ALGEBRAIC GROUPS WHOSE ORBIT CLOSURES CONTAIN ONLY FINITELY MANY ORBITS

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ABSTRACT. We classify all connected affine algebraic groups G such that there are only finitely many G-orbits in every algebraic G-variety containing a dense open G-orbit. We also prove that G enjoys this property if and only if every irreducible algebraic G-variety X is modality-regular, i.e., the modality of X (in the sense of V. Arnol'd) equals to that of a family which is open in X.

1. Introduction. The phenomenon of finiteness of the sets of orbits in orbit closures of an algebraic group G, that arises under certain conditions on G and actions, has been known for a long time and plays an essential role in several mathematical theories. For instance, if G is an affine torus, then every orbit closure contains only finitely many orbits, which is a key fact of the theory of toric embeddings [TE73] (see, e.g., also [Ful93]). Historically, the next example, generalizing the previous one, is the class of all equivariant embeddings of a fixed homogeneous space $\mathcal{O} = G/H$ of a connected reductive group G: every such embedding contains only finitely many orbits if and only if $\mathcal{O}$ is spherical [Akh85]. This fact is a key ingredient of the theory of spherical embeddings [LV83] (see, e.g., also [Tim11]). Historically, the first manifestation of the latter example was the case of parabolic H: then every equivariant embedding of $\mathcal{O}$ coincides with $\mathcal{O}$. One more example is obtained if G is a connected unipotent group: in this case, every quasiaffine orbit closure of G coincides with this orbit [Ros61, Thm. 2], which is an important fact of the algebraic transformation group theory.

In this paper, we consider the absolute case, i.e., that, in which no conditions on G and actions are imposed. Namely, we explore the problem of classifying the connected affine algebraic groups G such that every orbit closure of G contains only finitely many orbits. The answer we found turned out to be rather unexpected for us: we prove that, apart from the aforementioned class of affine tori, there is one more class of groups enjoying this property, and only this class, namely, that of all products of affine tori and $\mathbf{G}_a$.

The finiteness of the set of orbits of an algebraic group action can be equivalently expressed in terms of the notion of the modality of this action that goes back to Arnol'd's works on the theory of singularities [Arn75]. The modality is the maximal number of parameters, on which a family of orbits may depend. The finiteness of the set of orbits is equivalent to the condition that the modality is 0. It turns out that the answer to the above problem can be equivalently reformulated in terms of modality. Namely, we prove that every orbit closure of G contains only finitely many orbits if and only if every G-variety X is modality-regular, i.e., its modality equals to that of a family which is open in X.

Our main result, Theorem 2, is formulated in Section 4 and proved in Section 7. Section 3 contains the materials about the modality necessary for formulating Theorem 2. In Sections 5 and 6 are collected the auxiliary results on property (F) from the formulation of Theorem 2 and on the modality, which we use in the proof of this theorem. In Section 2 are collected the conventions, notation, and terminology.

2. Conventions, notation, and terminology. We fix an algebraically closed field k. In what follows, as in [Bor91], [Spr98], [PV94], the word *variety* means an algebraic variety over k in the sense of [Ser55] (so an algebraic group means an algebraic group over k). We assume that $\operatorname{char} k = 0$ as we use the classification of commutative unipotent algebraic groups valid only in characteristic 0. We use freely the standard notation and conventions of [Bor91], [Spr98], [PV94], where also the proofs of unreferenced claims and/or the relevant references can be found.

If all irreducible components of a variety X have the same dimension, then X is called *equidimensional*.

Below all actions of algebraic groups on varieties are algebraic (morphic).

If an algebraic group G acts on a variety X, we say that X is a *G-variety*.

If an algebraic group is isomorphic to $\mathbf{G}_m^d$ for some d, we call it a *torus*.

3. Modality. Let H be a connected algebraic group. Any irreducible H-variety F such that all H-orbits in F have the same dimension d is called a *family* of H-orbits depending on

$$\operatorname{mod}(H : F) := \dim F - d \tag{1}$$

parameters; the integer $\operatorname{mod}(H : F)$ is called the *modality* of F. If $F \dashrightarrow F/\!\!/H$ is a rational geometric quotient of this action (it exists by the Rosenlicht theorem [PV94, Thm. 4.4]), then

$$\operatorname{mod}(H : F) = \dim F/\!\!/H = \operatorname{tr}\deg_k k(F)^H \tag{2}$$

and $F/\!\!/H$ may be informally viewed as the variety parametrizing typical H-orbits in F.

