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Polynomial-Time Solvability of the Independent Set Problem in a Certain Class of Subcubic Planar Graphs

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Abstract—The independent set problem for a given simple graph consists in computing the size of a largest subset of its pairwise nonadjacent vertices. In this article, we prove the polynomial solvability of the problem for the subcubic planar graphs with no induced tree obtained by identifying the ends of three paths of lengths 3, 3, and 2 respectively.

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INTRODUCTION

The article is a continuation of the series of papers [3–6, 8–10], where the algorithmic complexity of the independent set problem (the IS problem) was studied. Recall that an independent set (i.s.) in an simple graph is a set of its pairwise nonadjacent vertices. A *maximum independent set* (m.i.s.) in G is an independent set with the greatest number of vertices; its size is called the *independence number* of G and denoted by $\alpha(G)$. The IS problem for a given graph G and a natural number k consists in finding whether $\alpha(G) \geq k$.

Several algorithmic instruments for graph reduction are known for solving the IS problem. For example, the so-called *adjacent absorption law*. A vertex a in a graph G *adjacently absorbs* a vertex b if $ab \in E(G)$ and $N(a) \supseteq N(b) \setminus \{a\}$. In this event, $\alpha(G) = \alpha(G \setminus \{a\})$. Adjacent absorption is a particular representative of the so-called *compressions* [1]; i.e., the mappings of the vertex set of a graph into itself that are not automorphisms and under which every two distinct nonadjacent vertices go to distinct nonadjacent vertices. Thus, compression transforms a graph into its induced subgraph, and the independence number is obviously preserved. Recall that a graph H is called an *induced subgraph* of a graph G if H is obtained by removing some vertices of G. A graph H is called a *minor* of a graph G if H is obtained from G by removing vertices and edges and also by contracting edges.

A graph *class* is an arbitrary set of ordinary graphs closed under isomorphisms. A graph class is called IS-*simple* if the IS problem is polynomially solvable for the graphs of this class. A graph class with NP-hard IS-problem will be called IS-*hard*.

A class is called *hereditary* if it is closed under vertex removal. It is known that every hereditary class $\mathcal{X}$ can be defined by the set $\mathcal{S}$ of its minimal forbidden induced subgraphs; here the notation $\mathcal{X} = \mathrm{Free}(\mathcal{S})$ is adopted. A hereditary class is called *finitely defined* if the set of its minimal forbidden induced subgraphs is finite. A *minor closed* graph class is a class that, together with its every graph, contains all minors of this graph. Every minor closed class can be defined by the set of its forbidden minors. For example, the class of planar graphs $\mathcal{P}$ is minor closed, the set of its forbidden minors consists of the graphs $K_{3,3}$ and K_5 by Wagner's Criterion.

A *triod* $T_{i,j,k}$ is a tree obtained by identifying three endvertices of paths P_{i+1}, P_{j+1}, and P_{k+1} respectively. The class $\mathcal{T}$ consists of all possible graphs whose each connected component is a tree

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with at most three leaves (i.e., a triod). It is proved in [5] that every finitely defined class $\mathcal{X}$ containing $\mathcal{T}$ is IS-hard. The same is true if, instead of $\mathcal{X}$, we consider the class $\mathcal{P}(3) \cap \mathcal{X}$, where $\mathcal{P}(3)$ is the set of *subcubic planar graphs*, i.e., of planar graphs with all vertex degrees at most 3. In the same article [5], there was made the conjecture that each finitely defined class not containing $\mathcal{T}$ is IS-easy. To this end, it suffices to show that, for every graph $G \in \mathcal{T}$, the class $\mathrm{Free}(G)$ is IS-easy. At present, this is proved for every graph $G \in \mathcal{T}$ with at most 5 vertices. The complexity status of the IS problem is unknown already for the class $\mathrm{Free}(P_6)$.

At the same time, it would be of interest to study the complexity of the IS problem for the classes of the form $\mathcal{Y} \cap \mathrm{Free}(G)$, $G \in \mathcal{T}$, where $\mathcal{Y}$ is a proper hereditary subset of the set of all graphs. Some authors earlier proved that the following classes of graphs free of triods of a given type are IS-simple: the class $\mathcal{D}(d) \cap \mathrm{Free}(T_{1,i,i})$ [8] for every $d, i \in \mathbb{N}$, where $\mathcal{D}(d)$ is the class of graphs with all vertex degrees at most d; for each $i \in \mathbb{N}$, the classes $\mathcal{P} \cap \mathrm{Free}(T_{1,2,i})$ [3, 9], $\mathcal{P} \cap \mathrm{Free}(T_{1,i,i})$ [6], and $\mathcal{P}(3) \cap \mathrm{Free}(T_{2,2,i})$ [4], and also the class $\mathcal{D}(3) \cap \mathrm{Free}(T_{2,2,2})$ [10].

In this article, we prove that the graph class $\mathcal{P}(3) \cap \mathrm{Free}(T_{3,3,2})$ is IS-simple.

1. NOTATIONS

We use the following notations: P_n is a simple path with n vertices, K_n is a complete graph with n vertices, $K_{n,m}$ is a complete bipartite graph with n vertices in one part and m vertices in the other, $\overline{a,b}$ is the set of naturals $\{a, a+1, \dots, b\}$, and $N(x)$ is a neighborhood of x. The graph $G \setminus V'$ is obtained from a graph G by removing all vertices of $V' \subseteq V(G)$, and $G[V']$ is the subgraph of G induced by $V' \subseteq V(G)$.

The notation $[a, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2]$ means that the vertices involved generate the triod $T_{3,3,2}$ with the edge set $\{ab_1, b_1b_2, b_2b_3, ac_1, c_1c_2, c_2c_3, ad_1, d_1d_2\}$.

The domain in a plane embedding of a planar graph bounded by a induced cycle $(v_1, \dots, v_k)$ of this graph will be denoted by $D(v_1, \dots, v_k)$.

2. THE REPLACEMENT OPERATION AND ITS IMPORTANCE

In this article, we will use some local graph transformations that are particular cases of the so-called replacement schemes proposed in [2]. In [2], the sufficiently general class of transformations is considered under which the independence number is exactly preserved but it is observed that admitting the change of the independence number by some constant would give nothing principally new.

Let H_1 and H_2 be graphs and let $A \subseteq V(H_1) \cap V(H_2)$. We say that H_1 and H_2 are *α-similar with respect to A* if there exists a constant c such that $\alpha(H_1 \setminus X) = \alpha(H_2 \setminus X) + c$ for every $X \subseteq A$.

Let G be a graph and let H be its induced subgraph. Call a subset $A \subseteq V(H)$ *H-separating* if none of the vertices in the graph $H \setminus A$ is adjacent to any of the vertices of $G \setminus V(H)$.

Suppose that graphs H_1 and H_2 are α-similar with respect to $A \subseteq V(H_1) \cap V(H_2)$. Assume that G contains a induced subgraph H_1 with H_1-separating set A. The *replacement of the subgraph H_1 in G by the graph H_2* consists in formation of the graph G^* with the vertex set $(V(G) \setminus V(H_1)) \cup V(H_2)$ and the edge set $(E(G) \setminus E(H_1)) \cup E(H_2)$.

Lemma 1. *If G^* is the graph obtained by replacing H_1 with H_2 in G then*

$$\alpha(G^*) = \alpha(G) + \alpha(H_2) - \alpha(H_1).$$

Proof. Let S be a m.i.s. in G,

$$M = S \setminus V(H_1), \qquad X = \bigcup_{x \in M} (N(x) \cap V(H_1)).$$

Since $X \subseteq A$, we have $\alpha(G) = |M| + \alpha(H_1 \setminus X)$. If we add to M a m.i.s. of $H_2 \setminus X$ in G^* then we obtain an i.s. of size $|M| + \alpha(H_2 \setminus X)$. Consequently,

$$\alpha(G^*) \geq |M| + \alpha(H_2 \setminus X) = \alpha(G) - \alpha(H_1 \setminus X) + \alpha(H_2 \setminus X) = \alpha(G) - \alpha(H_1) + \alpha(H_2).$$

The reverse inequality is proved by analogy. Lemma 1 is proved. $\square$

The replacement is the most important instrument for obtaining the main result of this article.

3. IRREDUCIBLE GRAPHS AND THEIR PROPERTIES

3.1. Compressible Subgraphs with Separating Set

Suppose that H is a graph, $A \subseteq V(H)$, and $B = V(H) \setminus A$. Let $\mathfrak{M}(H, A)$ denote the family of all sets $X \subseteq A$ such that $\alpha(G[B \cup Y]) < \alpha(G[B \cup X])$ for each $Y \subset X$. The definition of α-similarity shows that $c = \alpha(H_1) - \alpha(H_2)$, and so $\alpha(H_1) - \alpha(H_1 \setminus X) = \alpha(H_2) - \alpha(H_2 \setminus X)$ for every $X \subseteq A$. Hence, after removing the vertices of every set $X \subseteq A$ from H_1 and H_2, the independence numbers change identically. It follows that $\mathfrak{M}(H_1, A) = \mathfrak{M}(H_2, A)$ for α-similar graphs H_1 and H_2 with respect to A. The converse is also obvious: H_1 and H_2 are α-similar with respect to $A \subseteq V(H_1) \cap V(H_2)$ if and only if $\mathfrak{M}(H_1, A) = \mathfrak{M}(H_2, A)$.

Call a pair (H, A) *degenerate* if the union of all elements in $\mathfrak{M}(H, A)$ is not equal to A.

