# Homology Spheres, Acyclic Groups and Kan-Thurston Theorem

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# Basic Notions

## Definitions

1. A *homology sphere* is a smooth closed $n$-manifold $\Sigma^n$ such that
$$H_*(\Sigma^n) \cong H_*(\mathbb{S}^n).$$

2. A discrete group $G$ is called *perfect* if $H_1(G) = 0$.

3. A discrete group $G$ is called *superperfect* if $H_1(G) = H_2(G) = 0$.

E.g., $\forall n \geqslant 3$ the group $\mathrm{SL}(n, \mathbb{F}_q)$ is superperfect except for $\mathrm{SL}(3, \mathbb{F}_2)$, $\mathrm{SL}(4, \mathbb{F}_2)$, $\mathrm{SL}(3, \mathbb{F}_4)$.

# Kervaire and Novikov's Result

**Proposition (folklore)**

Let $\Sigma^n$ be a homology $n$-sphere ($n \geqslant 3$) with fundamental group $G$. Then $G$ is superperfect.

**Theorem (M. Kervaire'69, S. Novikov'62 (unpubl.))**

If $G$ is a finitely presented superperfect group then for any $n \geqslant 5$ *there exists* a homology $n$-sphere $\Sigma^n$ with $\pi_1(\Sigma^n) = G$.

# Poincaré Sphere

Recall, it is

$$\mathbb{S}^3/2I,$$

where

$$2I \cong \langle a, b \mid a^2 = b^3 = (ab)^5 \rangle \cong \mathrm{SL}(2, \mathbb{Z}/5)$$

is called the *binary icosohedral group* acting by some quaternionic representation on $\mathbb{S}^3 \subset \mathbb{H}$ by the left multiplication.

**Theorem (J. Milnor'57)**

The Poincaré sphere is *the only homology 3-sphere* with a finite nontrivial fundamental group up to homeomorphism.

The Poincaré sphere is the Brieskorn sphere $\Sigma(2, 3, 5)$.

# Deficiency of a Group

## Definition

The *deficiency* def$(G)$ of a finitely presented group $G$:

$$\max\{f - r \mid f = |F|, r = |R|, G \cong \langle F \mid R \rangle\}$$

over all representations of $G$ with finite $f$ and $r$.

E.g., def$(2I) = 0$, since we have

## Proposition (folklore)

If a superperfect group has a balanced representation with $|F| = |R|$, then its deficiency vanishes.

# Deficiency of a Group

The Proposition above follows from

## Epstein's Lemma

For any finitely presented group $G = \langle F \mid R \rangle$ we have

$$|F| - |R| \leqslant \operatorname{rank} H_1(G) - s(H_2(G)),$$

where $s(H_2(G))$ denotes the minimal number of generators of the group $H_2(G)$.

Thus, we get

## Proposition (folklore)

The fundamental groups of any homology 3-sphere has zero deficiency.

# Brieskorn Spheres $\Sigma_a$

## Definition

For positive integers $a_1, a_2$ and $a_3$ a *3-dimensional Brieskorn manifold*

$$M(a_1, a_2, a_3) := \{z \in \mathbb{C}^3 \mid z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0\} \cap \{z \in \mathbb{C}^3 \mid |z| = 1\}.$$

There is an equivalent characterization as some Seifert manifold (W. Neumann, F. Raymond'77).

$M_a$ is a homology sphere $\Leftrightarrow$ $a_1, a_2$ and $a_3$ are pairwise coprime (Brieskorn'66).

A triple $a = (a_1, a_2, a_3)$ is a complete topological invariant for Brieskorn spheres $\Sigma_a$ (W. Neumann'70).

# Fundamental Group of $\Sigma_a$

$$\pi_1(\Sigma_a) \cong \Gamma'(a_1, a_2, a_3)$$

$$1 \to C \to \Gamma(p,q,r) \to D(p,q,r) \to 1 \text{ -- central}$$

$$D(p,q,r) = \langle x, y, z \mid x^p = y^q = z^r = xyz = 1 \rangle$$

This is by J. Milnor'75.

From Seifert's characterization, one gets:

$$\pi_1(\Sigma_a) = \langle x_1, x_2, x_3, h \mid \forall i : \; [h, x_i] = 1, \; x_1 x_2 x_3 = 1, \; x_i^{a_i} h = 1 \rangle$$

From Milnor's results, one can derive

**Proposition (K.)**

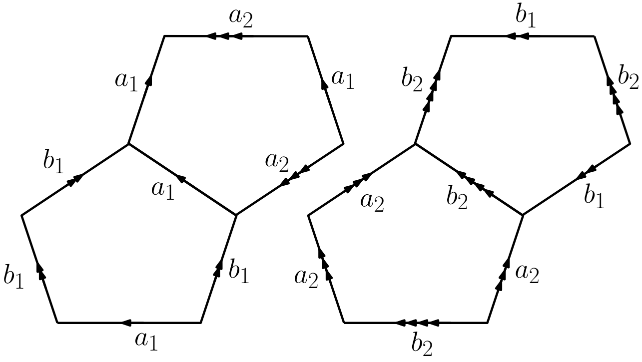
$\Gamma(p,q,r)$ has zero deficiency for any $p, q, r \geqslant 1$.

