Acyclic groups and Kan-Thurston theorem

Misha Korneff

Toric topology and combinatorics

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1. *The map t_X:*

$$H_*(TX; A) \cong H_*(X; A), \ \ H^*(TX; A) \cong H^*(X; A)$$

for all $\pi_1(X)$-module A.

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   for all $\pi_1(X)$-module A.

2. *TX is aspherical, i. e. $TX \simeq K(\pi, 1)$ for some π and*

$$1 \to \ker \pi_1(X) \to \pi_1(TX) \overset{\pi_1 t_X}{\to} \pi_1(X) \to 1.$$

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  defines a functor T into category of aspherical spaces
- Map t_X is a natural transformation between T and Id

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Following equivalence holds:

$$\mathrm{Ho}\,\mathscr{CW} \cong \mathscr{GP}[\Gamma^{-1}]$$

where $\mathrm{Ho}\,\mathscr{CW}$ *is a category of CW-complexes and homotopy classes of continuous maps between them as morphisms. Objects of* $\mathscr{GP}$ *are pairs of discrete groups* (G, P)*, where* P *is a normal perfect subgroup of* G*, morphisms are homomorphisms* $f : (G, P) \to (G', P')$ *for which* $f(P) \subset P'$

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   - $\mathscr{G}\mathscr{P}$ is a category of pairs (G, P), $P \lhd G$, P is perfect
2. After certain localizations all of arrows will be invertible

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4. $T(\partial\sigma)$ is $K(\pi, 1)$ by induction
5. Group π can be embedded into $C\pi$ $\Rightarrow$ there is induced mapping

$$g : T(\partial\sigma) \to K(C\pi, 1).$$

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Comment

If our space X is a finite simplicial complex, space TX can be chosen as finite space with the same dimension

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Construction of monoid class. space:

$$\bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \rightarrow \ldots \rightarrow \bullet \xrightarrow{g_n} \bullet$$

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- Otherwise: $[d^{-1}c] = [yx^{-1}] = [b^{-1}a] = [vu^{-1}]$
- Hence, $[cu] = [dv]$

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Moreover,

Theorem (O. U. Lenz, 2011)

Let M be commut. monoid and $\varphi : M \to G$ group.-n. Then $|\varphi|$ is homotopy equivalence.

Higman's Groups Hig_n

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Definition

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The second construction of Hig_n

- Suppose $K_i \cong K$, $L_i \cong L$, where

$$K = \langle x, h \mid [h, x] = x \rangle, \; L = \langle K_0, K_1 \mid x_0 = h_1 \rangle$$

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- $\mathrm{Hig}_n = G_{n-1} \star_{\mathbb{Z}\star\mathbb{Z}} L$, $G_{n-1} = \langle K_2, \ldots, K_{n-1} | x_2 = h_3, \ldots, x_{n-2} = h_{n-1} \rangle$

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- $\mathrm{Hig}_n = G_{n-1} \star_{\mathbb{Z}\star\mathbb{Z}} L$, $G_{n-1} = \langle K_2, \ldots, K_{n-1} | x_2 = h_3, \ldots, x_{n-2} = h_{n-1} \rangle$ here we use so-called free product with amalgamation such that $\langle h_0, x_1 \rangle$ and $\langle x_{n-1}, h_2 \rangle$ are identified by $h_0 \sim x_{n-1}$, $x_1 \sim h_2$

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$$K = \langle x, h \mid [h, x] = x \rangle, \ \ L = \langle K_0, K_1 \mid x_0 = h_1 \rangle$$

$$G_{n-1} = \langle K_2, ..., K_{n-1} | x_2 = h_3, ..., x_{n-2} = h_{n-1} \rangle$$

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- It is easy to see that

$$G_n = G_{n-1} \star_{\mathbb{Z}} K$$

with identification $x_{n-1} \simeq h_n$ (we added variables x_n и h_n)

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- Thus, we construct Higman's groups by means of free product with amalgamation and HNN-extensions
- HNN-extension of the group $\langle b\rangle \cong \mathbb{Z}$ by isomorphic subgroups $\langle b\rangle \cong \langle b^2\rangle$ gives Baumslag-Solitar group

$$K = \mathrm{BS}(1,2) = \langle a, b \mid a^{-1}ba = b^2\rangle.$$

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- The group K has one-relator presentation and this relator is not a proper power of irreducible word (as the sum of exponents equals ± 1 for the corresponding word-relator)

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- These observations allow us to compute homology of Hig_n
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Theorem (E. Dyer, A. T. Vasquez, 1972)

Let P be a one relator presentation of the group G. If the relator is not a proper power, then the geometric dimension of G is less than or equal to 2, more concretely $K(G, 1)$ is homotopic to usual Van Kampen 2-complex of G.

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Groups G_k have following homology (in $\mathbb{Z}$)

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Proposition

Higman's groups Hig_n are acyclic and for $K(\mathrm{Hig}_n, 1)$ we can take 2-complex with 1 zero-cells, n one-cells, n two-cells

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where $F_i \cong F,\ C_i \cong C$,

$$F = \langle a, b\rangle,$$

$C = \langle u = a, v = b^{-1}a^{-1}bab, w = b^{-2}ab^{-1}a^{-2}bab^2, x = b^{-3}ab^{-1}a^{-2}bab^3\rangle$

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Group $\mathfrak{A}$ is acyclic

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- And for $n = 2$:

$$0 \to \pi_2(BG) \to \mathbb{Z} \to A \to G \to 0$$

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Extensions of $\mathbb{S}^1$ by Acyclic Groups

- For $A = \mathrm{Hig}_n$ and $A = \mathfrak{A}$ there is no torsion
- Consequently, $\pi_2(\mathbb{B}G) = 0$, i. e. G is aspherical

Fibration of Polyhedral Product

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Theorem

For any simplicial complex $\mathcal{K}$ *on* $[m]$ *there is the fibration*

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- $(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I = \bigcup_{I \in \mathcal{K}} \left(\prod_{i \in I} X_i \times \prod_{i \neq I} A_i \right)$

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- If we set $X = \mathbb{B}\mathbb{S}^1 = \mathbb{C}P^\infty$, then

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- $(PX, \Omega X) \simeq (Y, G)$, where Y is G-space

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Question

Is there a homotopy of pairs $(PX, \pi) \simeq (Y, \pi)$, where Y is some π-space (on which π acts freely)?

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