if there exists a homeomorphism $h : M^2 \to M^2$ such that $h(Sing\ (\mathcal{F}_1)) = Sing\ (\mathcal{F}_2)$ and h sends every leaf of $\mathcal{F}_1$ onto a leaf of $\mathcal{F}_2$. One says that h *maps the foliation* $\mathcal{F}_1$ *onto the foliation* $\mathcal{F}_2$. Orientable foliations (flows) $\mathcal{F}_1$, $\mathcal{F}_2$ are *orbitally topologically equivalent* if the homeomorphism $h : M^2 \to M^2$ above keeps the orientation of leaves (resp., trajectories). In general, the classification assumes the following steps:

(1) Find a constructive topological invariant which takes the same values for topologically equivalent foliations.
(2) Describe all topological invariants which are admissible, i.e. may be realized in the chosen class of foliations.
(3) Find a standard representative in each equivalence class, i.e. given any admissible invariant, one constructs a foliation whose invariant is the admissible one.

An invariant is called *complete* if it takes the same value if and only if two foliations are topologically equivalent. The 'if' part only gives a *relative* invariant.

5.1. Classification of irrational flows and foliations. For completeness, we begin with the classical results on the classification of irrational flows on the torus $\mathbb{T}^2$. After that we present the classification of strongly irrational foliations on a closed hyperbolic surface. Let us recall that an irrational foliation is a foliation with no fake saddles such that every one-dimensional leaf is dense. A strongly irrational foliation is an irrational one with no thorns (saddle type singularities of the index $\frac{1}{2}$). We see that an irrational flow which can be considered as an orientable irrational foliation is a strongly irrational foliation automatically. Note that an irrational flow on $\mathbb{T}^2$ is a transitive (even minimal) fixed-point-free flow.

Irrational flows on 2-torus. A classical example of constructing an effective topological invariant is given by the Poincaré rotation number for fixed-point-free flows on $\mathbb{T}^2$. Let f^t be a flow on $\mathbb{T}^2$. Suppose that f^t has a nontrivially recurrent trajectory l. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the covering projection, and $\bar{l}$ a lift of l. By Weil's theorem, $\bar{l}$ has an asymptotic direction with the co-asymptotic geodesic a straight line $y = kx$. Since any straight line divides the plane $\mathbb{R}^2$ into two half-planes, all nontrivially recurrent trajectories of f^t have the same co-asymptotic geodesic $y = kx$. The number k is called the *rotation number* of f^t, denoted by $rot\ (f^t)$. The existence of nontrivially recurrent trajectory implies the nonexistence of periodic trajectories that non-homotopy to zero. Therefore, the rotation number $k = rot\ (f^t)$ is irrational.

THEOREM 5.1. *Let f_1^t and f_2^t be flows on $\mathbb{T}^2$ such that the both f_1^t and f_2^t have nontrivially recurrent trajectories. If f_1^t and f_2^t are topologically equivalent, then there is an integer unimodular matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that*

$$rot\ (f_2^t) = \frac{-c + a \cdot rot\ (f_1^t)}{d - b \cdot rot\ (f_1^t)}, \quad d - b \cdot rot\ (f_1^t) \neq 0. \tag{4}$$

THEOREM 5.2. *Let f^t be an irrational flow on $\mathbb{T}^2$. Then f^t is orbitally topologically equivalent to a linear flow of the form $\dot{x} = 1$, $\dot{y} = \mu$ where $\mu = rot\ (f^t)$.*

As a consequence, we get the following classical classification result.

THEOREM 5.3. *Let f_1^t and f_2^t be irrational flows on $\mathbb{T}^2$. Then f_1^t and f_2^t are topologically equivalent if and only if there is the integer unimodular matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that (4) holds. Moreover, given any irrational $\mu \in \mathbb{R}$, the flow of the form $\dot{x} = 1$, $\dot{y} = \mu$ is irrational and $\mu = rot\ (f^t)$ (clearly, every number calculated by (4) is irrational).*

Irrational foliations on hyperbolic surfaces. Let $\mathcal{F}$ be a strongly irrational foliation on a closed orientable surface M^2. By Theorem 4.9, the geodesic framework $G(\mathcal{F})$ of $\mathcal{F}$ is a minimal strongly nontrivial geodesic lamination such that each component of $M^2 - G(\mathcal{F})$ is an open geodesic polygon with a finite number of sides and ideal vertices. Thus, $G(\mathcal{F})$ is a strongly irrational geodesic lamination.

The following four theorems obtained by Aranson and Grines [**19**] give a complete classification of strongly irrational foliations on M^2 (see the survey [**21**]). This theorems correspond to the three steps of the topological classification. The first and second theorems produce a constructive topological invariant which takes the same values for topologically equivalent foliations. The third theorem describes all topological invariants which are admissible, i.e. may be realized in the chosen class of foliations. The fourth theorem shows that for any admissible invariant there is a strongly irrational foliation whose invariant is the admissible one.

THEOREM 5.4. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be strongly irrational foliations on a closed orientable hyperbolic surface M. Then $\mathcal{F}_1$, $\mathcal{F}_2$ are topologically equivalent via a homeomorphism $M^2 \to M^2$ homotopic to identity if and only if their geodesic frameworks coincide, $G(\mathcal{F}_1) = G(\mathcal{F}_2)$.*

The *generalized mapping class group GM* is the quotient

$$Homeo\ (M^2)/Homeo_0\ (M^2),$$

where $Homeo\ (M^2)$ is the group of self-homeomorphisms of M^2 and $Homeo_0\ (M^2)$ is the subgroup of homeomorphisms homotopic to the identity. It is known that any homeomorphism $f : M^2 \to M^2$ induces a one-to-one map $f_* : \mathcal{L} \to \mathcal{L}$, $f_* \in GM$ [**32**]. Given $\lambda \in \mathcal{L}$, the family

$$GM(\lambda) = \{f_*(\lambda) \mid f_* \in GM\}$$

is called an *orbit of the geodesic lamination* λ.