Lemma 2. *If a graph G has a induced subgraph H with H-separating set A and (H, A) is a degenerate pair then, for some vertex $x \in A$, we have $\alpha(G \setminus \{x\}) = \alpha(G)$.*

Proof. Suppose that a vertex $x \in A$ belongs to no set of the family $\mathfrak{M}(H, A)$. Assume that S is an m.i.s. in G and $x \in S$. The set $X = A \cap S$ does not belong to $\mathfrak{M}(H, A)$. Then there exists $Y \subset X$ such that $x \notin Y$ and

$$\alpha(H[B \cup X]) = \alpha(H[B \cup Y]) = |S \cap V(H)|.$$

Let Z be an m.i.s in $H[B \cup Y]$. Then $(S \setminus V(H)) \cup Z$ is an i.s. of size $\alpha(G)$ in $G \setminus \{x\}$. Lemma 2 is proved. $\square$

Below in Section 3.1, we assume that the pair (H, A) is nondegenerate. If $A = \{v_1, v_2\} \subseteq V(H)$ then there are three possible cases:

$$\text{(I)}\, \mathfrak{M}(H, A) = \{\{v_1, v_2\}\}, \quad \text{(II)}\, \mathfrak{M}(H, A) = \{\{v_1\}, \{v_2\}\}, \quad \text{(III)}\, \mathfrak{M}(H, A) = \{\{v_1\}, \{v_2\}, \{v_1, v_2\}\}.$$

In each of these cases, define a graph H' as follows:

(I) H' is a simple path (v_1, v_3, v_2); (II) H' is the complete graph on the two vertices v_1, v_2;
(III) H' is the empty graph on the two vertices v_1 and v_2.

In each of these cases, the graphs H and H' are α-similar with respect to A.

Lemma 3. *Suppose that $H = (V, E)$ is a connected induced subgraph in G including an H-separating set $A = \{v_1, v_2\}$ and $|V(H)| \geq 3$. Let $G^*(t)$ be the result of replacing H by $H'(t)$ in G, where $H'(t)$ is defined by the rule number t. Then, for every t, $G^*(t)$ belongs to $\mathcal{P}(3) \cap \mathrm{Free}(T_{3,3,2})$ if G does not contain separating cliques and lies in the same set.*

Proof. Obviously, $G^*(t) \in \mathcal{P}(3)$. Assume that $G^*(t)$ has a induced triod $T_{3,3,2}$. Since G does not include separating cliques and $|V(H)| \geq 3$, therefore v_1v_2 is not an edge in G. This and the connectedness of H imply that H contains a induced path of length at least 2 between v_1 and v_2. Therefore, G has a induced subgraph $T_{3,3,2}$ in each of the cases (I)–(III) by the definition of $H'(t)$; a contradiction. Hence, the assumption was erroneous, and Lemma 3 is proved. $\square$

Refer to a induced connected subgraph H in a graph 2-*compressible* if H has an H-sepa- rating set with exactly two vertices and $|V(H)| \geq 4$.

Let H be a graph and let $A = \{v_1, v_2, v_3\} \subseteq V(H)$. Depending on the family $\mathfrak{M}(H, A)$, in each of the cases, define the graph H' as follows:

(I) if $\mathfrak{M}(H, A) = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}\}$ then H' is a simple path $(v_1, v_3, v_4, v_5, v_2)$;

(II) if $\mathfrak{M}(H, A) = \{\{v_1, v_2, v_3\}\}$ then put H' to be a simple path $(v_1, v_4, v_2, v_5, v_3)$;

(III) if $\mathfrak{M}(H, A) = \{\{v_1\}, \{v_2, v_3\}\}$ then H' is a simple path (v_1, v_2, v_4, v_3);

(IV) if $\mathfrak{M}(H, A) = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}\}$ then H' is the tree with vertices v_1, v_2, v_3, v_4, v_5 and edges v_1v_4, v_2v_4, v_4v_5, and v_5v_3;

(V) if $\mathfrak{M}(H, A) = \{\{v_1, v_2\}, \{v_1, v_3\}\}$ then H' is the graph with vertices v_1, v_2, v_3, v_4, v_5, v_6 and edges v_1v_4, v_4v_5, v_4v_2, v_2v_5, v_5v_6, and v_6v_3;

(VI) if $\mathfrak{M}(H, A) = \{\{v_1\}, \{v_2\}, \{v_3\}\}$ then put $H' = K_3$ with vertices v_1, v_2, and v_3;

(VII) if $\mathfrak{M}(H, A) = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}\}$ then H' is a simple path $(v_1, v_4, v_5, v_3, v_2)$.

In each of these cases, H and H' are α-similar with respect to A.

A induced subgraph H in a graph is called $(3, t)$-*compressible* if the H-separating set contains exactly three vertices and $|V(H)| \geq 4$ (if $t = \mathrm{VI}$), or $|V(H)| \geq 5$ (if $t = \mathrm{III}$), or $|V(H)| \geq 6$ (if t in $\{\mathrm{I}, \mathrm{II}, \mathrm{IV}, \mathrm{VII}\}$), or $|V(H)| \geq 7$ (if $t = \mathrm{V}$).

In what follows, apply each of the above-described seven compressions to the graphs of the class $\mathcal{P}(3) \cap \mathrm{Free}(T_{3,3,2})$ so that the result again belong to the same class (as a rule, the result of applying a replacement to G is a induced subgraph of G).

Suppose now that H is a graph and $A = \{v_1, v_2, v_3, v_4\} \subseteq V(H)$. Again, in each of the cases of the family $\mathfrak{M}(H, A)$, define a graph H':

(I) if $\mathfrak{M}(H, A) = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_3, v_4\}\}$, then as H' consider the simple cycle (w_1, w_3, w_2, w_4);

(II) if $\mathfrak{M}(H, A) = \big\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}\big\}$ then H' is the graph with vertices w_1, w_2, w_3, w_4 and edges w_1w_2, w_3w_4;

(III) if $\mathfrak{M}(H, A) = \{\{v_2, v_4\}, \{v_1, v_3, v_4\}\}$ then H' is the graph with vertices $V(H') = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$ and edges $E(H') = \{w_1w_5, w_5w_6, w_6w_2, w_6w_7, w_7w_3, w_3w_8, w_8w_2, w_8w_4\}$;

(IV) if $\mathfrak{M}(H, A) = \{\{v_2, v_4\}, \{v_1, v_3\}\}$ then H' is the graph with vertices $V(H') = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$ and edges $E(H') = \{w_1w_5, w_5w_6, w_6w_2, w_6w_7, w_7w_3, w_3w_4, w_4w_8, w_2w_8\}$.

In each of these cases, H and H' are α-similar with respect to A.

A induced subgraph H of a graph is called $(4, t)$-*compressible* if the H-separating set contains exactly four vertices and $|V(H)| \geq 5$ (if $t \in \{\mathrm{I}, \mathrm{II}\}$) or $|V(H)| \geq 9$ (if $t \in \{\mathrm{III}, \mathrm{IV}\}$).

In what follows, apply each of the above-described four compressions to the graphs of the class $\mathcal{P}(3) \cap \mathrm{Free}(T_{3,3,2})$ so that the result again belong to the same class (as a rule, the result of applying a replacement to G is a induced subgraph of G).

Let us write a H-separating set as a collection that is called an H-*separator* (as distinct from the set). We assume that under replacement the ith element of a collection goes to the vertex v_i.

3.2. The Notion of Irreducible Graph and Its Meaning

Call a connected graph G *irreducible* if the following are fulfilled simultaneously:

1°. G belongs to $\mathcal{P}(3) \cap \mathrm{Free}(T_{3,3,2})$.

2°. G has no separating cliques.

3°. G does not posses a induced subgraph H and an H-separating set A such that $|V(H)| \leq 12$, $|A| \leq 4$, and the pair (H, A) is degenerate.

4°. G has no connected induced subgraph H_1 with more than 12 vertices such that to G one can apply the replacement of the subgraph H_1 by some graph H_2 such that $|V(H_2)| < |V(H_1)|$ and the result belongs to the class $\mathcal{P}(3) \cap \mathrm{Free}(T_{3,3,2})$.

It is known that, for a hereditary graph class $\mathcal{X}$, the IS problem is polynomially reducible to the same problem for the part $\mathcal{X}$ constituted by all connected graphs of $\mathcal{X}$ without separating cliques [5]. Suppose that $G \in \mathcal{P}(3) \cap \mathrm{Free}(T_{3,3,2})$. Exhausting all subsets of its vertices of size at most 12 and exhausting all graphs with at most 11 vertices and also solving at most $2^4 = 16$ IS problems for each of the graphs with at most 12 vertices, we can check in time $O(|V(G)|^{12})$ whether G satisfies conditions 3° and 4°. The membership of a graph with n vertices and m edges in the class $\mathcal{P}(3)$ is recognized in time $O(n + m)$ [7]. The membership of a graph with n vertices in the class $\mathrm{Free}(T_{3,3,2})$ is recognized in time $O(n^9)$ by exhausing all 9-element subsets of vertices and checking the inducedness of the subgraph $T_{3,3,2}$ by one of these subsets. The above-listed facts and Lemma 2 imply that the IS problem for the graphs of class $\mathcal{P}(3) \cap \mathrm{Free}(T_{3,3,2})$ is polynomially reducible to the IS problem for the irreducible graphs of this class.

3.3. Some Auxiliary Results

Lemma 4. *For all induced* 5*-cycle in an irreducible graph G, at least four vertices of the cycle have degree* 3.

Proof. Suppose the contrary: The graph G has an induced cycle $(x_1, x_2, x_3, x_4, x_5)$ in which either $\deg(x_3) = \deg(x_5) = 2$ or $\deg(x_4) = \deg(x_5) = 2$. In the former case, the pair $(G[x_1, x_2, x_3, x_4, x_5], \{x_1, x_2, x_4\})$ is degenerate. In the latter case, we obtain the $(3, \mathrm{VI})$-compressible subgraph $H = G[x_1, x_2, x_3, x_4, x_5]$ with H-separator (x_1, x_2, x_3). The result of compression, denoted by G^*, is a minor of G since G^* is obtained by contracting the edges x_1x_5 and x_4x_3; therefore, $G^* \in \mathcal{P}(3)$.

The graph G^* has no induced triod $T_{3,3,2}$: for this it suffices to show that no induced triod $T_{3,3,2}$ in G^* includes the edge x_1x_3. Indeed, if such a triod $T_{3,3,2}$ in G^* exists then G includes an induced triod $T_{3,3,2}$ one of whose edges is x_1x_5.