Given an H-variety Y, we denote by $\mathscr{F}(Y)$ the set of all locally closed H-stable subsets of Y, which are the families. The integer

$$\operatorname{mod}(H : Y) := \max_{F \in \mathscr{F}(Y)} \operatorname{mod}(H : F), \tag{3}$$

is then called the *modality* of the H-variety Y.

The definition of modality implies that (3) still holds if $\mathscr{F}(Y)$ is replaced by the set of all maximal (with respect to inclusion) families in Y, i.e., by the *sheets* of Y [PV94, Sect. 6.10]. Recall that there are only finitely many sheets of Y. If Y is irreducible, then $Y^{\rm reg}$ is a sheet, called *regular*, which is open and dense in Y. By (2),

$$\operatorname{mod}(H : Y^{\rm reg}) = \operatorname{tr\,deg}_k k(Y)^H. \tag{4}$$

Similarly, (3) still holds if $\mathscr{F}(Y)$ is replaced by the set of all H-stable irreducible locally closed (or closed) subsets of Y, and $\operatorname{mod}(H : F)$ by $\operatorname{tr\,deg}_k k(F)^H$.

Let G be a (not necessarily connected) algebraic group and let X be a G-variety. Then by definition,

$$\operatorname{mod}(G : X) := \operatorname{mod}(G^0 : X),$$

where G^0 is the identity component of Ge.

It readily follows from the definition that if Z is a locally closed G-stable subset of X, then

$$\operatorname{mod}(G : X) \geqslant \operatorname{mod}(G : Z).$$

Recall that, for every integer s, the set $\{x \in X \mid \dim G\cdot x \leqslant s\}$ is closed in X. Whence, for every locally closed (not necessarily G-stable) subset Z in X,

$$Z^{\rm reg} := \{z \in Z \mid \dim G\cdot z \geqslant \dim G\cdot x \text{ for every } x \in Z\} \tag{5}$$

is a nonempty open subset of Z.

The aforesaid shows that $\operatorname{mod}(G : X) = 0$ if and only if the set of all G-orbits in X is finite.

The existence of regular sheets leads to defining the following distinguished class of algebraic group actions:

Definition 1. An irreducible G-variety X and the action of G on X are called *modality-regular* if $\operatorname{mod}(G : X) = \operatorname{mod}(G : X^{\rm reg})$.

4. Main result: formulation.

Theorem 2. *For any connected affine algebraic group G, the following properties are equivalent:*

- (F) *there are only finitely many G-orbits in every irreducible G-variety containing a dense open G-orbit;*
- (M) *every irreducible G-variety is modality-regular;*
- (G) *G is either a torus or a product of a torus and a group isomorphic to $\mathbf{G}_a$.*

Remark. For any connected affine algebraic group G, the following properties are equivalent:

- (i) G is a product of a torus and a group isomorphic to $\mathbf{G}_a$;
- (ii) G is nilpotent and its unipotent radical is one-dimensional.

5. Auxiliary results: property (F). This section contains some auxiliary results on property (F) from the formulation of Theorem 2 that will be used in its proof. First we explore its behaviour under passing to a subgroup and a quotient group. Then we explore it for two-dimensional connected solvable, and, in conclusion, for semisimple affine algebraic groups.

Lemma 3. *Let G be a connected affine algebraic group and let H be its closed subgroup. If G enjoys property* (F)*, then*

(a) *H enjoys property* (F)*;*
(b) *G/H, for normal H, enjoys property* (F)*.*

Proof. (a) Arguing on the contrary, suppose there exists an irreducible H-variety Y with infinitely many H-orbits, one of which, say, $\mathcal{O}$, is open in Y. Since the action canonically lifts to the normalization [Ses63], we may (and shall) assume that Y is normal. Then, by [Sum74, Lemma 8], we have $Y = \bigcup_{i\in I} U_i$, where each U_i is an H-stable quasi-projective open subset of Y. As Y is irreducible, each U_i contains $\mathcal{O}$. Since in the Zarisky topology any open covering contains a finite subcovering, there is $i_0 \in I$ such that U_{i_0} contains infinitely many H-orbits. Therefore replacing Y by U_{i_0}, we may (and shall) assume that Y is quasi-projective. Then, by [Ser58, 3.2] (see also [PV94, Thm. 4.9]), the homogeneous fiber space $X := G \times^H Y$ over G/H with the fiber Y is an algebraic variety. Since for the action of H on Y there are infinitely many orbits one of which is open, the natural action of G on X enjoys these properties as well; see [PV94, Thm. 4.9]. This contradicts the condition that G enjoys property (F), thereby proving (a).