THEOREM 5.5. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be strongly irrational foliations on a closed orientable hyperbolic surface M^2. Then $\mathcal{F}_1$, $\mathcal{F}_2$ are topologically equivalent if and only if the orbits of their geodesic frameworks coincide.*

We see that the orbit of a geodesic framework is a complete topological invariant for strongly irrational foliations. Thus the geodesic framework is an analog of Poincaré rotation number for the class of strongly irrational foliations (flows) on $\mathbb{T}^2$. The next theorem shows that the geodesic framework of a strongly irrational foliation is a strongly irrational geodesic lamination. This is completely similar to an irrational Poincare rotation number.

THEOREM 5.6. *Let $\mathcal{F}$ be a strongly irrational foliation on a closed orientable hyperbolic surface M^2. Then its geodesic framework $G(\mathcal{F})$ is a strongly irrational geodesic lamination, $G(\mathcal{F}) \in \Lambda^{irr}$.*

THEOREM 5.7. *Given any strongly irrational geodesic lamination $G \in \Lambda^{irr}$ on a closed orientable hyperbolic surface M, there is a strongly irrational foliation $\mathcal{F}$ on M^2 such that $G(\mathcal{F}) = G$.*

As a consequence, one gets the classification of irrational flows. Note that the classification of irrational flows on closed non-orientable surfaces was obtained in [**18**].

5.2. Classification of nontrivial minimal sets. Recall that a *minimal set* of a flow is a nonempty closed set that is invariant (i.e., consists of trajectories of the flow) and does not contain proper subsets with the above-described properties. A similar definition applies to foliations, provided that "invariant" means a union of leaves and singularities. The trivial minimal sets of flows include fixed points, periodic trajectories, and the minimal set that coincides with a closed surface, which is the torus in this case. The situation for foliations is analogous. Nontrivial minimal sets are nowhere dense and locally homeomorphic to the product of a segment and a Cantor set. A nontrivial minimal set consists of nonclosed trajectories that are recurrent in the Birkhoff sense, in short B-recurrent. Moreover, every B-recurrent trajectory is everywhere dense in the minimal set [**17**].

Nontrivial minimal sets on $\mathbb{T}^2$. We present here results from [**25**]. It is obvious that the geodesic framework of a nontrivial minimal set on the torus $\mathbb{T}^2$ is a linear irrational flow.

LEMMA 5.1. *Let N be a nontrivial minimal set of a flow f^t on $\mathbb{T}^2$ and $G(N)$ the geodesic framework of N. Then there exists a continuous mapping $h : \mathbb{T}^2 \to \mathbb{T}^2$ that is homotopic to the identity with the following properties: 1) $h(N) = \mathbb{T}^2$; 2) each trajectory from N is homeomorphically mapped by h onto a geodesic of $G(N)$; 3) if w is the component of the set $\mathbb{T}^2 \setminus N$ then $h(w)$ is a geodesic of $G(N)$.*

Denote by $\delta(N)$ the boundary of the set N that is accessible from $\mathbb{T}^2 \setminus N$. It can be shown that $\delta(N)$ is invariant and consists of a finite or a countable family of trajectories of N. Therefore, by Lemma 5.1, $h(\delta(N))$ is a finite or a countable family of geodesics from $G(N)$. This family of geodesics is called a *distinguished family of the minimal set* N and is denoted by $R(N)$. Of course, this family depends on the transformation h from Lemma 5.1 and is determined by the set N up to a translation, i.e., up to a homeomorphism of $\mathbb{T}^2$ whose covering is given by $x \mapsto x + x_0$, $y \mapsto y + y_0$, where x_0 and y_0 are certain constants. The following theorem gives a topological classification of nontrivial minimal sets of flows on $\mathbb{T}^2$.

THEOREM 5.8. *Let N_1 and N_2 be nontrivial minimal sets of flows f_1^t and f_2^t, respectively, on $\mathbb{T}^2$. Then, N_1 and N_2 are orbitally topologically equivalent via a homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity if and only if their geodesic frameworks coincide (with regard to the orientation of geodesics) and there exists a translation of $\mathbb{T}^2$ that sends the distinguished family of one minimal set to the distinguished family of the other minimal set. The geodesic framework of a nontrivial minimal set on $\mathbb{T}^2$ is a linear irrational flow. For any finite or countable family N_0 of trajectories of a linear irrational flow, there exists a flow with a nontrivial minimal set N such that $R(N) = N_0$.*

Note that the geodesic framework and the cardinality of the set of distinguished geodesics alone do not provide a complete topological invariant. Moreover, it can be shown that there exists a continuum of pairwise topologically non-equivalent nontrivial minimal sets with the same geodesic framework and any prescribed fixed cardinality ≥ 2 of the set of distinguished geodesics.

Nontrivial minimal sets on a hyperbolic surface. The classification below was obtained in [**20**]. Let N be a nontrivial minimal set of a flow f^t on an orientable closed hyperbolic surface M^2. A component of the set $M^2 \setminus N$ is called a *Denjoy cell* if it is simply connected and its boundary accessible from $M^2 \setminus N$ consists of exactly two trajectories of N. These two trajectories have the same co-asymptotic geodesic called a *distinguished* geodesic. Similar to the case of the torus, we'll call a family of distinguished geodesics a *distinguished family of the geodesic framework of the minimal set N*. Since the generation or elimination of Denjoy cells do not change the geodesic framework of a nontrivial minimal set, the presence of these cells can be considered, in a sense, artificial. Therefore, we first consider a classification of nontrivial minimal sets without Denjoy cells. Let us recall that a minimal strongly nontrivial geodesic lamination is called weakly irrational.

