НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ

Axiomatic analysis of the union of minimal externally stable sets concerning application and implementation of this novel social choice correspondence

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Alternatives, preferences, choices

X – the *general set* of alternatives.

A – the *feasible set* of alternatives: $A \subseteq X \wedge A \neq \varnothing$. The feasible set is a variable.

R – *social preferences*, $R \subseteq X \times X$.

R is presumed to be complete: $\forall\, x \in X, \forall\, y \in X, (x, y) \in R \vee (y, x) \in R$.

P – *strict* social preferences, $P \subseteq R$: $(x, y) \in P \Leftrightarrow ((x, y) \in R \wedge (y, x) \notin R)$.

A *social preference-based choice correspondence* is a mapping S: $2^X \backslash \varnothing \times 2^{X \times X} \to 2^X$

with arguments A and P and values in the set of subsets of A.

It is presumed that S depends on A and P only through restriction of P on A:

$$S = S(A, P) = S(P|_A) \subseteq A$$

i.e. social choices are dependent on social preferences for available alternatives only.

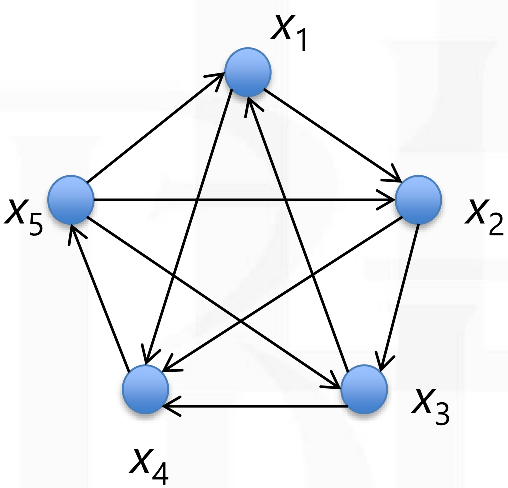
Tournament solutions

A *tournament solution* is a social preference based-choice correspondence *S* that has the following properties:

1. *Nonemptiness*: $\forall A, \forall P, \; S(P|_A) \neq \varnothing$;
2. *Neutrality*: permutation of alternatives' names and social choice commute;
3. *Condorcet consistency*: if there is the Condorcet winner *w* for $P|_A$ then $S(P|_A) = \{w\}$.

	x_1	x_2	x_3	x_4	x_5
x_1	0	1	0	1	0
x_2	0	0	1	1	0
x_3	1	0	0	1	0
x_4	0	0	0	0	1
x_5	1	1	1	0	0

Tournament matrix

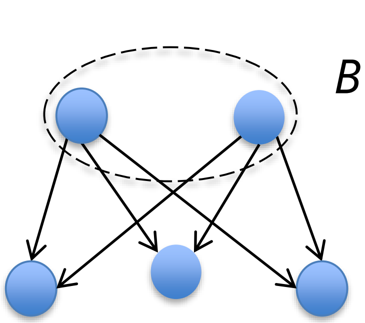


Tournament digraph

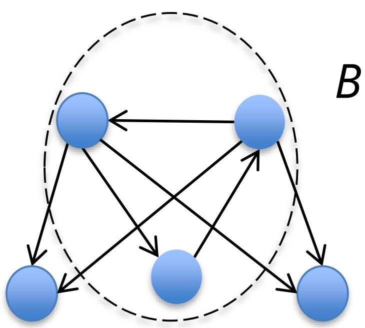
Stable sets

A nonempty subset B of A is called

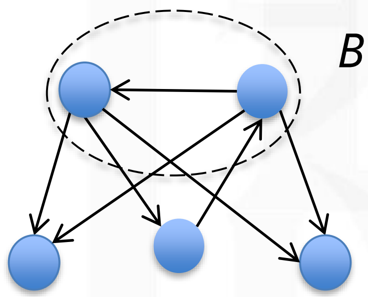
R-dominant	if	$\forall\, x \in A \backslash B,$	$\forall\, y \in B$: yRx
P-dominant	if	$\forall\, x \in A \backslash B,$	$\forall\, y \in B$: yPx
P-dominating	if	$\forall\, x \in A,$	$\exists\, y \in B$: yPx
P-externally stable	if	$\forall\, x \in A \backslash B,$	$\exists\, y \in B$: yPx
R-externally stable	if	$\forall\, x \in A \backslash B,$	$\exists\, y \in B$: yRx
Weakly stable	if	$\forall\, x \in A \backslash B,$	$(\exists\, y \in B: yPx) \vee (\forall\, y \in B, yRx)$