(b) Assume, again arguing on the contrary, that there is an irreducible algebraic G/H-variety X with infinitely many G/H-orbits, one of which is open in X. Since the canonical homomorphism $G \to G/H$ determines an action of G on X whose orbits coincide with the G/H-orbits, this contradicts the condition that G enjoys property (F), thereby proving (b). □

We now consider the two-dimensional connected solvable affine algebraic groups. Let S be such a group. Then $S = T \ltimes S_u$, where T is a maximal torus and S_u is the unipotent radical of S. There are only the following possibilities:

(S1) S_u is trivial. Then S is a two-dimensional torus.

(S2) T is trivial. Then S isomorphic to $\mathbf{G}_a \times \mathbf{G}_a$.

Indeed, as S is unipotent, there is a one-dimensional closed subgroup C lying in its center; see [Spr98, 6.3.4]. If $g \in S \setminus C$, from $\dim S = 2$ we infer that S is the closure of the subgroup generated by C and g. As this subgroup is commutative, S is commutative as well. Since $\operatorname{char} k = 0$, this entails the claim; see [Spr98, 3.4.7].

(S3) $\dim T = \dim S_u = 1$, i.e., T and S_u are isomorphic to respectively to $\mathbf{G}_m$ and $\mathbf{G}_a$. Then there is $n \in \mathbf{Z}$ such that S is isomorphic to the group

$S(n) := \mathbf{G}_m \ltimes \mathbf{G}_a$, in which the group operation is defined by the formula

$$(t_1, u_1)(t_2, u_2) := (t_1 t_2, t_2^n u_1 + u_2). \tag{6}$$

Indeed, as $\dim S_u = 1$, there is an isomorphism $\theta\colon \mathbf{G}_a \to S_u$. For any $t \in T$, the map $S_u \to S_u$, $u \mapsto tut^{-1}$, is an automorphism; whence there is a character $\chi\colon T \to \mathbf{G}_m$ such that $t\theta(u)t^{-1} = \theta(\chi(t)u)$ for all $u \in \mathbf{G}_a$, $t \in T$; whence the claim.

The group $S(n)$ is commutative if and only if $n = 0$.

Proposition 4. *The group $\mathbf{G}_a^d$ does not enjoy property* (F) *for every $d \geqslant 2$.*

Proof. The action action of $\mathbf{G}_a^d$ on itself by left translations is its action on the affine space $\mathbf{A}^d$ defined by the formula

$$u \cdot a := (a_1 + u_1, \dots, a_d + u_d)$$
$$\text{for } u = (u_1, \dots, u_d) \in \mathbf{G}_a^d,\ \ a = (a_1, \dots, a_d) \in \mathbf{A}^d.$$

We indentify $\mathbf{A}^d$ with the affine chart

$$\mathbf{P}_d^d := \{(p_0 : \dots : p_d) \in \mathbf{P}^d \mid p_d \neq 0\}$$

of the projective space $\mathbf{P}^d$. Then the following formula extends this action up to the action of $\mathbf{G}_a^d$ on $\mathbf{P}^d$:

$$u \cdot p := (p_0 + u_1 p_d : \dots : p_{d-1} + u_d p_d : p_d)$$
$$\text{for } u = (u_1, \dots, u_d) \in \mathbf{G}_a^d,\ p = (p_0 : \dots : p_d) \in \mathbf{P}^d.$$

For the latter action, $\mathbf{P}_d^d$ is an open orbit, and the hyperplane $\mathbf{P}^d \setminus \mathbf{P}_d^d$ is pointwise fixed. As $\dim \mathbf{P}^d \setminus \mathbf{P}_d^d > 0$ for $d \geqslant 2$, this completes the proof. □

Proposition 5. *Every group $S(n)$ for $n \neq 0$ does not enjoy property* (F)*.*