THEOREM 5.9. *Let N be a nontrivial minimal set of a flow f^t on a closed orientable hyperbolic surface M^2. Suppose that N does not contain Denjoy cells. Then, N is orbitally topologically equivalent, via a homeomorphism $M^2 \to M^2$ homotopic to the identity, to its own geodesic framework $G(N)$ that is an orientable weakly irrational geodesic lamination, $G(N) \in \Lambda_{or}$. For any orientable weakly irrational geodesic lamination $\Lambda \in \Lambda_{or}$, there exists a nontrivial minimal set N without Denjoy cells of a certain flow f^t such that $G(N) = \Lambda$.*[3]

We now, consider nontrivial minimal sets with Denjoy cells and describe the type of geodesics that form distinguished families of these minimal sets. Recall that a nontrivially recurrent geodesic may be either left or right improper; i.e., it may approach itself to an indefinitely close distance from either the left or the right side. If a nontrivially recurrent geodesic is improper only from one side, then it is called a *boundary* one. Otherwise (i.e., if a geodesic is improper from both sides), it is called *internal*.

A weakly irrational geodesic lamination on a closed hyperbolic surface has a finite nonzero number of boundary nontrivially recurrent geodesics and a continuum set of internal ones. The definition of a Denjoy cell and the density of each geodesic in a minimal geodesic lamination imply that each geodesic from a distinguished family is internal. The following two theorems give topological classification of nontrivial minimal sets of flows on a closed orientable hyperbolic surface.

[3]In fact, one can make it so that $N = \Lambda$.

THEOREM 5.10. *Let N_1 and N_2 be nontrivial minimal sets of flows f_1^t and f_2^t, respectively, on a closed orientable hyperbolic surface M^2. Then N_1 and N_2 are orbitally topologically equivalent via a homeomorphism $M^2 \to M^2$ homotopic to the identity if and only if they have identical geodesic frameworks (with regard to the orientation of geodesics) and the same family of distinguished geodesics.*

Disregarding the orientation of geodesics, we obtain a criterion for topological equivalence.

THEOREM 5.11. *Let N be a nontrivial minimal set of a flow f^t on a closed orientable hyperbolic surface M^2. Then the geodesic framework $G(N)$ is an orientable weakly irrational geodesic lamination that contains at most a countable distinguished family that consists of internal geodesics. Conversely, let Λ be an orientable weakly irrational geodesic lamination on M^2 and let $\mathcal{N}$ be at most a countable family of internal geodesics of Λ. Then there exists a nontrivial minimal set N of a certain flow f^t such that $G(N) = \Lambda$ and the distinguished family of the geodesic framework $G(N)$ coincides with $\mathcal{N}$.*

5.3. Classification of irrational 2-webs. A 2-*web* on a surface is a pair of foliations such that they have a common set of singularities and are topologically transversal at all non-singular points. Suppose that two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on a surface M^2 form a 2-web denoted by $(\mathcal{F}_1, \mathcal{F}_2)$. The set of singularities of the foliation $\mathcal{F}_i$ (for any i) is called the *set of singularities* of $(\mathcal{F}_1, \mathcal{F}_2)$ denoted by *Sing* $(\mathcal{F}_1, \mathcal{F}_2)$. A 2-web is *irrational or strongly irrational* if it consists of a pair of irrational or strongly irrational foliations respectively.

2-webs $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{F}_1', \mathcal{F}_2')$ are *topologically equivalent* if there is a homeomorphism $\varphi : M^2 \to M^2$ that maps the foliations $\mathcal{F}_i$ $(i = 1, 2)$ to the corresponding foliations $\mathcal{F}_i'$ and $\varphi(Sing\ (\mathcal{F}_1, \mathcal{F}_2)) = Sing\ (\mathcal{F}_1', \mathcal{F}_2')$. All classification results of this subsection was obtained in [**23**].

On the torus $\mathbb{T}^2$, a strongly irrational 2-web consists of a pair of transversal irrational foliations without singularities.

THEOREM 5.12. *Let $(\mathcal{F}_1, \mathcal{F}_2)$ be a strongly irrational 2-web on $\mathbb{T}^2$. Then $(\mathcal{F}_1, \mathcal{F}_2)$ is topologically equivalent via a homeomorphism homotopic to the identity to its own geodesic framework, which is a pair of linear transversal irrational foliations. Two strongly irrational 2-webs on $\mathbb{T}^2$ are topologically equivalent via a homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity if and only if their geodesic frameworks coincide.*

Let us pass on to strongly irrational 2-webs on a closed orientable hyperbolic surface. The geodesic framework of a strongly irrational foliation on a closed orientable hyperbolic surface M_h^2, $h \geq 2$, is a strongly irrational geodesic lamination. If foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ form a strongly irrational 2-web, then their geodesic frameworks must satisfy the following *consistency* conditions:

- The sets $M_h^2 \setminus supp\ G(\mathcal{F}_1)$ and $M_h^2 \setminus supp\ G(\mathcal{F}_2)$ have the same number of simply connected components, which is equal to the number of singularities of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ (which is the same for these foliations).
- For each simply connected component P_1 of the set $M_h^2 \setminus supp\ G(\mathcal{F}_1)$, there exists a simply connected component P_2 of the set $M_h^2 \setminus supp\ G(\mathcal{F}_2)$ such that there exist lifts $\overline{P}_1$ and $\overline{P}_2$ of these components that are polygons with alternating ideal vertices on S_∞, Fig. 8.

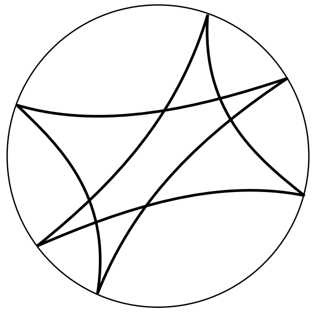

FIGURE 8. The polygons $\overline{P}_1$ and $\overline{P}_2$

THEOREM 5.13. *Let $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{F}'_1, \mathcal{F}'_2)$ be strongly irrational 2-webs on a closed orientable hyperbolic surface M^2. Then $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{F}'_1, \mathcal{F}'_2)$ are topologically equivalent via a homeomorphism $M^2 \to M^2$ that is homotopic to the identity if and only if their geodesic frameworks coincide. For any pair of consistent strongly irrational geodesic laminations, there exists a strongly irrational 2-web whose geodesic framework is equal to this pair of laminations.*

Sketch of proof. . We restrict ourselves to the first part of the statement. A homeomorphism of M^2 that is homotopic to the identity has a lift that is extended to the identity homeomorphism of S_∞. Therefore, if the webs are topologically equivalent via a homeomorphism homotopic to the identity, then their geodesic frameworks coincide.