Lemma 4 is proved. □

Lemma 5. *Let G be a planar subcubic graph and let C_1^*, C_2^*, and C_3^* be its three pairwise distinct cycles such that each of the sets*

$$E(C_1^*) \cap E(C_2^*), \qquad E(C_2^*) \cap E(C_3^*), \qquad E(C_1^*) \cap E(C_3^*)$$

generates a simple path in G. Suppose also that G contains three edges e_1^*, e_2^*, and e_3^* that constitute an induced path in given order, where

$$e_1^* \in (E(C_1^*) \cap E(C_2^*)) \setminus E(C_3^*), \qquad e_2^* \in E(C_1^*) \cap E(C_2^*) \cap E(C_3^*),$$
$$e_3^* \in (E(C_2^*) \cap E(C_3^*)) \setminus E(C_1^*).$$

Then, for every planar embedding of G, we have one of the inclusions

$$D(C_3^*) \subset D(C_2^*), \qquad D(C_1^*) \subset D(C_2^*).$$

Proof. Consider an arbitrary planar embedding of G. It suffices to consider the two cases: either $D(C_2^*) \subset D(C_1^*)$ or $D(C_2^*) \not\subseteq D(C_1^*)$ and $D(C_1^*) \not\subseteq D(C_2^*)$. Since, for all distinct i and j, $E(C_i^*) \cap E(C_j^*)$ generates a simple path in G and G is subcubic planar; therefore, $D(C_i^*) \cap D(C_j^*)$ is a Jordan curve. Let P be the simple path in G generated by the set of edges $E(C_1^*) \cap E(C_2^*)$. Clearly, $e_1^*, e_2^* \in E(P)$. Since $e_3^* \notin E(C_1^*)$, we have $e_3^* \notin E(P)$. Obviously, e_2^* is a final edge of P; otherwise, e_2^*, e_3^*, and the edges of P adjacent to e_2^* and different from e_1^* have a common vertex and lie in C_2^*.

Recall that $D(C_2^*) \cap D(C_3^*)$ is a Jordan curve of which the edges e_2^* and e_3^* are parts and e_1^* is not; moreover, G is a subcubic planar graph. This obviously implies $D(C_3^*) \subset D(C_2^*)$ in both that cases. Lemma 5 is proved. □

4. NONEXISTENCE OF IRREDUCIBLE GRAPHS WITH SUFFICIENTLY LARGE GENERATED TRIODS

This section aims at proving that there is no irreducible graph including an induced triod $T_{2,2,10}$. Suppose that such a graph $G = (V, E)$ exists. Consider its induced triod $T_{2,2,10}$. Denote the vertex of the triod of degree three by o, designate the vertices of the branch with ten vertices as $a_1, a_2, \ldots, a_{10}$ (in receding order from o) and the vertices of the other two branches as b_1, b_2 and c_1, c_2 (also in receding order from o).

In the next three lemmas, we demonstrate the impossibility of the equality

$$N(b_2) \setminus \{b_1\} = N(c_2) \setminus \{c_1\}.$$

We will prove these lemmas by way of contradiction on setting $N' = N(b_2) \setminus \{b_1\}$. Note that $N' \neq \varnothing$; otherwise, b_1 constitutes a separating clique of G. Obviously, N' consists either of a single vertex x or of two vertices x and y. In proving the next three lemmas, we will assume that x and y are nonadjacent; otherwise, the subgraph $G[b_2, c_2, x, y]$ is 2-compressible. Finally, we will suppose that $\deg(b_1) \geq \deg(c_1)$ since this assumption does not diminish generality.

Lemma 6. *Each element of $N' = N(b_2) \setminus \{b_1\}$ is not adjacent to no one of the vertexes of $\{a_1, a_2, a_3\}$.*

Proof. Suppose that some $x \in N'$ has a neighbor $a_{i'} \in \{a_1, a_2, a_3\}$. Obviously, this can be only a_1; otherwise, $[a_{i'}, a_{i'+1}, a_{i'+2}, a_{i'+3}, a_{i'-1}, a_{i'-2}, a_{i'-3}, x, c_2]$, where $a_0 = o$ and $a_{-1} = b_1$.

Let $N' = \{x, y\}$. If $yb_1 \in E$ then $[a_1, a_2, a_3, a_4, x, b_2, y, o, c_1]$. Clearly, y has a neighbor $a_{i''}$ in $\{a_2, a_3, a_4\}$; otherwise, $[a_1, a_2, a_3, a_4, x, c_2, y, o, b_1]$; but then

$$[a_{i''}, a_{i''+1}, a_{i''+2}, a_{i''+3}, a_{i''-1}, a_{i''-2}, a_{i''-3}, y, c_2].$$

Now, let $N' = \{x\}$. There exists a vertex $b_1' \in N(b_1) \setminus \{b_2, o\}$; otherwise, the vertices x and a_1 form a separating clique of G. If $b_1'c_1 \in E$ then, obviously, b_1' must be adjacent to some vertex $a_{i'''}$ in $\{a_2, a_3, a_4\}$; otherwise, $[a_1, a_2, a_3, a_4, o, b_1, b_1', x, c_2]$. It is easy to see that $i''' \neq 2$; otherwise, $\{b_1', a_2\}$ is a separating clique in G. Then

$$[a_{i'''}, a_{i'''+1}, a_{i'''+2}, a_{i'''+3}, b_1', b_1, b_2, a_{i'''-1}, a_{i'''-2}].$$

Consider the case when b_1' and c_1 are nonadjacent. Clearly, b_1' is adjacent at least with one of the vertices a_2 and a_3; otherwise, $[x, a_1, a_2, a_3, b_2, b_1, b_1', c_2, c_1]$. We may assume that b_1' is nonadjacent to a_2; otherwise, $\{x, a_1\}$ is a separating clique in G (if $\deg(c_1) = 2$) or c_1 has a neighbor not belonging to $\{o, c_2, b_1'\}$ and adjacent to a_3; moreover, b_1' and this neighbor are similar as regards arguments. In other words, we may assume that $b_1'a_3 \in E$.

Suppose that c_1 has a neighbor $c_1' \notin \{o, c_2, b_1'\}$. In this event, either $b_1'c_1' \in E$ or $c_1'a_2 \in E$; otherwise, $[x, b_2, b_1, b_1', c_2, c_1, c_1', a_1, a_2]$. Suppose first that $b_1'c_1' \in E$. If $c_1'a_2 \in E$ then $\{b_1', a_3\}$ is a separating clique in G; otherwise, $[b_1', c_1', c_1, c_2, a_3, a_2, a_1, b_1, b_2]$. Suppose on the contrary that $b_1'c_1' \notin E$. Then c_1' is adjacent to a_2; otherwise, $[x, b_2, b_1, b_1', c_2, c_1, c_1', a_1, a_2]$. Clearly, c_1' must be adjacent to some vertex a_i for $i \in \{4, 5\}$; otherwise, $[a_2, a_3, a_4, a_5, a_1, x, b_2, c_1', c_1]$. Hence, $[a_1, a_2, c_1', a_i, o, b_1, b_1', x, c_2]$. It remains to consider the case when $\deg(c_1) = 2$. The vertex a_2 has a neighbor $a_2' \notin \{a_1, a_3\}$ since otherwise b_1' and a_3 constitute a separating clique in G. The vertices a_2' and b_1' are nonadjacent; otherwise, $[a_3, a_4, a_5, a_6, b_1', b_1, b_2, a_2, a_1]$. But then $[x, b_2, b_1, b_1', a_1, a_2, a_2', c_2, c_1]$.

We have a contradiction in all cases. Therefore, the assumption of the existence of a vertex x fails. Lemma 6 is proved. □

Lemma 7. *If* $b_1' \in N(b_1) \setminus \{o, b_2\}$ *and* $c_1' \in N(c_1) \setminus \{o, c_2\}$ *then* $b_1' = c_1'$.

Proof. Suppose the contrary. Let v be an arbitrary vertex in $N' = N(b_2) \setminus \{b_1\}$. By Lemma 6, none of the vertices $a_1, \dots, a_{10}$ is a neighbor of v. Without loss of generality, we may consider the two cases: (1) $b_1'v \notin E$ and $c_1'v \notin E$; (2) $b_1'v \notin E$ and $c_1'v \in E$.

(1) Each of the vertices b_1' and c_1' has a neighbor in the set $\{a_1, a_2, a_3\}$; otherwise,

$$[o, c_1, c_2, v, a_1, a_2, a_3, b_1, b_1'] \quad \text{or} \quad [o, b_1, b_2, v, a_1, a_2, a_3, c_1, c_1'].$$

Assume that $b_1'a_3 \in E$. Then

$$\begin{aligned} &[b_1, b_2, v, c_2, b_1', a_3, a_4, o, a_1], && \text{if } b_1'a_4 \notin E,\ b_1'a_1 \notin E, \\ &[b_1, b_2, v, c_2, b_1', a_4, a_5, o, a_1], && \text{if } b_1'a_4 \in E,\ b_1'a_1 \notin E, \\ &[b_1, o, c_1, c_1', b_1', a_3, a_4, b_2, v], && \text{if } b_1'a_1 \in E. \end{aligned}$$

Suppose that $b_1'a_2 \in E$, $c_1'a_1 \in E$, $b_1'a_3 \notin E$, and $c_1'a_3 \notin E$. Under these conditions, b_1' and c_1' are nonadjacent; otherwise, $[b_1', b_1, b_2, v, a_2, a_3, a_4, c_1', c_1]$. Then $[b_1, b_1', a_2, a_3, o, c_1, c_1', b_2, v]$. This exhausts Case 1.

(2) If $N' = \{v\}$ then the subgraph $H_1 = G[o, b_1, b_2, c_1, c_2, c_1', v]$ with H_1-separator (o, c_1', b_1) is $(3, \mathrm{I})$-compressible. The result of the compression is obtained by removing c_1 and c_2 from G.