Proof. It follows from (6) that

$$S(n) \to \mathrm{GL}_2, \quad (t, u) \mapsto \begin{pmatrix} t^n & 0 \\ u & 1 \end{pmatrix},$$

is a representation of $S(n)$. It determines the following linear action of $S(n)$ on $\mathbf{A}^2$:

$$g \cdot a := (a_1 t^n, a_1 u + a_2), \ \text{ where } g = (t, u) \in S(n) \text{ and } a = (a_1, a_2) \in \mathbf{A}^2. \tag{7}$$

From (7) and $n \neq 0$ we immediately infer that the fixed point set of this action is the line $\ell := \{(a_1, a_2) \in \mathbf{A}^2 \mid a_1 = 0\}$ whose complement $\mathbf{A}^2 \setminus \ell$ is a single orbit. This completes the proof. □

In conclusion, we consider semisimple algebraic groups.

Proposition 6. *Every nontrivial connected semisimple algebraic group G does not enjoy property* (F)*.*

Proof. Let α be a root of G with respect to a maximal torus and let G_α be the centralizer of the torus $(\ker\alpha)^0$ in G. The commutator group (G_α, G_α) is isomorphic to either SL_2 or PSL_2 (see, e.g., [Spr98, 7.1.2, 8.1.4]). Correspondingly, the Borel subgroups of (G_α, G_α) are isomorphic to either $S(1)$ or $S(2)$. Hence, by Proposition 5, they do not enjoy property (F). The claim now follows from Lemma 3. □

6. Auxiliary results: modality. The following lemma helps to practically determine the modality and will be used in the proof of Theorem 2.

Lemma 7. *Let G be an algebraic group, let X be a G-variety, and let $\{C_i\}_{i\in I}$ be a collection of the subsets of X such that*

(i) *I is finite;*
(ii) *$\bigcup_{i\in I} C_i = X$;*
(iii) *the closure $\overline{C_i}$ of C_i in X is irreducible for every $i \in I$;*
(iv) *every C_i is G-stable;*
(v) *all G-orbits in C_i have the same dimension d_i for every $i \in I$.*

Then the following hold:

(a) $\mathrm{mod}(G : X) = \max_{i\in I}\big(\dim \overline{C_i} - d_i\big)$*;*
(b) *if X is irreducible, then $X = \overline{C_{i_0}}$ for some i_0, and* $\mathrm{mod}(G : X^{\mathrm{reg}}) = \dim X - d_{i_0}$.

Proof. By (iii), we have a family $\overline{C_i}^{\mathrm{reg}}$, and (v) implies $C_i \subseteq \overline{C_i}^{\mathrm{reg}}$. Whence

$$\mathrm{mod}(G : \overline{C_i}^{\mathrm{reg}}) = \dim \overline{C_i} - d_i. \tag{8}$$

From (3) and (8), we infer that $\mathrm{mod}(G : X) \geqslant \max\limits_{i\in I}(\dim \overline{C}_i - d_i)$. To prove the opposite inequality let $Z \in \mathscr{F}(X)$ be a family of s-dimensional G-orbits such that $\mathrm{mod}(G : X) = \dim Z - s$ and let $J := \{i \in I \mid Z \cap C_i \neq \varnothing\}$. By (ii), we have $Z = \bigcup_{j\in J}(Z \cap \overline{C_j})$. Since Z is irreducible and, by (i), J is finite, there is $j_0 \in J$ such that $Z \subseteq \overline{C_{j_0}}$. As $Z \cap C_{j_0} \neq \varnothing$, we have $s = d_{j_0}$. Therefore, $\mathrm{mod}(G : X) = \dim Z - s \leqslant \dim \overline{C_{j_0}} - d_{j_0}$. This proves (a).

By (ii), $\bigcup_{i\in I} \overline{C_i} = X$. If X is irreducible, then, in view of (i), this equality implies the existence of i_0 such that $X = \overline{C_{i_0}}$. This and (8) prove (b). □

Lemma 8. *Let G be a connected algebraic group and let $\varphi\colon X \dashrightarrow Y$ be a rational G-equivariant map of the irreducible G-varieties.*

(i) *If φ is dominant, then* $\mathrm{mod}(G : X^{\mathrm{reg}}) \geqslant \mathrm{mod}(G : Y^{\mathrm{reg}})$. *If, moreover, $\dim X = \dim Y$, then* $\mathrm{mod}(G : X^{\mathrm{reg}}) = \mathrm{mod}(G : Y^{\mathrm{reg}})$.
(ii) *If φ is a surjective morphism, then* $\mathrm{mod}(G : X) \geqslant \mathrm{mod}(G : Y)$.