Suppose that the geodesic frameworks $(G(\mathcal{F}_1), G(\mathcal{F}_2))$ and $(G(\mathcal{F}'_1), G(\mathcal{F}'_2))$ coincide, $G(\mathcal{F}_1) = G(\mathcal{F}'_1)$, $G(\mathcal{F}_2) = G(\mathcal{F}'_2)$. Consider the lifts $(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2)$ and $(\overline{\mathcal{F}}'_1, \overline{\mathcal{F}}'_2)$ of the 2-webs $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{F}'_1, \mathcal{F}'_2)$, respectively. Let $\overline{m} \in \Delta$ be a point that is not a singularity of the 2-web $(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2)$. According to Theorem 4.9, semileaves, say $\bar{l}_1$ and $\bar{l}_2$, of $\overline{\mathcal{F}}_1$ and $\overline{\mathcal{F}}_2$ passing through $\overline{m}$ have the asymptotic directions defined by some points σ_1 and σ_2 of S_∞ respectively. Since the foliations $\overline{\mathcal{F}}_1$ and $\overline{\mathcal{F}}_2$ are transversal outside the set of singularities, $\sigma_1 \neq \sigma_2$. The points σ_1 and σ_2 are reached by the geodesic frameworks of the foliations $\overline{\mathcal{F}}'_1$ and $\overline{\mathcal{F}}'_2$, respectively. Therefore, by Theorem 4.9, there exist semileaves $\bar{l}'_1$ and $\bar{l}'_2$ of these foliations that reach the points σ_1 and σ_2, respectively. Note that according to Theorem 4.9, if $\bar{l}_i$ does not belong to a separatrix of a singularity, then $\bar{l}'_i$ does not belong to a separatrix of any singularity; and conversely, if $\bar{l}_i$ belongs to a separatrix of a singularity, then $\bar{l}'_i$ also belongs to a separatrix of a singularity, $i = 1, 2$. Since the co-asymptotic geodesics of the corresponding leaves or semileaves that contain $\bar{l}_i$ and $\bar{l}'_i$ coincide and the geodesic frameworks of the foliations $\overline{\mathcal{F}}'_1$ and $\overline{\mathcal{F}}'_2$ are transversal, the semileaves $\bar{l}'_1$ and $\bar{l}'_2$ intersect at some point denoted by $\overline{m}'$. Since $\overline{\mathcal{F}}'_1$ and $\overline{\mathcal{F}}'_2$ form a 2-web, the point $\overline{m}'$ is unique. Denote the mapping $\overline{m} \to \overline{m}'$ by $\overline{\phi}$. By virtue of Theorem 4.9, $\overline{\phi}$ is extended to all the singularities of the 2-web $(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2)$ and maps a singularity to a singularity of the 2-web $(\overline{\mathcal{F}}'_1, \overline{\mathcal{F}}'_2)$ with the same number of separatrices that reach the same points on S_∞.

One can verify that $\overline{\phi}$ covers a certain homeomorphism $\phi : M^2 \to M^2$ that realizes a topological equivalence of the 2-webs $(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2)$ and $(\overline{\mathcal{F}}'_1, \overline{\mathcal{F}}'_2)$. Since by the construction, $\overline{\phi}$ is extended to the absolute as the identity mapping, ϕ is homotopic to the identity. □

5.4. Homeomorphisms with invariant local laminations. Let $f : M^2 \to M^2$ be a homeomorphism of a surface M and $\mathcal{F}$ a foliation on M^2 that is invariant under f (i.e. $f(Sing\ (\mathcal{F})) = Sing\ (\mathcal{F})$ and f maps every leaf onto a leaf). $\mathcal{F}$ is said to be *contracting* if, given any points a and b that belong to same leaf, the distance between $f^n(a)$ and $f^n(b)$ tends to zero as $n \to +\infty$ in the interior metric on the leaves. A foliation $\mathcal{F}$ is called *expanding* if it is contracting under f^{-1}.

Following Anosov and Zhuzhoma [**12**] a homeomorphisms $f : M^2 \to M^2$ is called *almost pseudo-Anosov* (AP-homeomorphism) if it satisfies the conditions:

- f has invariant foliations $\mathcal{F}^s$, $\mathcal{F}^u$ that form a strongly irrational 2-web.
- $\mathcal{F}^s$ is contractive and $\mathcal{F}^u$ is expanding under f.

AP-homeomorphisms are in sense non-uniform pseudo-Anosov homeomorphisms. The class of AP-homeomorphisms includes pseudo-Anosov ones for which the contraction and expansion satisfy some uniform estimates.

Homeomorphisms of $\mathbb{T}^2$ *and hyperbolic surfaces.* Let us recall that on $\mathbb{T}^2$ a strongly irrational 2-web actually is a 2-web consisting of a pair of transversal irrational foliations without singularities. The following theorem says that an AP-homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$ is Anosov hyperbolic automorphism up to conjugacy (see [**44**]).

THEOREM 5.14. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be AP-homeomorphism. Then f is conjugate to an Anosov hyperbolic automorphism.*

Let $\overline{f} : \Delta \to \Delta$ be a lift for $f : M^2 \to M^2$ where M^2 is a closed orientable hyperbolic surface. Due to [**56**] (see also [**46**]), $\overline{f}$ extends continuously to a homeomorphism $\Delta \cup S_\infty \to \Delta \cup S_\infty$ denoted again by $\overline{f}$. The crucial step in a classification of AP-homeomorphisms is the following theorem (see [**39**]).