# Acyclic Groups

## Definition

A discrete group $G$ is called *acyclic* if

$$H_*(G, \mathbb{Z}) \cong H_*(\{e\}, \mathbb{Z}).$$




$$\mathrm{Hig}_4 = \{a_1, b_1, a_2, b_2 \mid b_1^{a_1} = b_1^2,\ a_1^{a_2} = a_1^2,\ a_2^{b_2} = a_2^2,\ b_2^{b_1} = b_2^2\}$$

# Higman Groups

More generally, Higman defined

$$\mathrm{Hig}_n = \langle x_i,\ i \in \mathbb{Z}/n \mid x_i^{x_{i-1}} = x_i^2 \rangle,$$

where $x_i^{x_{i-1}} := x_{i-1}^{-1} x_i x_{i-1}$.

**Proposition (N. Monod'17)**

$\mathrm{Hig}_n$ is acyclic for all $n \geqslant 4$.

# Universal Acyclic Groups

**Definition**

A finitely presented group $G$ is called *universal* if it contains all recursively enumerable finitely generated groups.

The combination of Higman and Chiodo–Hill's results gives

**Proposition (K.)**

Every finitely presented group can be embedded into a universal acyclic group with 12 generators and 38 relations.

# $\pi_1(\Sigma_3)$ Is Not Acyclic

**Proposition (A. Berrick, J. Hillman'03)**

The fundamental group (nontrivial) of any 3-manifold cannot be acyclic.

**Corollary**

The groups $\mathrm{Hig}_n$ ($n \geqslant 4$) cannot serve as fundamental groups of homology 3-spheres.

# $\Sigma^3$ and Smooth Structures

## Theorem (F. Gonzalez-Acuña'70)

If $n \neq 3$ then

$$\Theta^n \cong \Theta^n_{\mathbb{Z}},$$

where $\Theta^n$ is the cobordism group of *homotopy* $n$-spheres, $\Theta^n_{\mathbb{Z}}$ is the one of *homology* $n$-spheres.

In particular, the groups $\Theta^n_{\mathbb{Z}}$ are finite for all $n \neq 3$.

## Theorem (K. Hendricks et al.'21)

Let $\Theta^3_{\mathrm{SF}}$ be the subgroup of $\Theta^3_{\mathbb{Z}}$ generated by the homology Seifert spheres. Then the quotient group $\Theta^3_{\mathbb{Z}}/\Theta^3_{\mathrm{SF}}$ has $\mathbb{Z}^\infty$ as a subgroup.

# Kervaire and Novikov's Result for $\Sigma^4$

**Theorem (M. Kervaire'69, S. Novikov'62 (unpubl.))**

Every balanced superperfect finitely presented group can be the fundamental group of a homology 4-sphere.

E.g., 2I, $\mathrm{Hig}_n$ ($n \geqslant 4$) serve as fundamental groups of homology 4-spheres.

**Corollary**

The fundamental group of any $\Sigma^3$ can serve as the fundamental group of some $\Sigma^4$.

Another way to get the corollary: the Suciu *1-spin* $\sigma_1(\Sigma^3)$.

# Deficiency of $\pi_1(\Sigma^4)$

**Question**

Is every acyclic group the fundamental group of some homology 4-sphere?

It seems that the universal acyclic finitely presented group would serve as a counterexample.

**Theorem (J. Hillman'02)**

There is a homology 4-sphere with the fundamental group of deficiency $-1$.

**Theorem (C. Livingston'03)**

For any $N > 0$, there exists a homology 4-sphere whose fundamental group of deficiency smaller than $-N$.

# Finite $\pi_1(\Sigma^4)$

Suppose the homology 4-sphere $\Sigma^4$ has finite nontrivial fundamental group.

According to Donaldson's theory $\widetilde{\Sigma^4}$ is homeomorphic to

$$(\mathbb{C}P^2)^{\#m},\ (\overline{\mathbb{C}P^2})^{\#n},\ \text{or } (\pm\mathcal{M}_{E_8})^{\#p}\#(\mathbb{S}^2\times\mathbb{S}^2)^{\#q},$$

where # — the connected sum of manifolds, $\mathcal{M}_{E_8}$ — Milnor's $E_8$-manifold.

For the signatures we have $\sigma(\widetilde{\Sigma^4}) = |\pi_1(\Sigma)|\cdot\sigma(\Sigma^4) = 0.$ Hence,

## Proposition (K.)

For $\Sigma^4$ with finite nontrivial $\pi_1(\Sigma^4)$, we have

$$\widetilde{\Sigma^4} \cong \left(\mathbb{S}^2\times\mathbb{S}^2\right)^{|\pi_1(\Sigma^4)|-1}.$$

# Finite $\pi_1(\Sigma^4)$

## Question

Is $2I$ the only nontrivial fundamental group of $\Sigma^4$? If not, how much?