Proof. The inequality in (i) follows from (4) because φ determines a G-equivariant field embedding $\varphi^*\colon k(Y) \hookrightarrow k(X)$.

Assume that $\dim X = \dim Y$. Then, by the fiber dimension theorem, the fibers of φ over the points of an open subset of Y are finite. Whence, for every point x of an open subset of X, we have $\dim G\cdot x = \dim G\cdot\varphi(x)$. This implies that $m_X := \max_{x\in X} \dim G \cdot x = m_Y := \max_{y\in Y} \dim G \cdot y$. From this equality and (1) we infer $\mathrm{mod}(G : X^{\mathrm{reg}}) = \dim X - m_X = \dim Y - m_Y = \mathrm{mod}(G : Y^{\mathrm{reg}})$. This proves (i).

To prove (ii), consider a family F in Y such that

$$\mathrm{mod}(G : Y) = \mathrm{mod}(G : F). \tag{9}$$

If φ is a surjective morphism, then $\varphi\colon \varphi^{-1}(F) \to F$ is a surjective morphism. As F is irreducible, there is an irreducible component $\widetilde{F}$ of $\varphi^{-1}(F)$ such that $\varphi\colon \widetilde{F} \to F$ is a surjective morphism. Since φ is G-equivariant and G is connected, $\widetilde{F}$ is G-stable, so the latter morphism is G-equivariant. Hence $\mathrm{mod}(G : X) \geqslant \mathrm{mod}(G : \widetilde{F}^{\mathrm{reg}}) \geqslant \mathrm{mod}(G : F^{\mathrm{reg}}) = \mathrm{mod}(G : F) = \mathrm{mod}(G : Y)$. (the first inequality follows from (3), and the second from (i); the first equality follows from $F = F^{\mathrm{reg}}$, and the second from (9)) This proves (ii). □

Recall [Ses63] that any action of an algebraic group G on an irreducible algebraic variety X canonically lifts to the normalization $X^{(\mathrm{n})} \to X$ making the latter G-equivariant. Lemma 8(i) and (3) entail

Corollary 9.

(i) $\mathrm{mod}(G : X^{(\mathrm{n})}) = \mathrm{mod}(G : X)$;
(ii) *the action of G on X is modality-regular if and only if that on $X^{(\mathrm{n})}$ is.*

Lemma 10. *For any action of a torus T on an irreducible variety Y, the following properties hold:*

(i) *the stabilizer of any point of an open subset of Y coincides with the kernel of this action;*
(ii) *this action is modality-regular.*

Proof. First, we may (and shall) assume that T acts of Y faithfully. Next, by Corollary 9, replacing Y by $Y^{(\mathrm{n})}$, we may (and shall) assume that Y is normal. By [Sum74, Cor. 2, p. 8], then Y is covered by T-stable affine open subsets. Whence, moreover, we may (and shall) assume that Y is affine.

(i) As Y is affine, we may (and shall) assume that Y is a closed T-stable subset of a finite-dimensional algebraic T-module and Y does not lie in a proper T-submodule of V (see [PV94, Thm. 1.5]). The action of G on V is faithful because that on Y is. As T is a torus, V is the direct sum of the T-weight subspaces. Let U be the complement in V to the union of these subspaces. The stabilizer of any point of U coincides with the kernel of the action on V, hence is trivial. As, by construction, $Y \cap U \neq \varnothing$, this proves (i).

(ii) Given Definition 1, the proof of [Vin86, Prop. 1] can be viewed as that of (ii). For the sake of completeness, below is a somewhat different argument.

We may (and shall) assume that T acts on Y faithfully; by (i), we then have $\mathrm{mod}(T : Y^{\mathrm{reg}}) = \dim Y - \dim T$. As in the proof of (i), we may (and shall) assume that Y is affine. Let S be a sheet in the T-variety Y, and let T_0 be the kernel of the action of T on S. Then $\mathrm{mod}(T : S) = \dim S - (\dim T - \dim T_0)$.