THEOREM 5.15. *Let f_1, $f_2 : M^2 \to M^2$ be AP-homeomorphisms of a closed orientable hyperbolic surface $M = \Delta/\Gamma$. Then f_1 and f_2 are conjugate via a homotopy trivial homeomorphism if and only if there exist the lifts $\overline{f}_1$, $\overline{f}_2 : \Delta \to \Delta$ of f_1, f_2 respectively whose extensions on S_∞ coincide, $\overline{f}_1|_{S_\infty} = \overline{f}_2|_{S_\infty}$.*

Let G be a group and ϕ_1, ϕ_2 automorphisms of G. Recall that ϕ_1, ϕ_2 are conjugate if there is an automorphism $\xi : G \to G$ such that $\phi_2 \circ \xi = \xi \circ \phi_1$. It is well known that a homeomorphism $f : M^2 \to M^2$ induces an automorphism $f_* : \pi_1(M^2) \to \pi_1(M^2)$ of the fundamental group $\pi_1(M)$. Two homeomorphisms f_1, $f_2 : M^2 \to M^2$ are called π_1-*conjugate* if f_{1*}, f_{2*} are conjugate automorphisms of the group $\pi_1(M)$.

If $h \circ f_1 = f_2 \circ h$ then $h_* \circ f_{1*} = f_{2*} \circ h_*$. Therefore, two conjugate homeomorphisms are necessarily π_1-conjugate. Moreover theorem 5.15 and Nielsen [**56**] imply that the π_1-conjugacy is also a sufficient condition of conjugacy for AP-homeomorphisms (see [**15, 16, 39**] and also [**35**]).

THEOREM 5.16. *Let $f_1, f_2 : M^2 \to M^2$ be AP-homeomorphisms of a closed orientable hyperbolic surface $M = \Delta/\Gamma$. Then f_1 and f_2 are conjugate if and only if they are π_1-conjugate.*

Note that in [**14–16**], necessary and sufficient conditions for the conjugacy of homeomorphisms $f : M \to M$ of a closed hyperbolic surface were obtained in the case when one of the invariant foliations is irrational and the other is of Denjoy type, as well as in the case when both invariant foliations are of Denjoy type.

Classification of one-dimensional basic sets. Let Ω be a one-dimensional basic set of an A-diffeomorphism $f : M^2 \to M^2$ of a closed orientable hyperbolic surface M^2. Then Ω is either an attractor or a repeller [**58, 67**]. Assume, for definiteness, that Ω is an attractor. In this case, Ω is an *expanding attractor*: its topological dimension coincides with the dimension of unstable manifolds. Profound results on the structure and dynamics of expanding attractors belong to Williams [**67**]. However, solving the problem of the classification of one-dimensional expanding attractors, one should take into account the character of the embedding of expanding attractors into the surface.

Recall that a closed subset Ω_c of a basic set Ω is called C-dense if both intersections $W^s(m) \cap \Omega_c$ and $W^u(m) \cap \Omega_c$ are everywhere dense in Ω_c for any point $m \in \Omega_c$. It is well known [**1, 31**] that a basic set consists of a finite number of C-dense components that are cyclically mapped to each other by the diffeomorphism. Passing to an iterate of the diffeomorphism, we can make it so that the diffeomorphism has only C-dense basic sets. Below, unless otherwise stated, we will assume that *expanding attractors are C-dense.*

If Ω is a one-dimensional expanding attractor then the unstable manifolds $\{W^u(m) : m \in \Omega\} \stackrel{\text{def}}{=} \mathcal{W}^u(\Omega)$ form a local C^1 lamination that consists of nontrivially recurrent leaves. Each leaf of $\mathcal{W}^u(\Omega)$ is everywhere dense in $\mathcal{W}^u(\Omega)$. This, combined with Theorem 4.8, implies the following proposition.

THEOREM 5.17. *Let $f : M^2 \to M^2$ be an A-diffeomorphism of a closed orientable hyperbolic surface M^2, and Ω a one-dimensional widely disposed (in particular, orientable) expanding attractor of f. Then*

(1) *the geodesic framework $G(\mathcal{W}^u(\Omega))$ of $\mathcal{W}^u(\Omega)$ is a weakly irrational geodesic lamination;*
(2) *any geodesic of $G(\mathcal{W}^u(\Omega))$ is a co-asymptotic geodesic of a leaf belonging to $\mathcal{W}^u(\Omega)$.*

Similar to Section 5.2, we introduce the concept of a *distinguished* geodesic as a geodesic that is co-asymptotic for more than one leaf of the lamination $\mathcal{W}^u(\Omega)$. The family of distinguished geodesics forms the distinguished set.

The next theorem follows from results obtained by Grines [**37**], and R. Plykin [**61**] (see also [**22**], [**42**], [**43**]) and give necessary and sufficient conditions for the conjugacy of one-dimensional basic sets via a homotopically trivial homeomorphism.

THEOREM 5.18. *Let $f_1, f_2 : M^2 \to M^2$ be two A-diffeomorphisms of a closed orientable hyperbolic surface M^2, and let Ω_1 and Ω_2 be two one-dimensional widely disposed (in particular, orientable) expanding attractors of these diffeomorphisms, respectively. Then f_1 and f_2 are conjugate on Ω_1 and Ω_2 via a homotopically trivial homeomorphism $M^2 \to M^2$ if and only if the geodesic frameworks $G(\mathcal{W}^u(\Omega_1))$ and $G(\mathcal{W}^u(\Omega_2))$ are equal (without regard to the orientation on the geodesics), and they have the same family of distinguished geodesics, and there exist lifts $\overline{f}_1, \overline{f}_2 : \Delta \to \Delta$ of these diffeomorphisms whose extensions to S_∞ coincide, $\overline{f}_1|_{S_\infty} = \overline{f}_2|_{S_\infty}$.*

Two homeomorphisms of a hyperbolic surface are homotopic if and only if they have lifts with identical extensions to the absolute. Therefore, Theorem 5.18 can be reformulated as follows.