Let $N' = \{v, u\}$. If $uc_1' \in E$ then the subgraph $G[b_2, c_2, v, u, c_1, c_1']$ is 2-compressible. Otherwise, $uc_1' \notin E$. Then, for $ub_1' \notin E$, we obtain a contradiction by analogy with Case 1, and, for $ub_1' \in E$, the subgraph $H_2 = G[o, b_1, b_2, c_1, c_2, v, u, b_1', c_1']$ with H_2-separator (b_1', o, c_1') is $(3, \mathrm{II})$-contractible. The result of the compression is obtained by removing the vertices u, v, b_2, and c_2 from G.

We have a contradiction in all cases. Therefore, the assumption $b_1' \neq c_1'$ fails. Lemma 7 is proved. □

Lemma 8. $N(b_2) \setminus \{b_1\} \neq N(c_2) \setminus \{c_1\}$.

Proof. Suppose the contrary. As above, put $N' = N(b_2) \setminus \{b_1\}$. The two cases are possible: (1) $N' = \{x\}$; (2) $N' = \{x, y\}$ and the vertices x and y are nonadjacent.

(1) Let $\deg(x) = 2$. Then there exist vertices $b_1' \in N(b_1) \setminus \{o, b_2\}$ and $c_1' \in N(c_1) \setminus \{o, c_2\}$; otherwise, $\{o, b_1\}$ or $\{o, c_1\}$ is a separating clique of G. By Lemma 7, we have $b_1' = c_1'$. Then the subgraph $G[b_1', o, b_1, c_1, b_2, c_2, x]$ is 2-compressible.

We will assume that $\deg(x) = 3$. Consider the possible subcases:

1.1: $\deg(b_1) \in \{2, 3\}$ and $\deg(c_1) = 2$. Under these conditions, the induced subgraph

$$H_1 = G[o, b_1, c_1, b_2, c_2, x]$$

with H_1-separator (o, b_1, x) is $(3, \mathrm{III})$-compressible. The result of the compression is obtained by removing the vertices c_1 and c_2 from G.

1.2: $\deg(b_1) = 3$ and $\deg(c_1) = 3$. In this subcase, by Lemma 7, the vertices b_1 and c_1 are adjacent to the same vertex $p \neq o$. Clearly, here the vertices p and x are nonadjacent; otherwise, $\{o\}$ is a separating clique of G. Then the subgraph $H_2 = G[o, p, b_1, c_1, b_2, c_2, x]$ with H_2-separator (o, p, x) is $(3, \mathrm{IV})$-compressible. The result of the compression is obtained by removing c_1 and c_2 from G.

(2) Without loss of generality, assume that $\deg(x) \geq \deg(y)$. The three subcases are possible:

2.1: $\deg(b_1) = \deg(c_1) = 2$. If $\deg(x) = 2$ and $\deg(y) = 2$ then o constitutes a separating clique in G. If $\deg(x) = 3$ then this variant is completely equivalent to Case 1.2.

2.2: $\deg(b_1) = 3$ and $\deg(c_1) = 2$. If $\deg(x) = \deg(y) = 2$ then $\{o, b_1\}$ is a separating clique in G. If $\deg(x) = 3$ and $\deg(y) = 2$ then the subgraph $H_3 = G[o, b_1, c_1, b_2, c_2, x, y]$ with H_3-separator (b_1, x, o) is $(3, \mathrm{I})$-compressible. The result of compression is obtained by removing b_2 and y from G.

Suppose that $\deg(x) = \deg(y) = 3$. For symmetry reasons, we may assume that, in some planar embedding of G, the vertex y lies inside $D' = D(o, c_1, c_2, x, b_2, b_1)$. Clearly, if $b_1' \in N(b_1) \setminus \{o, b_2\}$ then b_1' is adjacent to at least one of the vertices a_1, a_2, and a_3; otherwise, $[o, c_1, c_2, x, a_1, a_2, a_3, b_1, b_1']$. Consequently, the vertices $b_1', a_1, \dots, a_{10}$ are simultaneously either inside D' or outside D'. If they are inside D' then they must lie inside the domain $D'' = D(o, c_1, c_2, y, b_2, b_1)$. Hence, a neighbor of y different from b_2 and c_2 must belong to D''; otherwise, $\{b_1, o\}$ is a separating clique in G. Therefore, $\{x\}$ is a separating clique in G. If $b_1, a_1, \dots, a_{10}$ do not belong to D' then the neighbor of x different from b_2 and c_2 must also lie outside D'; otherwise, $\{b_1, o\}$ is a separating clique in G. Hence, y forms a separating clique in G.

2.3: $\deg(b_1) = \deg(c_1) = 3$. By Lemma 7, b_1 and c_1 are adjacent to a common vertex $q \neq o$. If one of the edges qx or qy belongs to E then the induced subgraph $G[o, q, b_1, c_1, b_2, c_2, x, y]$ is 2-compressible. If $\{o, q, x, y\}$ is an independent set in G then the induced subgraph $H_4 = G[o, q, b_1, c_1, b_2, c_2, x, y]$ with H_4-separator (o, q, x, y) is $(4, \mathrm{I})$-compressible. The result of compression is a minor of G since it is obtained by contracting the edges b_1b_2, c_1c_2, b_2y, and xc_2. Therefore, this minor is a subcubic planar graph. It is not hard to check that if the obtained graph contains an induced triod $T_{3,3,2}$ then G also contains an induced triod $T_{3,3,2}$.

We have a contradiction in all three cases. Thus, there is no irreducible graph G with

$$N(b_2) \setminus \{b_1\} = N(c_2) \setminus \{c_1\}.$$

Lemma 8 is proved. □

Thus, we proved that there exists a vertex d in G not belonging to $T_{2,2,10}$ and adjacent to exactly one of the vertices b_2 and c_2. Assume without loss of generality that this vertex is adjacent to b_2. It is clear that d must be adjacent to at least one of the vertices in the set $\{b_1, c_1, a_1, a_2, a_3\}$; otherwise, $[o, a_1, a_2, a_3, b_1, b_2, d, c_1, c_2]$.

Lemma 9. *The equality* $N(d) \cap \{b_1, c_1, a_1, a_2, a_3\} = \{b_1\}$ *is impossible.*

Proof. Suppose the contrary; i.e., the equality holds. Clearly, $\deg(b_2) = \deg(d) = 3$; otherwise, G has a separating clique $\{b_1, b_2\}$. So, we may assume that there exist vertices $b_2' \in N(b_2) \setminus \{b_1, d\}$ and $d' \in N(d) \setminus \{b_1, b_2\}$. If $d' = b_2'$ then the subgraph $G[b_1, b_2, d, d']$ is 2-compressible; therefore, we assume that $b_2' \neq d$. Each of the vertices b_2' and d' must be adjacent to at least one of the vertices c_1, c_2, a_1, a_2, and a_3; otherwise,

$$[o, a_1, a_2, a_3, b_1, d, d', c_1, c_2] \quad \text{or} \quad [o, a_1, a_2, a_3, b_1, b_2, b_2', c_1, c_2].$$

Thus, at least one of the vertices b_2' and d' must be adjacent to at least one of the vertices a_1, a_2, and a_3; otherwise, $\{b_2', d'\}$ contains a vertex (say, d') having exactly one neighbor in $\{c_1, c_2\}$, the vertex c_2. Hence,

$$k = \max(\{i \in \overline{1,3} \mid a_i b_2' \in E\} \cup \{i \in \overline{1,3} \mid a_i d' \in E\})$$

is defined. Assume that d' is a neighbor of a_k and consider one of the possible cases: $k = 1$, $k = 2$, and $k = 3$.

1. Let $k = 1$. Obviously, $b_2' c_1 \in E$ since otherwise $b_2' c_2 \in E$, $b_2' c_1 \notin E$, and $[o, a_1, a_2, a_3, c_1, c_2, b_2', b_1, d]$. Therefore, $[a_1, o, c_1, c_2, d', d, b_2, a_2, a_3]$ (if $c_2 d' \notin E$), or $[d', c_2, c_1, b_2', a_1, a_2, a_3, d, b_1]$ (if $c_2 d' \in E$ and $c_2 b_2' \notin E$), or $[a_1, a_2, a_3, a_4, d', c_2, b_2', o, b_1]$ (if $c_2 d' \in E$ and $c_2 b_2' \in E$).

2. Let $k = 2$. If $a_1 d' \in E$ then the subgraph $H = G[o, b_1, b_2, d, d', a_1, a_2]$ with H-separator (o, b_2, a_2) is $(3, \mathrm{V})$-compressible. The result of the compression is obtained by remoning a_1 from G. Assume that $d' a_1 \notin E$. The vertex d' must be adjacent to at least one of the vertices a_3, a_4, and a_5; otherwise, $[a_2, d', d, b_2, a_3, a_4, a_5, a_1, o]$. Moreover, the vertex d' is adjacent namely to a_3 since otherwise $[d', a_2, a_1, o, a_4, a_5, a_6, d, b_2]$ (if $d' a_4 \in E$) or $[d', a_2, a_1, o, a_5, a_6, a_7, d, b_2]$ (if $d' a_5 \in E$). If $\deg(a_1) = 2$ then the pair $(G[o, a_1, a_2, a_3, b_1, b_2, d, d'], \{o, b_2, a_3\})$ is degenerate. Consider the two subcases, namely, $b_2' a_1 \in E$ and $b_2' a_1 \notin E$.

2.1: $b_2' a_1 \in E$. If, moreover, $b_2' c_1 \in E$ then we have the 2-compressible subgraph

$$G[o, a_1, a_2, a_3, c_1, b_1, b_2, b_2', d, d'].$$

If $b_2' c_1 \notin E$ and $b_2' c_2 \in E$ then $\deg(c_1) = 2$. Indeed, suppose that $c_1' \in N(c_1) \setminus \{o, c_2\}$, then

$$[a_1, o, c_1, c_1', b_2', b_2, d, a_2, a_3].$$

Thus, for $b_2' c_1 \notin E$ and $b_2' c_2 \in E$, we obtain a 2-compressible subgraph

$$G[o, a_1, a_2, a_3, c_1, c_2, b_1, b_2, b_2', d, d'].$$

Finally, if $b_2' c_1 \notin E$ and $b_2' c_2 \notin E$ then $[a_1, o, c_1, c_2, b_2', b_2, d, a_2, a_3]$.