Let $\pi\colon Y \to Y/\!\!/T_0 =: \operatorname{Spec} k[Y]^{T_0}$ be the categorical quotient for the action of T_0 on Y. As T_0 act on Y faithfully, (i) yields $\dim Y/\!\!/T_0 \leqslant \dim Y - \dim T_0$.

Since S is pointwise fixed by T_0 and π separates closed T_0-orbits, we have $\dim \pi(S) = \dim S$; whence $\dim S \leqslant \dim Y/\!\!/T_0 \leqslant \dim Y - \dim T_0$. Combining this information, we complete the proof: $\operatorname{mod}(T : S) = \dim S - \dim T + \dim T_0 \leqslant \dim Y - \dim T_0 - \dim T + \dim T_0 = \operatorname{mod}(T : Y^{\rm reg})$. □

7. Main result: proof.

We shall prove the implications (M)⇒(F)⇒(G)⇒(M).

1. The implication (M)⇒(F) is clear.

2. We now turn to the proof of the implication (F)⇒(G).

Let the group G enjoys property (F). Let R be the radical of G. Since the group G/R is semisimple, our assumption, Lemma 3, and Proposition 6 entail that G/R is trivial, i.e., G is solvable. Whence $G = T \ltimes U$, where T is a maximal torus and U is the unipotent radical of G. We should show that either U is trivial or U is isomorphic to $\mathbf{G}_a$ and G is commutative. Arguing on the contrary, we suppose that this is not so.

Then U is a nontrivial unipotent group. Hence there exists a chain $\{e\} = U_1 \subsetneqq U_2 \subsetneqq \cdots \subsetneqq U_d = U$ of closed connected subgroups, normal in G, such that $d \geqslant 2$, and the successive quotients are one-dimensional; see [Bor91, 10.6].

We claim that $d = 2$. Indeed, if this is not the case, the above chain contains U_3. Since $\dim U_3 = 2$, arguing as case (S3) of Section 5, we obtain that U_3 is isomorphic to $\mathbf{G}_a \times \mathbf{G}_a$. By Proposition 4 and Lemma 3, this is impossible since G enjoys property (F). Thus $d = 2$; whence U is isomorphic to $\mathbf{G}_a$.

Next, the assumption that G is not commutative means that the conjugating action of T on U is nontrivial. As T is generated by its one-dimensional subtori, there is such a subtorus T' not lying in the kernel of this action. Then $T'U$ is a noncommutative closed connected two-dimensional subgroup of G; see [Bor91, 2.2]. Hence it is isomorphic to $S(n)$ for some $n \neq 0$; see case (S3) in Section 5. By Proposition 5 and Lemma 3, this is impossible since G enjoys property (F). This contradiction proves the implication (F)⇒(G).

3. Now we turn to the proof of the last implication (G)⇒(M).

Assume that (G) holds and G acts on an irreducible variety X. We should show that this action is modality-regular. In view of Lemma 10(ii), we should consider only the case, where G is the product of two subgroups:

$$G = T \times U, \quad T \text{ is a torus}, U \text{ is isomorphic to } \mathbf{G}_a. \tag{10}$$

Below, exploring the actions of the subgroups of G on X, we always mean the actions obtained by restricting the given action of G on X.

We may (and shall) assume that G acts on X faithfully. In view of Corollary 9(ii), replacing X by $X^{(\rm n)}$, we also may (and shall) assume that X is normal.

Notice that since the elements of T (respectively, U) are semisimple (unipotent), and the G-stabilizers of points of X, being closed in G, contain the

Jordan decomposition components of their elements, we have, in view of (10), for these stabilizers:

$$G_x = T_x \times U_x \quad \text{for every } x \in X. \tag{11}$$

As G acts on X faithfully, from (11), (10), and Lemma 10(i) we infer that

$$G_x \text{ is finite for every } x \in X^{\mathrm{reg}}. \tag{12}$$

Let S be a sheet of the action of T on X. As T and U commute and both are connected, S is U-stable and every sheet C of the action of U on S is T-stable, hence G-stable. Consider the set of all C's, obtained in this way when S runs over all sheets of the action of G on X. This set is finite; we fix a numbering of its elements: $C_1, \ldots, C_n$. The construction and $\dim U = 1$ yield the following:

(C1) $X = C_1 \cup \ldots \cup C_n$;
(C2) every C_i is a locally closed irreducible G-stable subset of X;
(C3) all T-orbits in C_i have the same dimension d_i for every i;
(C4) for every i, either $C_i^U = C_i$ or $\dim U \cdot x = 1$ for all $x \in C_i$.