THEOREM 5.19. *Let $f_1, f_2 : M^2 \to M^2$ be two homotopic (to each other) A-diffeomorphisms of a closed orientable hyperbolic surface M, and let Ω_1 and Ω_2 be their one-dimensional widely disposed (in particular, orientable) expanding attractors, respectively. Then f_1 and f_2 are conjugate on Ω_1 and Ω_2 via a homotopically trivial homeomorphism $M^2 \to M^2$ if and only if the geodesic frameworks $G(\mathcal{W}^u(\Omega_1))$ and $G(\mathcal{W}^u(\Omega_2))$ are equal (without regard to the orientation on the geodesics) and have the same family of distinguished geodesics.*

Note that it is possible to get a generalization of the last two theorems for nonorientable closed surfaces using results from [**45**].

References

[1] D. V. Anosov, On a Class of Invariant Sets of Smooth Dynamical Systems. *Proc. Fifth Int. Conf. on Nonlinear Oscillations*, Vol. 2: Qualitative Methods, Kiev, **1970**, 39-45.

[2] D. V. Anosov, *On the behavior of trajectories, in the Euclidean or Lobachevskiĭ plane, covering the trajectory of flows on closed surfaces. I* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), no. 1, 16–43, 207; English transl., Math. USSR-Izv. **30** (1988), no. 1, 15–38. MR887599

[3] D. V. Anosov, *On the behavior of trajectories, in the Euclidean or Lobachevskiĭ plane, covering the trajectory of flows on closed surfaces. II* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), no. 3, 451–478, 670; English transl., Math. USSR-Izv. **32** (1989), no. 3, 449–474. MR954292

[4] D. V. Anosov, *On infinite curves on a torus and on closed surfaces of negative Euler characteristic* (Russian), Trudy Mat. Inst. Steklov. **185** (1988), 30–53. Translated in Proc. Steklov Inst. Math. **1990**, no. 2, 33–58; Optimal control and differential games (Russian). MR979299

[5] D. V. Anosov, *How can curves on the universal covering plane that cover non-self-intersecting curves on a closed surface go to infinity?* (Russian), Trudy Mat. Inst. Steklov. **191** (1989), 34–44. Translated in Proc. Steklov Inst. Math. **1992**, no. 2, 35–45; Statistical mechanics and the theory of dynamical systems (Russian). MR1029036

[6] D. V. Anosov, *Infinite curves on the Klein bottle* (Russian), Mat. Sb. **180** (1989), no. 1, 39–56, 142; English transl., Math. USSR-Sb. **66** (1990), no. 1, 41–58. MR988845

[7] D. V. Anosov, *Flows on surfaces* (Russian), Trudy Mat. Inst. Steklov. **193** (1992), 10–14; English transl., Proc. Steklov Inst. Math. **3 (193)** (1993), 7–11. MR1265977

[8] D. V. Anosov, *On the behavior of trajectories, in the Euclidean or Lobachevskiĭ plane, covering the trajectory of flows on closed surfaces. III* (Russian, with Russian summary), Izv. Ross. Akad. Nauk Ser. Mat. **59** (1995), no. 2, 63–96, DOI 10.1070/IM1995v059n02ABEH000012; English transl., Izv. Math. **59** (1995), no. 2, 287–320. MR1337159

[9] D. V. Anosov, *On lifts to the plane of semileaves of foliations on a torus with a finite number of singularities* (Russian, with Russian summary), Tr. Mat. Inst. Steklova **224** (1999), no. Algebra. Topol. Differ. Uravn. i ikh Prilozh., 28–55; English transl., Proc. Steklov Inst. Math. **1 (224)** (1999), 20–45. MR1721353

[10] D. V. Anosov, *Flows on closed surfaces and related geometric problems* (Russian, with Russian summary), Tr. Mat. Inst. Steklova **236** (2002), no. Differ. Uravn. i Din. Sist., 20–26; English transl., Proc. Steklov Inst. Math. **1 (236)** (2002), 12–18. MR1931002

[11] D. V. Anosov and E. V. Zhuzhoma, *Asymptotic behavior of covering curves on the universal coverings of surfaces* (Russian, with Russian summary), Tr. Mat. Inst. Steklova **238** (2002), no. Monodromiya v Zadachakh Algebr. Geom. i Differ. Uravn., 5–54; English transl., Proc. Steklov Inst. Math. **3 (238)** (2002), 1–46. MR1969302

[12] D. V. Anosov and E. V. Zhuzhoma, *Nonlocal asymptotic behavior of curves and leaves of laminations on universal coverings* (Russian, with English and Russian summaries), Tr. Mat. Inst. Steklova **249** (2005), 239; English transl., Proc. Steklov Inst. Math. **2 (249)** (2005), 1–219. MR2200607

[13] S. H. Aranson, *Trajectories on nonorientable two-dimensional manifolds* (Russian), Mat. Sb. (N.S.) **80 (122)** (1969), 314–333. MR0259284

[14] S. Kh. Aranson, *Topological equivalence of foliations with singularities and homeomorphisms with invariant foliations on two-dimensional manifolds* (Russian), Uspekhi Mat. Nauk **41** (1986), no. 3(249), 167–168. MR854244

[15] S.Kh. Aronson, *Topological Classification of Foliations with Singularities and Homeomorphisms with Invariant Foliations on Closed Surfaces*, Part 1: Foliations, 1988, 6887 V-88; Part 2: Homeomorphisms, 1989, 1043 V-89. DEP VINITI, Gorkii.