2.2: $b_2' a_1 \notin E$. In this case, $b_2' c_1 \in E$; otherwise,

$$b_2' c_2 \in E, \quad b_2' c_1 \notin E, \quad \text{and} \quad [o, a_1, a_2, a_3, c_1, c_2, b_2', b_1, d].$$

Thus we have the degenerate pair $(G[o, a_1, a_2, a_3, b_1, b_2, b_2', d, d', c_1], \{a_3, a_1, c_1, b_2'\})$.

3. Let $k = 3$. By Case 2, we will assume that $d' a_2 \notin E$. Clearly, $d' a_1 \notin E$; otherwise,

$$[d', a_3, a_4, a_5, a_1, o, c_1, d, b_2].$$

The vertex d' must be adjacent to one of the vertices a_4 or a_5; otherwise, $[a_3, a_2, a_1, o, d', d, b_2, a_4, a_5]$. The vertex d' must be adjacent exactly to a_4; otherwise,

$$d' a_5 \in E, \qquad d' a_4 \notin E, \qquad [d', a_5, a_6, a_7, a_3, a_2, a_1, d, b_2].$$

Consider the subcases of $\deg(a_2) = 2$ and $\deg(a_2) = 3$.

3.1: Suppose that there exists $a_2' \in N(a_2) \setminus \{a_1, a_3\}$. First, put $a_2' a_1 \notin E$. Then a_2' must have a neighbor in the set $\{c_1, c_2\}$; otherwise, $[o, a_1, a_2, a_2', b_1, d, d', c_1, c_2]$. If $a_2' c_2 \in E$ but $a_2' a_5 \notin E$ then $[a_2, a_3, a_4, a_5, a_1, o, b_1, a_2', c_2]$. If E contains $a_2' c_2$ and $a_2' a_5$ then $[a_2, a_2', a_5, a_6, a_3, d', d, a_1, o]$. Finally, if $a_2' c_2 \notin E$ then $a_2' c_1 \in E$ and also $b_2' c_2 \in E$ and $b_2' a_1 \in E$ (otherwise, $[o, a_1, a_2, a_3, b_1, b_2, b_2', c_1, c_2]$). Hence, $[a_1, a_2, a_3, a_4, b_2', b_2, d, o, c_1]$.

Suppose that $a_2'a_1 \in E$. Then $\deg(a_2') = 3$; otherwise, G contains the separating clique $\{a_1, a_2\}$. Clearly, $b_2'c_1 \in E$; otherwise,

$$b_2'c_2 \in E, \qquad b_2'c_1 \notin E, \qquad [o, a_1, a_2, a_3, c_1, c_2, b_2', b_1, d].$$

Let $a_2'' \in N(a_2') \setminus \{a_1, a_2\}$. If $a_2'' = b_2'$ then the subgraph $G[o, a_1, a_2, a_3, a_4, a_2', b_1, b_2, d, d', c_1, b_2']$ is 2-compressible. If $a_2'' \neq b_2'$ then b_2' and a_2'' are adjacent (otherwise, $[o, a_1, a_2', a_2'', b_1, d, d', c_1, b_2']$); hence, $[b_2', a_2'', a_2', a_2, b_2, d, d', c_1, o]$.

3.2: $\deg(a_2) = 2$. In this case, we obtain the degenerate pair

$$(G[o, a_1, a_2, a_3, a_4, b_1, b_2, d, d'], \{a_1, o, b_2, a_4\}).$$

Lemma 9 is proved. □

Lemma 10. *The equality $N(d) \cap \{b_1, c_1, a_1, a_2, a_3\} = \{c_1\}$ is impossible.*

Proof. Suppose the contrary, i.e., that the equality holds. Obviously, $\deg(c_2) \geq 2$; otherwise, c_1 constitutes a separating clique in G. Consider an arbitrary vertex $e \in N(c_2) \setminus \{c_1\}$. It is easy to see that

$$N(e) \cap \{a_1, a_2, a_3, b_1, b_2\} \neq \varnothing$$

since otherwise $[o, a_1, a_2, a_3, c_1, c_2, e, b_1, b_2]$. If $ea_3 \in E$, $ea_1 \notin E$, and $ea_2 \notin E$ then e must be adjacent to some a_i for $i \in \overline{4,6}$ (otherwise, $[a_3, a_4, a_5, a_6, a_2, a_1, o, e, c_2]$); but then $[c_1, c_2, e, a_i, o, a_1, a_2, d, b_2]$. If

$$ea_2 \in E, \qquad ea_1 \notin E, \qquad ea_3 \notin E, \qquad eb_1 \notin E$$

then e must be adjacent to some a_i for $i \in \{4, 5\}$ (otherwise $[a_2, a_1, o, b_1, a_3, a_4, a_5, e, c_2]$), and so $[e, a_i, a_{i+1}, a_{i+2}, c_2, c_1, d, a_2, a_1]$. If $ea_1 \in E$, but $ea_2 \notin E$, $ea_3 \notin E$, $eb_1 \notin E$, and $eb_2 \notin E$ then $ea_4 \in E$ (otherwise $[a_1, a_2, a_3, a_4, o, b_1, b_2, e, c_2]$); and then $[e, a_4, a_5, a_6, c_2, c_1, d, a_1, a_2]$.

These arguments imply that the following cases are possible:

(1) $ea_1 \in E, ea_2 \in E$; (2) $ea_1 \in E, ea_3 \in E$; (3) $ea_2 \in E, ea_3 \in E$;
(4) $ea_2 \in E, eb_1 \in E$; (5) $ea_1 \in E, eb_1 \in E$; (6) $ea_1 \in E, eb_2 \in E$;
(7) $eb_1 \in E, ed \in E$; (8) $eb_2 \in E, ed \in E$; (9) $eb_1 \in E, eb_2 \in E$;
(10) $N(e) \cap \{a_1, a_2, a_3, b_1, b_2\} = \{b_1\}$; 11) $N(e) \cap \{a_1, a_2, a_3, b_1, b_2\} = \{b_2\}$.

Firstly consider the cases when e is adjacent to two elements of $\{b_1, b_2, d\}$. Clearly, in each of these cases, the degrees of the vertices $o, b_1, b_2, c_1, c_2, d, e$ are equal to 3; otherwise, the graph generated by them is 2-compressible. Consider a planar embedding of G. Obviously, the vertices c_2 and e simultaneously lie either in the domain $D(o, b_1, b_2, d, c_1)$ or outside it.

Suppose that $eb_1 \in E$ and $eb_2 \in E$. Assume additionally that c_2 and e lie inside $D(o, b_1, b_2, d, c_1)$. Obviously, an element of $N(c_2) \setminus \{c_1, e\}$ either belongs to $D(c_1, o, b_1, e, c_2)$ or lies in $D(c_1, c_2, e, b_2, d)$. In the former case, $a_1, \ldots, a_{10}$ belong to $D(c_1, o, b_1, e, c_2)$ (otherwise $\{c_2\}$ is a separatong clique in G) and then $\{d\}$ is a separating clique in G. In the latter case, an element of the set $N(d) \setminus \{c_1, b_2\}$ belongs to $D(c_1, c_2, e, b_2, d)$ (otherwise, $\{c_2\}$ is a separating clique in G), but then $\{o\}$ is a separating clique in G. The case when the vertices c_2 and e lie outside $D(o, b_1, b_2, d, c_1)$ is considered similarly.

By analogy with the arguments of the previous paragraph, it is demonstrated that c_1 and b_2 cannot have degree 3 simultaneously if $eb_1 \in E$ and $ed \in E$.

If $eb_2 \in E$ and $ed \in E$ then the subgraph $H_1 = G[o, b_1, b_2, c_1, c_2, d, e]$ with H_1-separator (b_1, o, c_2) is $(3, \mathrm{III})$-compressible. The result of compression is obtained by removing the vertices e, b_2, and d from G.

Throughout the sequel, we will assume that none of the vertices in $N(c_2) \setminus \{c_1\}$ is adjacent to two elements of $\{b_1, b_2, d\}$.

(a) Suppose first that $N(e) \cap \{b_1, b_2\} \neq \varnothing$. A possible neighbor e' of c_2 different from c_1 and e has equal rights with the vertex e. In particular, e' must be adjacent to at least one of the vertices b_1, b_2, a_1, a_2, and a_3. Symmetry reasons imply that only three variants are possible:

(i) there exists a vertex in $N(c_2) \setminus \{c_1\}$ adjacent at least to one of $a_1, \ldots, a_{10}$ or $\deg(c_2) = 2$ and e has a neighbor adjacent to at least one of the vertices $a_1, \ldots, a_{10}$;

(ii) $\deg(c_2) = 2$, and neither e nor any of its neighbors is adjacent to any of the vertices $a_1, \ldots, a_{10}$;

(iii) $N(c_2) = \{e, e', c_1\}$ and the vertices e and e' satisfy the equalities

$$N(e) \cap \{a_1, a_2, a_3, b_1, b_2\} = \{b_1\}, \qquad N(e') \cap \{a_1, a_2, a_3, b_1, b_2\} = \{b_2\}.$$

Suppose the fulfillment of (i). Then G has an induced cycle C_1' containing the edges a_1o, oc_1, and c_1c_2. Consider the cycles

$$C_2' = \begin{cases} (o, c_1, c_2, e, b_1) & \text{if } \ eb_1 \in E, \\ (o, c_1, c_2, e, b_2, b_1) & \text{if } \ eb_1 \notin E, \ eb_2 \in E, \end{cases} \qquad C_3' = (o, b_1, b_2, d, c_1).$$

The edges c_2c_1, c_1o, ob_1 and the cycles C_1', C_2', C_3' satisfy the conditions of Lemma 5. Consider a planar embedding of G. Then, by Lemma 5, $D(C_3') \subset D(C_2')$ or $D(C_1') \subset D(C_2')$.