The construction implies that X^{reg} is one of these subsets; we assume that

$$X^{\mathrm{reg}} = C_1. \tag{13}$$

In view of (C3) and (12), we have

$$\begin{aligned} \operatorname{mod}(T : C_i) &= \dim C_i - d_i \ \text{ for every } i, \\ d_1 &= \dim T. \end{aligned} \tag{14}$$

By Lemma 10(ii), the action of T on X is modality-regular, so (14) yields

$$\dim X - \dim T \geqslant \dim C_i - d_i \ \text{ for every } i. \tag{15}$$

From (11), (C3), (C4), we deduce that

$$\operatorname{mod}(G : C_i) = \begin{cases} \dim C_i - d_i & \text{if } C_i^U = C_i \ , \\ \dim C_i - d_i - 1 & \text{if } C_i^U = \varnothing. \end{cases} \tag{16}$$

In particular, (13), (14), (16), and the faithfulness of the action of G on X yield

$$\operatorname{mod}(G : X^{\mathrm{reg}}) = \dim X - \dim T - 1. \tag{17}$$

Arguing on the contrary, we now suppose that the action of G on X is not modality-regular, i.e.,

$$\operatorname{mod}(G : X) > \operatorname{mod}(G : X^{\mathrm{reg}}). \tag{18}$$

Then, as a first step, we shall find a certain C_{i_0} that has some special properties. The next step will be analysing these properties which eventually will lead to a sought-for contradiction.

Namely, by (18) and Lemma 7, there is i_0 such that

$$\operatorname{mod}(G : X^{\mathrm{reg}}) < \operatorname{mod}(G : C_{i_0}). \tag{19}$$

Combining (15), (16), (17), (19), we obtain

$$\begin{aligned}\dim C_{i_0} - d_{i_0} - 1 &\overset{(15)}{\leqslant} \dim X - \dim T - 1\\ &\overset{(17)}{=\!=} \operatorname{mod}(G : X^{\mathrm{reg}}) \overset{(19)}{<} \operatorname{mod}(G : C_{i_0})\\ &\overset{(16)}{=\!=} \begin{cases}\dim C_{i_0} - d_{i_0} & \text{if } C_{i_0}^U = C_{i_0},\\ \dim C_{i_0} - d_{i_0} - 1 & \text{if } C_{i_0}^U = \varnothing.\end{cases}\end{aligned} \tag{20}$$

In turn, from (20) we infer the following:

$$C_{i_0}^U = C_{i_0}, \tag{21}$$

$$\dim C_{i_0} - d_{i_0} = \dim X - \dim T. \tag{22}$$

Denote by T_{i_0} be the identity component of the kernel of the action of T on C_{i_0} and consider in G the closed subgroup

$$H := T_{i_0} \times U. \tag{23}$$

By (C3) and Lemma 10(i), we have

$$\dim T_{i_0} = \dim T - d_{i_0}, \tag{24}$$

$$\dim H = \dim T - d_{i_0} + 1. \tag{25}$$

From (21) and the definitions of T_{i_0} and H we infer that

$$C_{i_0}^H = C_{i_0}. \tag{26}$$

By [Sum74, Cor. 2, p. 8], as X is normal, Y is covered by the T_{i_0}-stable affine open subsets. Whence there is a T_{i_0}-stable affine open subset A in X such that

$$A \cap C_{i_0} \text{ is a dense open subset of } C_{i_0}, \tag{27}$$

$$A \cap X^{\mathrm{reg}} \text{ is a dense open subset of } A. \tag{28}$$

Consider the categorical quotient for the affine T_{i_0}-variety A:

$$\pi\colon A \to A/\!\!/T_{i_0} =: \operatorname{Spec} k[A]^{T_{i_0}}.$$

By (12), we have $\dim T_{i_0} \cdot x = \dim T_{i_0}$ for every $x \in X^{\mathrm{reg}}$. This, the fiber dimension theorem, the T_{i_0}-equivariance of π, and the equality $\dim A = \dim X$ then yield:

$$\dim A/\!\!/T_{i_0} \leqslant \dim A - \dim T_{i_0} \overset{(24)}{=\!=} \dim X - \dim T + d_{C_{i_0}} \overset{(22)}{=\!=} \dim C_{i_0}. \tag{29}$$