[16] S. Kh. Aranson, *Topology of vector fields, of foliations with singularities, and of homeomorphisms with invariant foliations on closed surfaces* (Russian), Trudy Mat. Inst. Steklov. **193** (1992), 15–21; English transl., Proc. Steklov Inst. Math. **3 (193)** (1993), 13–18. MR1265978

[17] S. Kh. Aranson, G. R. Belitsky, and E. V. Zhuzhoma, *Introduction to the qualitative theory of dynamical systems on surfaces*, Translations of Mathematical Monographs, vol. 153, American Mathematical Society, Providence, RI, 1996. Translated from the Russian manuscript by H. H. McFaden. MR1400885

[18] S. Kh. Aranson, E. V. Zhuzhoma, and I. A. Tel′nykh, *Transitive and supertransitive flows on closed nonorientable surfaces* (Russian), Mat. Zametki **63** (1998), no. 4, 625–628, DOI 10.1007/BF02311259; English transl., Math. Notes **63** (1998), no. 3-4, 549–552. MR1680986

[19] S. H. Aranson and V. Z. Grines, *Certain invariants of dynamical systems on two-dimensional manifolds (necessary and sufficient conditions for the topological equivalence of transitive systems)* (Russian), Mat. Sb. (N.S.) **90(132)** (1973), 372–402, 479. MR0339275

[20] S. H. Aranson and V. Z. Grines, *The representation of minimal sets of flows on two-dimensional manifolds by geodesic lines* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), no. 1, 104–129, 215. MR0501166

[21] S. Kh. Aranson and V. Z. Grines, *Topological classification of flows on closed two-dimensional manifolds* (Russian), Uspekhi Mat. Nauk **41** (1986), no. 1(247), 149–169, 240. MR832412

[22] S. Kh. Aranson and V. Z. Grines, *Topological classification of cascades on closed two-dimensional manifolds* (Russian), Uspekhi Mat. Nauk **45** (1990), no. 1(271), 3–32, 222, DOI 10.1070/RM1990v045n01ABEH002322; English transl., Russian Math. Surveys **45** (1990), no. 1, 1–35. MR1050926

[23] S. Kh. Aranson, V. Z. Grines, and V. A. Kaimanovich, *Classification of supertransitive 2-webs on surfaces*, J. Dynam. Control Systems **9** (2003), no. 4, 455–468, DOI 10.1023/A:1025687817308. MR2001955

[24] S. Aranson, V. Grines, and E. Zhuzhoma, *On Anosov-Weil problem*, Topology **40** (2001), no. 3, 475–502, DOI 10.1016/S0040-9383(99)00071-3. MR1838992

[25] S.Kh. Aranson, E.V. Zhuzhoma, On the Topological Equivalence of Nowhere Dense Minimal Sets of Dynamical Systems on the Torus. *Izv. Vyssh. Uchebn. Zaved.*, ser. Matem., 1976 (5), 104-107.

[26] S. Kh. Aranson and E. V. Zhuzhoma, *Quasiminimal sets of foliations, and one-dimensional basic sets of A-diffeomorphisms of surfaces* (Russian), Dokl. Akad. Nauk **330** (1993), no. 3, 280–281; English transl., Russian Acad. Sci. Dokl. Math. **47** (1993), no. 3, 448–450. MR1241956

[27] S. Kh. Aranson and E. V. Zhuzhoma, *On the structure of quasiminimal sets of foliations on surfaces* (Russian, with Russian summary), Mat. Sb. **185** (1994), no. 8, 31–62, DOI 10.1070/SM1995v082n02ABEH003572; English transl., Russian Acad. Sci. Sb. Math. **82** (1995), no. 2, 397–424. MR1302622

[28] S. Aranson and E. Zhuzhoma, *Maier's theorems and geodesic laminations of surface flows*, J. Dynam. Control Systems **2** (1996), no. 4, 557–582, DOI 10.1007/BF02254703. MR1420359

[29] S. Kh. Aranson and E. V. Zhuzhoma, *On properties of the absolute that affect the smoothness of flows on closed surfaces* (Russian, with Russian summary), Mat. Zametki **68** (2000), no. 6, 819–829, DOI 10.1023/A:1026696213559; English transl., Math. Notes **68** (2000), no. 5-6, 695–703. MR1835180

[30] Ivar Bendixson, *Sur les courbes définies par des équations différentielles* (French), Acta Math. **24** (1901), no. 1, 1–88, DOI 10.1007/BF02403068. MR1554923

[31] Rufus Bowen, *Periodic points and measures for Axiom A diffeomorphisms*, Trans. Amer. Math. Soc. **154** (1971), 377–397. MR0282372

[32] Andrew J. Casson and Steven A. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, London Mathematical Society Student Texts, vol. 9, Cambridge University Press, Cambridge, 1988. MR964685

[33] T. M. Cherry, *Topological Properties of the Solutions of Ordinary Differential Equations*, Amer. J. Math. **59** (1937), no. 4, 957–982, DOI 10.2307/2371361. MR1507295

[34] P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973), 45–109. MR0336648

[35] Albert Fathi, François Laudenbach, and Valentin Poénaru, *Thurston's work on surfaces*, Mathematical Notes, vol. 48, Princeton University Press, Princeton, NJ, 2012. Translated from the 1979 French original by Djun M. Kim and Dan Margalit. MR3053012

[36] A. A. Glutsyuk, *Limit sets at infinity for liftings of non-self-intersecting curves on a torus to the plane* (Russian, with Russian summary), Mat. Zametki **64** (1998), no. 5, 667–679, DOI 10.1007/BF02316282; English transl., Math. Notes **64** (1998), no. 5-6, 579–589 (1999). MR1691209

[37] V. Z. Grines, *The topological conjugacy of diffeomorphisms of a two-dimensional manifold on one-dimensional orientable basic sets. I* (Russian), Trudy Moskov. Mat. Obšč. **32** (1975), 35–60. MR0418161

[38] V. Z. Grines, *The topological conjugacy of diffeomorphisms of a two-dimensional manifold on one-dimensional orientable basic sets. II* (Russian), Trudy Moskov. Mat. Obšč. **34** (1977), 243–252. MR0474417

[39] V. Z. Grines, Diffeomorphisms of Two-Dimensional Manifolds with Transitive Foliations. *Methods of the Qualitative Theory of Differential Equations*, AMS Transl., Ser. 2, **149**(1991), 193-199.