(i.1) Let $eb_1 \in E$. It is not hard to see that, in any planar embedding of G, in each of the cases $D(C_3') \subset D(C_2')$ and $D(C_1') \subset D(C_2')$, the vertices $a_1, \dots, a_{10}$ and the vertices b_2, d lie to different sides of C_2'. An edge of the cycle C_1' incident to c_2 or e and different from the edges ec_2 and c_2c_1 lies to one side of C_2' with the vertices $a_1, \dots, a_{10}$. If there exists $b_2' \in N(b_2) \setminus \{b_1, d\}$ then $[o, a_1, a_2, a_3, b_1, b_2, b_2', c_1, c_2]$. Therefore, by Lemma 4, $\deg(d) = 3$ since (o, b_1, b_2, d, c_1) is an induced cycle in G. Note that G contains an induced path $(e, v_1, \dots, v_k, d)$; otherwise, $\{d\}$ is a separating clique in G. Each of the vertices $a_1, \dots, a_{10}$ and any vertex of this path lie to different sides of C_2'; therefore, $k = 1$ (otherwise, $[o, a_1, a_2, a_3, c_1, d, v_k, b_1, e]$). Then the subgraph $H_2 = G[o, b_1, b_2, c_1, c_2, d, e, v_1]$ with H_2-separator (o, c_2, v_1) is $(3, \text{II})$-compressible. The result of the compression is obtained by removing the vertices b_1, b_2, and d from G.

(i.2) Suppose that $eb_2 \in E$. Owing to Cases 1–11, we have either $N(e) \cap \{a_1, a_2, a_3, b_1, b_2\} = \{b_2\}$ or $ea_1 \in E$. Consider the subcase when d lies inside the domain $D(C_2')$, i.e., $D(C_3') \subset D(C_2')$. The second subcase is considered similarly. Due to the presence of the cycle C_1', the vertices $a_1, \dots, a_{10}$ do not belong to $D(C_2')$. Therefore, a possible element e' of the set $N(c_2) \setminus \{c_1, e\}$ does not belong to $D(C_2')$ because of $N(e') \cap \{a_1, a_2, a_3, b_1, b_2\} \neq \varnothing$.

A possible element of $N(d) \setminus \{c_1, b_2\}$ does not belong to $D(C_3')$; i.e., it lies in the difference $D(C_2') \setminus D(C_3')$; otherwise, either $\{d\}$ is a separating clique in G or there is a vertex $b_1' \in N(b_1) \setminus \{o, b_2\}$ in $D(C_3')$. In the latter case, either

$$[a_1, a_2, a_3, a_4, o, b_1, b_1', e, c_2] \quad \text{if} \quad ea_1 \in E$$

or

$$[o, a_1, a_2, a_3, c_1, c_2, e, b_1, b_1'] \quad \text{if} \quad N(e) \cap \{a_1, a_2, a_3, b_1, b_2\} = \{b_2\}.$$

If $ea_1 \in E$ then $\deg(d) = 2$; otherwise $\{d\}$ is a separating clique in G. Moreover, the vertices b_1 and c_2 have degree 3; otherwise, the induced subgraph $G[o, a_1, c_1, c_2, e, b_1, b_2, d]$ is 2-compressible. Then at least one of the two elements of $(N(b_1) \cup N(c_2)) \setminus \{o, b_2, c_1, e\}$ belongs to the domain $D(o, c_1, c_2, e, a_1)$, and so either $\{c_2\}$ or $\{b_1\}$ is a separating clique of G.

If $N(e) \cap \{a_1, a_2, a_3, b_1, b_2\} = \{b_2\}$ then at least one of the vertices d or e has degree 3; otherwise, the pair $(G[o, b_1, b_2, c_1, c_2, d, e], \{o, b_1, c_2\})$ is degenerate.

(i.2.1) Let $\deg(c_2) = 2$. Clearly, $\deg(d) = 2$; otherwise, the set $(N(d) \cup N(e)) \setminus \{c_1, c_2, b_2\}$ contains two elements that lie in the difference $D(C_2') \setminus D(C_3')$ since $\{d\}$ is not a separating clique of G. Then the pair $(G[o, b_1, b_2, c_1, c_2, d, e], \{o, b_1, e\})$ is degenerate.

(i.2.2) Suppose that $\deg(c_2) = 3$ and there exists a vertex $e' \in N(c_2) \setminus \{e, c_1\}$. Then $e' \notin D(C_2')$. In this case, there is a vertex $x \in N(e) \setminus \{b_2, c_2\}$; otherwise, either $\{d\}$ is a separating clique in G or the degenerate pair $(G[o, b_1, b_2, c_1, c_2, d, e], \{o, b_1, c_2\})$ is formed. The vertices x and b_1 cannot be adjacent; otherwise, $\deg(d) = 2$ and the subgraph $H_3 = G[o, b_1, b_2, c_1, c_2, d, e, x]$ with H_3-separator (o, c_2, x) is $(3, \text{II})$-compressible. The result of the compression is obtained by removing the vertices b_1, b_2, and d from G.

The case when $N(e') \cap \{a_1, a_2, a_3, b_1, b_2\} = \{b_1\}$ will be treated in considering condition (iii). Therefore, we will assume that e' has a neighbor among the vertices a_1, a_2, and a_3. Define the numbers

$$i' = \max\{i \in \overline{1,3} \mid a_ie' \in E\}, \qquad i'' = \min\{i \in \overline{1,3} \mid a_ie' \in E\}.$$

If $e' = x$ and $i' = 1$ then $\deg(b) = 2$ (otherwise, $\{b\}$ is a separating clique in G) and the subgraph $G[o, a_1, b_1, b_2, c_1, c_2, e, d, x]$ is 2-compressible. If $e' = x$ and $i' > 1$ then

$$[a_{i'}, a_{i'+1}, a_{i'+2}, a_{i'+3}, a_{i'-1}, a_{i'-2}, a_{i'-3}, x, e],$$

where $a_0 = o$ and $a_{-1} = b_1$.

If $e' \neq x$ and $e'b_1 \notin E$ then $i' \neq i''$ in view of Cases 1–11. Then

$$\begin{aligned} &[c_2, c_1, o, b_1, e', a_{i'}, a_{i'+1}, e, x] && \text{if } a_{i'+1}x \notin E, \\ &[e, c_2, e', a_{i''}, x, a_{i'+1}, a_{i'+2}, b_2, b_1] && \text{if } a_{i'+1}x \in E,\ a_{i'+2}x \notin E, \\ &[e, c_2, e', a_{i''}, x, a_{i'+2}, a_{i'+3}, b_2, b_1] && \text{if } a_{i'+1}x \in E,\ a_{i'+2}x \in E. \end{aligned}$$

If $e' \neq x$ and $e'b_1 \in E$ then $i' \in \{1, 2\}$ by Cases 1–11. For $i' = 2$, we have

$$\begin{aligned} &[b_1, e', a_2, a_3, b_2, e, x, o, c_1] && \text{if } a_3x \notin E, \\ &[e, c_2, e', a_2, x, a_3, a_4, b_2, d] && \text{if } a_3x \in E,\ a_4x \notin E, \\ &[e, c_2, e', a_2, x, a_4, a_5, b_2, d] && \text{if } a_3x \in E,\ a_4x \in E. \end{aligned}$$

Note that, in the last two cases, $x \notin D(C_2')$ since $a_3x \in E$.

Finally, for $i' = 1$ we conclude that $\deg(b) = \deg(e) = 3$; otherwise, the 2-compressible subgraph $G[o, a_1, b_1, b_2, c_1, c_2, d, e, e', x]$ is formed. Consequently, $x \in D(C_2') \setminus D(C_3')$, and hence $\{e', a_1\}$ is a separating clique in G. This finishes the consideration of condition (i).

The edges c_1o, dc_1, b_2d and the cycles C_1'', C_2'', C_3'' satisfy the conditions of Lemma 5. Therefore, for an arbitrary planar embedding of G, we have $D(C_3'') \subset D(C_2'')$ or $D(C_1'') \subset D(C_2'')$. Consider $D(C_3'') \subset D(C_2'')$; the second inclusion is considered analogously. The vertices c_2, e, and x belong to the domain $D(o, c_1, d, b_2, b_1)$, and the vertices $a_1, \dots, a_{10}$ and y do not. There exists a vertex $b_1' \in N(b_1) \setminus \{o, b_2\}$ lying in the given domain (otherwise, $\{e\}$ is a separating clique in G), but then $[d, b_2, b_1, b_1', y, a_i, a_{i+1}, c_1, c_2]$ for some $i \in \{1, 2\}$.

(ii.2.2.2) Suppose that $ya_1 \notin E$ and $ya_2 \notin E$. If in addition $xy \notin E$ then $[c_1, o, a_1, a_2, c_2, e, x, d, y]$. If $xy \in E$ then the subgraph $H_6 = G[o, b_1, b_2, c_1, c_2, e, d, x, y]$ with H_6-separator (o, b_1, x, y) is $(4, \mathrm{II})$-compressible. The result of the compression is obtained by removing the vertices c_1, c_2, b_2, d, and e from G.

Pass to condition (iii). Recall that here $N(c_2) = \{e, e', c_1\}$, $N(e) \cap \{a_1, a_2, a_3, b_1, b_2\} = \{b_1\}$, and $N(e') \cap \{a_1, a_2, a_3, b_1, b_2\} = \{b_2\}$. We will assume that $ee' \notin E$; otherwise, a 2-compressible subgraph $G[o, b_1, b_2, c_1, c_2, d, e, e']$ is formed.

Note that, in every planar embedding of G, the vertices d and e lie to different sides of the cycle $(o, b_1, b_2, e', c_2, c_1)$. It is not hard to deduce from here that $\deg(d) = 2$ or $\deg(e) = 2$; otherwise,

$$[o, a_1, a_2, a_3, b_1, e', c_1, d, d'] \quad \text{or} \quad [o, a_1, a_2, a_3, b_1, e, e'', c_1, d],$$

where $d' \in N(d) \setminus \{c_1, b_2\}$ and $e'' \in N(e) \setminus \{b_1, c_2\}$.