On the other hand, since $k[A]^{T_{i_0}}$ separates disjoint closed T_{i_0}-stable subsets of A (see [PV94, Thm. 9.4]), we have

$$\dim C_{i_0} \overset{(26)}{=\!=} \dim \pi(C_{i_0}) \leqslant \dim A/\!\!/T_{i_0} \tag{30}$$

From (29), (30) we obtain the equalities

$$\dim C_{i_0} = \dim A/\!\!/T_{i_0} = \dim A - \dim T_{i_0}. \tag{31}$$

From (31), (27), (28), and the fiber dimension theorem, we then deduce the existence of a dense open subset Q of $A/\!\!/T_{i_0}$ that enjoys the following properties:

$$Q \subseteq \pi(A \cap C_{i_0}) \cap \pi(A \cap X^{\mathrm{reg}}), \tag{32}$$

$$\pi^{-1}(q) \text{ is equidimensional of dimension } \dim T_{i_0} \text{ for every } q \in Q. \tag{33}$$

Now take a point $x \in \pi^{-1}(Q) \cap X^{\mathrm{reg}}$. In view of (12), we have

$$\dim T_{i_0} \cdot x = \dim T_{i_0}. \tag{34}$$

As orbits are open in their closures, and $T_{i_0} \cdot x \subseteq \pi^{-1}(\pi(x))$, from (33), (34) we infer that $T_{i_0} \cdot x$ is a dense open subset of an irreducible component of the fiber $\pi^{-1}(\pi(x))$. In view of (32), this fiber contains a point $s \in C_{i_0}$, so we have

$$\pi^{-1}(\pi(x)) = \pi^{-1}(\pi(s)). \tag{35}$$

As, by (26), the point s is T_{i_0}-fixed, it lies in the closure of $T_{i_0} \cdot x$ in A (and a fortiori in X); see [PV94, Thm. 4.7]. Thus $T_{i_0} \cdot x$ belongs to the set $\mathscr{S}$ of all T_{i_0}-orbits $\mathcal{O}$ in X that enjoy the following properties:

(a) $\dim \mathcal{O} = \dim T_{i_0}$;

(b) the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ in X contains s.

We claim that $\mathscr{S}$ is finite. Indeed, if a T_{i_0}-orbit $\mathcal{O}$ belongs to $\mathscr{S}$, then $\overline{\mathcal{O}} \cap A$ is an open neighbourhood of s in $\overline{\mathcal{O}}$, therefore $\mathcal{O} \cap A \neq \varnothing$. Whence $\mathcal{O}$ lies in A and contains s in its closure in A. This and (35) show that $\mathcal{O}$ is a $\dim T_{i_0}$-dimensional T_{i_0}-orbit of $\pi^{-1}(\pi(x))$; whence, as above, $\mathcal{O}$ is a dense open subset of an irreducible component of $\pi^{-1}(\pi(x))$. The claim now follows from the finiteness of the set of irreducible components of $\pi^{-1}(\pi(x))$.

The finiteness of $\mathscr{S}$ implies that the union of all T_{i_0}-orbits from $\mathscr{S}$ is a locally closed subset Z of X whose irreducible components are these orbits. As we proved above, one of these components id $T_{i_0} \cdot x$. Since U commutes with T_{i_0} and, by (21), s is a U-fixed point, the subset Z is U-stable. The connectedness of U then entails that each irreducible component of this subset is U-stable. In particular, $T_{i_0} \cdot x$ is U-stable. Whence $T_{i_0} \cdot x$ is H-stable and therefore we have

$$H \cdot x = T_{i_0} \cdot x. \tag{36}$$

In view of (12), (23), (10), we now obtain the sought-for contradiction:

$$\dim T_{i_0} + 1 = \dim H = \dim H \cdot x \overset{(36)}{=\!=} \dim T_{i_0} \cdot x = \dim T_{i_0}. \tag{37}$$

This completes the proof.

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