[40] V. Z. Grines, *On the topological classification of structurally stable diffeomorphisms of surfaces with one-dimensional attractors and repellers* (Russian, with Russian summary), Mat. Sb. **188** (1997), no. 4, 57–94, DOI 10.1070/SM1997v188n04ABEH000216; English transl., Sb. Math. **188** (1997), no. 4, 537–569. MR1462029

[41] V. Z. Grines, *Structural stability and asymptotic behavior of invariant manifolds of A-diffeomorphisms of surfaces*, J. Dynam. Control Systems **3** (1997), no. 1, 91–110, DOI 10.1007/BF02471763. MR1436551

[42] V. Z. Grines, *Topological classification of one-dimensional attractors and repellers of A-diffeomorphisms of surfaces by means of automorphisms of fundamental groups of supports*, J. Math. Sci. (New York) **95** (1999), no. 5, 2523–2545, DOI 10.1007/BF02169053. Dynamical systems. 7. MR1712741

[43] V. Z. Grines, *On topological classification of A-diffeomorphisms of surfaces*, J. Dynam. Control Systems **6** (2000), no. 1, 97–126, DOI 10.1023/A:1009573706584. MR1738742

[44] V. Z. Grines, V. S. Medvedev, and E. V. Zhuzhoma, *On surface attractors and repellers in 3-manifolds* (Russian, with Russian summary), Mat. Zametki **78** (2005), no. 6, 813–826, DOI 10.1007/s11006-005-0181-1; English transl., Math. Notes **78** (2005), no. 5-6, 757–767. MR2249032

[45] V. Z. Grines and R. V. Plykin, *Topological classification of amply situated attractors of A-diffeomorphisms of surfaces*, Methods of qualitative theory of differential equations and related topics, Amer. Math. Soc. Transl. Ser. 2, vol. 200, Amer. Math. Soc., Providence, RI, 2000, pp. 135–148, DOI 10.1090/trans2/200/11. MR1769568

[46] Michael Handel and William P. Thurston, *New proofs of some results of Nielsen*, Adv. in Math. **56** (1985), no. 2, 173–191, DOI 10.1016/0001-8708(85)90028-3. MR788938

[47] P. Koebe, Riemannische Manigfaltigkeiten und nichteuklidiche Raumformen, IY. *Sitzung. der Preuss. Akad. der Wissenchaften*, **1929**, 414-457.

[48] A. Mayer, *De trajectoires sur les surfaces orientées* (French), C. R. (Doklady) Acad. Sci. URSS (N.S.) **24** (1939), 673–675. MR0002240

[49] A. Mayer, *Trajectories on the closed orientable surfaces* (Russian, with English summary), Rec. Math. [Mat. Sbornik] N.S. **12(54)** (1943), 71–84. MR0009485

[50] Nelson Groh Markley, *THE STRUCTURE OF FLOWS ON TWO-DIMENSIONAL MANIFOLDS*, ProQuest LLC, Ann Arbor, MI, 1966. Thesis (Ph.D.)–Yale University. MR2615823

[51] Nelson G. Markley, *The Poincaré-Bendixson theorem for the Klein bottle*, Trans. Amer. Math. Soc. **135** (1969), 159–165. MR0234442

[52] Nelson G. Markley, *Invariant simple closed curves on the torus*, Michigan Math. J. **25** (1978), no. 1, 45–52. MR497881

[53] N. G. Markley and M. H. Vanderschoot, *An exotic flow on a compact surface.* part 1, Colloq. Math. **84/85** (2000), no. part 1, 235–243. Dedicated to the memory of Anzelm Iwanik. MR1778853

[54] N. G. Markley and M. H. Vanderschoot, *Remote limit points on surfaces*, J. Differential Equations **188** (2003), no. 1, 221–241, DOI 10.1016/S0022-0396(02)00065-7. MR1954514

[55] Harold Marston Morse, *A One-to-One Representation of Geodesics on a Surface of Negative Curvature*, Amer. J. Math. **43** (1921), no. 1, 33–51, DOI 10.2307/2370306. MR1506428

[56] Jakob Nielsen, *Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen* (German), Acta Math. **50** (1927), no. 1, 189–358, DOI 10.1007/BF02421324. MR1555256

[57] Dmitri Panov, *Foliations with unbounded deviation on* $\mathbb{T}^2$, J. Mod. Dyn. **3** (2009), no. 4, 589–594, DOI 10.3934/jmd.2009.3.589. MR2587087

[58] R. V. Plykin, *The topology of basic sets of Smale diffeomorphisms* (Russian), Mat. Sb. (N.S.) **84 (126)** (1971), 301–312. MR0286134

[59] R. V. Plykin, *Sources and sinks of* A*-diffeomorphisms of surfaces* (Russian), Mat. Sb. (N.S.) **94(136)** (1974), 243–264, 336. MR0356137

[60] R. V. Plykin, *Hyperbolic attractors of diffeomorphisms* (Russian), Uspekhi Mat. Nauk **35** (1980), no. 3(213), 94–104. International Topology Conference (Moscow State Univ., Moscow, 1979). MR580625

[61] R. V. Plykin, *The geometry of hyperbolic attractors of smooth cascades* (Russian), Uspekhi Mat. Nauk **39** (1984), no. 6(240), 75–113. MR771099

[62] H. Poincaré, Sur les courbes définies par les equations differentielles. *J. Math. Pures Appl.* **2**(1886), 151-217.

[63] V. I. Pupko, Non-Self-Intersecting Curves on Closed Surfaces. *Sov. Math. Dokl.*, **8**(1967), 1405-1407.

[64] R. Clark Robinson and R. F. Williams, *Finite stability is not generic*, Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), Academic Press, New York, 1973, pp. 451–462. MR0331430

[65] A. Weil, On systems of curves on a ring-shaped surface. *J. Indian Math. Soc.*, **19**(1932), 5, 109-112.

[66] A. Weil, Les familles de curbes sur le tore. *Mat. Sbornik*, **43**(1936), 5, 779-781.

[67] R. F. Williams, *Expanding attractors*, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 169–203. MR0348794

National Research University Higher School of Economics, 25/12 Bolshaya Pecherskaya, 603005, Nizhni Novgorod, Russia

E-mail address: vgrines@yandex.ru

National Research University Higher School of Economics, 25/12 Bolshaya Pecherskaya, 603005, Nizhni Novgorod, Russia

E-mail address: zhuzhoma@mail.ru