For symmetry reasons, we can put $\deg(d) = 2$ and $\deg(e) = 3$. Then the subgraph

$$H_7 = G[o, b_1, b_2, c_1, c_2, d, e, e']$$

with H_7-separator (o, e, e') is $(3, \mathrm{II})$-compressible. The result of the compression is obtained by removing the vertices b_2, d, and c_1 from G.

(b) Suppose next that $N(e) \cap \{b_1, b_2\} = \varnothing$. In view of cases 1–11, the vertex e is adjacent exactly to two vertices in the set $\{a_1, a_2, a_3\}$. Observe also that $\deg(c_2) = 2$; otherwise, a possible element in $N(c_2) \setminus \{c_1, e\}$ having equal rights with e has exactly two neighbors in the set $\{a_1, a_2, a_3\}$. Consider all of the three possible variants of the intersection $N(e) \cap \{a_1, a_2, a_3\}$.

(b.1) In the case when $ea_1 \in E$ and $ea_2 \in E$, the degenerate pair $(G[o, a_1, a_2, c_1, c_2, e], \{a_2, o, c_1\})$ is formed in G.

(b.2) Assume that $ea_2 \in E$ and $ea_3 \in E$. Consider the two variants separately: $\deg(a_1) = 2$ and $\deg(a_1) = 3$.

(b.2.1) Let $\deg(a_1) = 2$. Then the subgraph $H_8 = G[o, a_1, a_2, a_3, c_1, c_2, e]$ with H_8-separator (a_3, o, c_1) is $(3, \mathrm{VI})$-compressible. The result of the compression is the minor G' of G obtained by contracting the edges a_2a_1, a_1o, c_2e, and c_1c_2. Consequently, G' belongs to the class $\mathcal{P}(3)$ and contains no induced triod $T_{3,3,2}$.

(b.2.2) Suppose that there exists a vertex $a_1' \in N(a_1) \setminus \{a_2, o\}$. Then $N(a_1') \cap \{b_2, d\} \neq \varnothing$; otherwise, $[c_1, c_2, e, a_3, o, a_1, a_1', d, b_2]$.

(b.2.2.1) If $a_1'b_2 \in E$ and $a_1'd \in E$ then we find a 2-compressible subgraph

$$G[o, a_1, a_2, a_3, b_1, b_2, d, c_1, c_2, e, a_1'].$$

(b.2.2.2) If $a_1'b_2 \in E$ and $a_1'd \notin E$ then we have a $(4, \mathrm{III})$-compressible subgraph

$$H_9 = G[o, a_1, a_2, a_3, b_1, b_2, d, c_1, c_2, e, a_1']$$

with H_9-separator (a_3, a_1', b_1, d). The result of the compression is obtained by removing the vertices c_1, c_2, and e from G.

(b.2.2.3) If $a_1'd \in E$, $a_1'b_2 \notin E$, then the degenerate pair

$$(G[o, a_1, a_2, a_3, b_1, b_2, d, c_1, c_2, e, a_1'], \{a_3, a_1', b_1, b_2\})$$

is formed.

(b.3) Suppose that $ea_1 \in E$ and $ea_3 \in E$. If in addition $\deg(a_2) = 2$ then the subgraph $H_{10} = G[o, a_1, a_2, a_3, c_1, c_2, e]$ with H_{10}-separator (a_3, o, c_1) is $(3, \mathrm{VI})$-compressible. The compression by contracting the edges a_2a_1, a_1o, c_2e, and c_1c_2 gives a minor G' of G. Consequently, G' belongs to the class $\mathcal{P}(3)$ and has no induced triod $T_{3,3,2}$.

If there exists a vertex $a_2' \in N(a_2) \setminus \{a_3, a_1\}$ then a_2' is adjacent to some vertex a_i, $i \in \overline{4,6}$; otherwise, $[a_3, a_4, a_5, a_6, e, c_2, c_1, a_2, a_2']$. The vertex a_2' is adjacent to one of the vertices b_1 and b_2; otherwise, $[a_1, a_2, a_2', a_i, o, b_1, b_2, e, c_2]$. Then

$$[a_2', a_i, a_{i+1}, a_{i+2}, b_1, b_2, d, a_2, a_1] \text{ if } a_2'b_1 \in E, \qquad [a_2', a_i, a_{i+1}, a_{i+2}, b_2, d, c_1, a_2, a_1] \text{ if } a_2'b_2 \in E.$$

Lemma 10 is proved. ∎

Lemma 11. *The vertex d cannot be adjacent to one of the vertices a_1, a_2, and a_3 or e simultaneously adjacent to the vertices b_1 and c_1.*

Proof. Suppose the contrary. If d is adjacent to b_1 then d is also adjacent to one of the vertices c_1, a_1, a_2, and a_3 by Lemma 9. If $da_i \in E$ for some $i \in \overline{1,3}$ then we obtain

$$[a_i, a_{i+1}, a_{i+2}, a_{i+3}, a_{i-1}, a_{i-2}, a_{i-3}, d, b_2],$$

where $a_0 = o$, $a_{-1} = c_1$, and $a_{-2} = c_2$.

Suppose that $db_1 \in E$ and $dc_1 \in E$. Then there exists a vertex $b_2' \in N(b_2) \setminus \{b_1, d\}$; otherwise, $\{b_1, d\}$ is a separating clique in G. If $b_2'c_2 \in E$ then the subgraph $H_1 = G[o, c_1, c_2, b_1, b_2, d, b_2']$ with H_1-separator (b_2', c_2, o) is $(3, \mathrm{III})$-compressible. The result of the compression is obtained by removing the vertices b_1, b_2, and d from G. If $b_2'c_2 \notin E$ then b_2' and d have equal rights as regards arguments and also $b_2'b_1 \notin E$.

Thus, throughout the sequel, put $db_1 \notin E$, and by Lemma 10 we infer that the vertex d has a neighbor among the vertices a_1, a_2, and a_3. Put $i' = \max\{i \mid da_i \in E,\ i \in \overline{1,3}\}$.

(a) Assume that there exists a vertex $e \in N(c_2) \setminus \{c_1\}$ nonadjacent to b_2. The vertices e and d have equal rights as regards arguments; therefore, we may assume that $ec_1 \notin E$ and e has a neighbor $a_{i''}$, where $i'' = \max\{i \mid ea_i \in E,\ i \in \overline{1,3}\}$. Assume for symmetry reasons that $i' \geq i''$. It is clear that $i'' \in \{1, 2\}$.

(a.1) Let $i'' = 1$. If $ea_4 \in E$ then $[a_4, a_5, a_6, a_7, e, c_2, c_1, a_3, a_2]$; if $ea_4 \notin E$ then necessarily $eb_1 \in E$; otherwise, $[a_1, a_2, a_3, a_4, o, b_1, b_2, e, c_2]$. If $\deg(c_2) = 2$ then the pair $(G[o, a_1, b_1, c_1, c_2, e], \{a_1, b_1, c_1\})$ is degenerate. If there exists a vertex $e' \in N(c_2) \setminus \{e, c_1\}$ then it is adjacent to one of the vertices b_2,

a_2, or a_3; otherwise, $[o, a_1, a_2, a_3, c_1, c_2, e', b_1, b_2]$ or $N(e') \cap \{c_1, b_1, b_2, a_1, a_2, a_3\} = \{c_1\}$. But this is impossible by Lemma 9 and due to the fact that the vertices d and e' have equal rights.

Thus, the subgraph $G[o, a_1, a_2, a_3, b_1, b_2, d, c_1, c_2, e, e']$ has $K_{3,3}$ as a minor. For obtaining it, we must contract the edge c_1c_2 and also contract the subgraph $G[a_2, a_3, b_2, d, e']$ into a vertex. Consequently, the graph G is not planar by Wagner's criterion.

(a.2) Let $i'' = 2$. In this case, $i' = 3$. If $da_i \in E$ for some $i \in \overline{5,8}$ then

$$[d, a_i, a_{i+1}, a_{i+2}, a_3, a_2, a_1, b_2, b_1].$$

If $ea_4 \in E$ then $[a_4, a_3, d, b_2, e, c_2, c_1, a_5, a_6]$, and if $ea_5 \in E$ then $[a_5, a_6, a_7, a_8, e, c_2, c_1, a_4, a_3]$. Therefore, necessarily $da_4 \in E$; otherwise, $[a_3, a_2, e, c_2, d, b_2, b_1, a_4, a_5]$. Consider the alternative: $ea_1 \notin E$ and $ea_1 \in E$.

(a.2.1) Under the conditions $ea_1 \notin E$, $ea_4 \notin E$, and $ea_5 \notin E$, find $eb_1 \in E$; otherwise, we have

$$[a_2, a_3, a_4, a_5, a_1, o, b_1, e, c_2].$$

If there exists a vertex x_1 belonging to the difference $(N(c_1) \cup N(c_2)) \setminus \{o, c_1, c_2, e\}$ then $x_1b_2 \notin E$; otherwise, the subgraph $G[o, a_1, a_2, a_3, b_1, b_2, c_1, c_2, d, e, x_1]$ contains $K_{3,3}$ as a minor. Then, necessarily, $x_1a_1 \in E$; otherwise,

$$[b_1, o, c_1, x_1, b_2, d, a_4, e, a_2] \quad \text{if } x_1c_1 \in E,$$
$$[o, a_1, a_2, a_3, c_1, c_2, x_1, b_1, b_2] \quad \text{if } x_1c_1 \notin E, \ \ x_1c_2 \in E.$$

Finally, either $[a_2, a_1, x_1, c_1, e, b_1, b_2, a_3, a_4]$ (if $x_1c_1 \in E$) or $[a_1, a_2, a_3, a_4, o, b_1, b_2, x_1, c_2]$ (if $x_1c_2 \in E$). Thus, $\deg(c_1) = \deg(c_2) = 2$, and we get a contradiction to Lemma 4: the graph G contains the induced 5-cycle (o, c_1, c_2, e, b_1).