## Theorem (I. Hambleton, M. Kreck'88)

Let $\pi$ and $\chi$ be a finite group and integer, respectively. Then, there is only a *finite number* of closed orientable 4-manifolds with the fundamental group $\pi$ and the Euler characteristic $\chi$ up to homeomorphism.

## Corollary

Let $\pi$ be a finite group. Then, there are *only finitely many* homology 4-spheres with the fundamental group $\pi$ up to homeomorphism.

# Homology Spheres With Fixed $\pi_1$

As follows from the results of A. Suciu'90, the corollary above doesn't hold for $|\pi| = \infty$.

For $M^n$ and $p > 0$ we get the *p-spin of M*:

$$\sigma_p M^n := \partial\left(M_0 \times \mathbb{D}^{p+1}\right) = M_0 \times \mathbb{S}^p \bigsqcup_{\mathbb{S}^{n-1} \times \mathbb{S}^p} \mathbb{S}^{n-1} \times \mathbb{D}^{p+1},$$

where $M_0 = M \setminus \operatorname{Int} \mathbb{D}^n$.

**Theorem (A. Suciu'90)**

For $n \geqslant 3$ and $N \geqslant 2$ there are $N$ homology $n$-spheres with isomorphic $\pi_1$'s and $\pi_2$'s as $\mathbb{Z}\pi_1$-modules, but with different $k$-invariants.

# Groups That Are Not $\pi_1(\Sigma^4)$

**Theorem (J.-C. Hausmann, Sh. Weinberger'85)**

There are nontrivial superperfect finitely presented groups, both with and without torsion, which cannot serve as fundamental groups of homology 4-spheres.

The case of a non-torsion group uses the *Kan–Thurston construction*.

# Kan-Thurston Construction

**Theorem (D. Kan, W. Thurston'76)**

*For every path connected $X \in s\mathcal{S}et_*$, there exists*

$$t : K(G_X, 1) \simeq TX \to X,$$

*such that*

$$H_*(TX; t^*\mathcal{A}) \cong H_*(X; \mathcal{A}), \ H^*(TX; t^*\mathcal{A}) \cong H^*(X; \mathcal{A})$$

*for every local coefficient system $\mathcal{A}$ on $X$.*

It is based on the *ad hoc* constructions of acyclic group cones.

# Categorical Meaning

## Corollary (A. Deleanu'82)

$$\pi_0 \mathcal{CW} \cong \mathcal{GP}[\Gamma^{-1}].$$

*Objects* of $\mathcal{GP}$ are pairs $(G, P)$, $P \lhd G$, $H_1(P) = 0$.
*Morphisms* of $\mathcal{GP}$ are homomorphisms of pairs $f : (G, P) \to (G', P')$ for which $f(P) \subset P'$.

*The set of morphisms* $\Gamma$ consists of those morphisms $f : (G, P) \to (G', P')$ such that $f : G/P \cong G'/P'$ and $f_* : H_*(G; \mathcal{A}) \cong H_*(G'; \mathcal{A})$ for any $G'/P'$–module $\mathcal{A}$.

Functors of the Quillen plus and Kan–Thurston construction are *inverse* in some sense.

# $\infty$-topos Meaning?

## Theorem (M. Hoyois'19)

*$\mathcal{X}$ — $\infty$-topos where $\pi_1$ preserves products. Then, there is an adjunction*

$$\mathcal{X} \underset{\mathrm{id}\times\{1\}}{\overset{(-)^+}{\underset{\perp}{\leftrightarrows}}} \mathcal{X}^{\lozenge},$$

*$\mathcal{X}^{\lozenge}$ — an $\infty$-category of pairs $(X, P)$, $X \in \mathcal{X}$ and $P$ is a perfect normal subgroup of $\pi_1(X)$.*

*Moreover, the map $\mathrm{pr}_1 \circ \eta : X \to (X, P)^+$ is acyclic, $\eta$ — the unit.*

## Question

Is there an $\infty$-topos meaning of the Kan-Thurston construction?

# Recognition of Spheres

**Question**

Is there an algorithm for recognition of the standard sphere $\mathbb{S}^n$?

**Theorem (H. Rubinstein, A. Thompson'94)**

Yes, there is for $n = 3$.

**Theorem (S. Novikov'62)**

The property of an $n$-dimensional manifold to be a standard $n$-dimensional sphere ($n \geqslant 5$) or the property of a contractible region in an $(n+1)$-dimensional Euclidean space with a smooth boundary to be the ordinary $(n+1)$-disk, *unrecognizable*.

**Question**

What about homology 4-spheres?

*Thank you!*