(a.2.2) Let $ea_1 \in E$. If $\deg(c_2) = 2$ then we have the degenerate pair

$$(G[o, a_1, a_2, c_1, c_2, e], \{a_2, o, c_1\}).$$

Suppose that there exists $e' \in N(c_2) \setminus \{c_1, e\}$. By Lemmas 9 and 10, either $e'b_2 \in E$ or $e'b_1 \in E$ and $e'c_1 \in E$. In the latter case, G contains a 2-compressible subgraph

$$G' = G[o, a_1, a_2, a_3, a_4, b_1, b_2, c_1, c_2, e, e', d]$$

. Let $e'b_2 \in E$. The presence of $e'b_1$ in E also leads to a 2-compressible subgraph G', and so $e'b_1 \notin E$. In this case, $[a_5, a_4, a_3, a_2, a_6, a_7, a_8, e', b_2]$ if $e'a_5 \in E$ and $[b_2, d, a_4, a_5, e', c_2, e, b_1, o]$ otherwise.

(b) Suppose that there exists $e \in N(c_2) \setminus \{c_1\}$ adjacent to b_2. Moreover, assume that there is a vertex $e' \in N(c_2) \setminus \{c_1, e\}$. The vertices d and e' have equal rights; therefore, $e'c_1 \in E$, $e'b_1 \in E$ by Lemmas 9 and 10 and by what was proved in (a). Consequently, the subgraph $H_2 = G[o, b_1, b_2, c_1, c_2, e, e']$ with H_2-separator (e, b_2, o) is $(3, \mathrm{III})$-compressible. The result of the compression is obtained by removing the vertices c_1, c_2, and e' from G.

Thus, we have shown that $\deg(c_2) = 2$. The vertex e cannot be adjacent to a_i for $i \in \overline{3,7}$; otherwise, $[a_i, a_{i+1}, a_{i+2}, a_{i+3}, e, c_2, c_1, a_{i-1}, a_{i-2}]$. Also, $eb_1 \notin E$; otherwise, we have the degenerate pair $(G[o, c_1, c_2, b_1, b_2, e], \{b_1, o, c_1\})$. Consider the possible values of i'.

(b.1) For $i' = 1$ we get $[a_4, a_5, a_6, a_7, d, b_2, b_1, a_3, a_2]$ if $da_4 \in E$ and $[a_1, a_2, a_3, a_4, o, c_1, c_2, d, b_2]$ if $da_4 \notin E$.

(b.2) Let $i' = 3$.

If $da_1 \in E$ then $[d, a_3, a_4, a_5, a_1, o, c_1, b_2, e]$. If $da_2 \in E$ and $ea_1 \notin E$ then $[b_2, b_1, o, a_1, d, a_3, a_4, e, c_2]$. If $da_2 \in E$ and $ea_1 \in E$ then the pair

$$(G[o, a_1, a_2, a_3, b_1, b_2, c_1, c_2, e, d], \{b_1, c_1, a_3\})$$

is degenerate.

If $da_1 \notin E$, $da_2 \notin E$, and $ea_1 \notin E$ then

$$[b_2, b_1, o, a_1, d, a_4, a_5, e, c_2] \quad \text{if } da_4 \in E,$$
$$[b_2, b_1, o, a_1, d, a_3, a_4, e, c_2] \quad \text{if } da_4 \notin E.$$

Finally, if $da_2 \notin E$, $da_1 \notin E$, and $ea_1 \in E$ then find the induced 5-cycle (a_1, o, c_1, c_2, e) in G. By Lemma 4, c_1 has a neighbor $v \notin \{o, c_2\}$. Since G is planar, $va_2 \notin E$ and $va_4 \notin E$ (otherwise, there is the minor $K_{3,3}$ appear). Then

$$\begin{aligned} &[e, c_2, c_1, v, a_1, a_2, a_3, b_2, b_1], \quad \text{if } vb_1 \notin E,\\ &[b_2, b_1, v, c_1, d, a_3, a_4, e, a_1], \quad \text{if } vb_1 \in E,\ da_4 \notin E,\\ &[b_2, b_1, v, c_1, d, a_4, a_5, e, a_1], \quad \text{if } vb_1 \in E,\ da_4 \in E. \end{aligned}$$

(b.3) Let $i' = 2$. In this case, if $ea_1 \notin E$ then

$$[d, a_2, a_1, o, a_4, a_5, a_6, b_2, e] \quad \text{for } da_4 \in E, \qquad [a_2, a_1, o, c_1, d, b_2, e, a_3, a_4] \quad \text{for } da_4 \notin E.$$

If $ea_1 \in E$ then (a_1, o, c_1, c_2, e) is an induced 5-cycle in G. By Lemma 4, c_1 has a neighbor $u \notin \{o, c_2\}$, and, by the planarity of G, we get $ua_3 \notin E$ and $ua_4 \notin E$. Then

$$\begin{aligned} &[e, c_2, c_1, u, a_1, a_2, a_3, b_2, b_1] \quad \text{if } ub_1 \notin E,\\ &[b_2, b_1, u, c_1, d, a_3, a_4, e, a_1] \quad \text{if } ub_1 \in E,\ da_4 \notin E,\\ &[b_2, b_1, u, c_1, d, a_4, a_5, e, a_1] \quad \text{if } ub_1 \in E,\ da_4 \in E. \end{aligned}$$

Pass to the case $da_1 \in E$. If $\deg(b_1) = \deg(c_1) = 2$ then the subgraph $H_3 = G[o, b_1, b_2, c_1, c_2, e]$ with H_3-separator (e, b_2, o) is $(3, \mathrm{III})$-compressible. The result of the compression is obtained by removing the vertices c_1 and c_2 from G. Thus, either $N(b_1) \setminus \{o, b_2\} \neq \varnothing$ or $\deg(b_2) = 2$ and $N(c_1) \setminus \{o, c_2\} \neq \varnothing$.

(b.3.1) Suppose that there exists $x_2 \in N(b_1) \setminus \{o, b_2\}$. If $x_2c_1 \notin E$ and $x_2e \notin E$ then

$$\begin{aligned} &[b_2, b_1, x_2, a_3, e, c_2, c_1, d, a_1] \quad \text{for } x_2a_3 \in E,\\ &[b_2, e, c_2, c_1, d, a_2, a_3, b_1, x_2] \quad \text{for } x_2a_3 \notin E. \end{aligned}$$

If $x_2c_1 \in E$ and $\deg(x_2) = 2$ then the subgraph $H_4 = G[o, b_1, b_2, c_1, c_2, e, x_2]$ with H_4-separator (o, b_2, e) is $(3, \mathrm{VII})$-compressible. The result of compression is obtained by removing b_1 and x_2 from G.

If $x_2c_1 \in E$ and there exists $y \in N(x_2) \setminus \{b_1, c_1\}$ then $y \neq a_3$ and $ya_3 \notin E$; otherwise, the subgraph $G[o, a_1, a_2, a_3, b_1, b_2, c_1, c_2, e, d, x_2, y]$ contains $K_{3,3}$ as a minor. However, $y \neq e$; otherwise, we have the 2-compressible subgraph $G[o, b_1, b_2, c_1, c_2, e, x_2]$. The vertices y and e are adjacent since, otherwise, $[b_2, b_1, x_2, y, d, a_2, a_3, e, c_2]$. Note that (c_1, c_2, e, y, x_2) is an induced 5-cycle in G; therefore, by Lemma 4, y is also adjacent to some vertex $z \in N(y) \setminus \{x_2, e\}$. Then $[e, c_2, c_1, o, b_2, d, a_2, y, z]$.

If $x_2e \in E$ and $\deg(x_2) = 2$ then the subgraph $H_5 = G[o, b_1, b_2, c_1, c_2, e, x_2]$ with H_5-separator (o, b_2, c_1) is $(3, \mathrm{I})$-compressible. The result of the compression is obtained by removing the vertices b_1 and x_2 from G.

Finally, if for $x_2e \in E$ there exists a vertex $y' \in N(x_2) \setminus \{b_1, e\}$ then $y' \neq a_3$ and $y'a_3 \notin E$; otherwise, G is not planar. Then

$$\begin{aligned} &[o, b_1, b_2, e, a_1, a_2, a_3, c_1, y'] \quad \text{if } y'c_1 \in E,\\ &[e, c_2, c_1, o, b_2, d, a_2, x_2, y'] \quad \text{if } y'c_1 \notin E. \end{aligned}$$

(b.3.2) Suppose that $\deg(b_1) = 2$ and there exists a vertex $x_3 \in N(c_1) \setminus \{o, c_2\}$. The vertex x_3 is necessary adjacent to e or a_3 since otherwise $[o, a_1, a_2, a_3, b_1, b_2, e, c_1, x_3]$. If $x_3e \in E$ then we obtain a $(3, \mathrm{II})$-compressible subgraph $H_6 = G[o, b_1, b_2, c_1, c_2, e, x_3]$ with H_6-separator (o, b_2, x_3). The result of the compression is obtained by removing the vertices c_1 and c_2 from G. If $x_3a_3 \in E$ and $x_3e \notin E$ then

$$\begin{aligned} &[c_1, x_3, a_4, a_5, c_2, e, b_2, o, a_1] \quad \text{for } x_3a_4 \in E,\\ &[c_1, x_3, a_3, a_4, c_2, e, b_2, o, a_1] \quad \text{for } x_3a_4 \notin E. \end{aligned}$$

Lemma 11 is proved. $\square$

5. THE MAIN RESULT

Theorem. *The class* $\mathcal{P}(3) \cap \mathrm{Free}(T_{3,3,2})$ *is IS-simple.*

Proof. It follows from Lemmas 8–11 that every irreducible graph belongs to $\mathrm{Free}(T_{2,2,10})$. Hence, for the graphs in $\mathcal{P}(3) \cap \mathrm{Free}(T_{3,3,2})$, the problem is polynomially reduced to the same problem for the graphs in $\mathcal{P}(3) \cap \mathrm{Free}(T_{2,2,10})$. The class $\mathcal{P}(3) \cap \mathrm{Free}(T_{2,2,10})$ is IS-simple [4]. Therefore, the graph class $\mathcal{P}(3) \cap \mathrm{Free}(T_{3,3,2})$ is also IS-simple. Theorem 1 is proved. $\square$

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