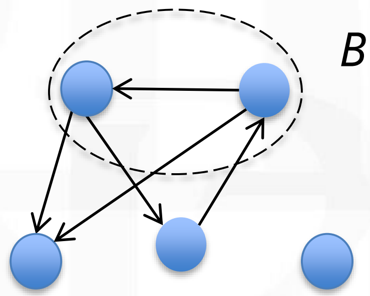
P-dominant



P-dominating



P-ext. stable



Weakly stable

Minimal stable sets

A set *B* is called *minimal* with respect to a given property if *B* has the property and none *of* B's proper nonempty subsets does.

Tournament solutions: the union of all minimal

R-dominant sets	*WTC*	a.k.a. the *weak top cycle* (Good 1971, Smith 1973)
P-dominant sets	*STC*	a.k.a. the *strong top cycle* (Schwartz 1970, 1972)
P-dominating sets	*D*	(Duggan 2013, Subochev 2016)
P-externally stable sets	*ES*	(Wuffl, Feld, Owen & Grofman 1989, Subochev 2008)
R-externally stable sets	*RES*	(Aleskerov & Subochev 2009, 2013)
Weakly stable sets	*WS*	(Aleskerov & Kurbanov 1999)

Properties a.k.a. Axioms

Extension (Sen's property γ, concordance): $\forall A$, $\forall B$, $S(A) \cap S(B) \subseteq S(A \cup B)$.

Idempotency: $\forall A$, $S(S(A)) = S(A)$.

The Aizerman-Aleskerov condition: $\forall A$, $\forall B$, $S(A) \subseteq B \subseteq A \Rightarrow S(B) \subseteq S(A)$.

generalized Nash independence of irrelevant alternatives (*i. of outcasts*):
$\forall A$, $\forall B$, $S(A) \subseteq B \subseteq A \Rightarrow S(B) = S(A)$.

NIIA $\Leftrightarrow$ *Idempotency* $\wedge$ *the Aizerman-Aleskerov condition*

Properties a.k.a. Axioms

When a social choice correspondence S is based on social preferences P,

P-monotonicity (monotonicity w.r.t. social preferences):

$\forall P_1,P_2 \subseteq X^2, \forall A \subseteq X, \forall x \in S(P_1|_A), (P_1|_{A\setminus\{x\}}=\mathrm{P}_2|_{A\setminus\{x\}} \wedge \forall y \in A, xP_1y \Rightarrow xP_2y) \Rightarrow x \in S(P_2|_A)$

Independence of social preferences for irrelevant alternatives (*independence of losers*):

$\forall P_1,P_2 \subseteq X^2, \forall A \subseteq X, (\forall x \in S(P_1|_A), \forall y \in A, ((xP_1y \Leftrightarrow xP_2y) \wedge (yP_1x \Leftrightarrow yP_2x)) \Rightarrow S(P_1|_A)=S(P_2|_A)$

Axiomatic analysis. Negative results

Theorem 1.

D does not satisfy any other axiom from the list.

WS does not satisfy any other axiom except *P*-monotonicity.

ES and *RES* do not satisfy the Extension axiom.

Axiomatic analysis. Positive results

Theorem 2.

ES and *RES* satisfy the (generalized) Nash independence of irrelevant alternatives and, consequently, both the Idempotency and the Aizerman-Aleskerov conditions.

ES and *RES* satisfy the Independence of social preferences for irrelevant alternatives.

ES, *RES* and *WS* satisfy *P*-monotonicity.

The society and the majority rule

The *society* is a group G of n individual decision-makers (voters, experts etc.), $n>1$.

Each member of G has preferences for alternatives from X: $\Pi_k \subseteq X \times X$, $k \in G$.

We suppose that all possible Π_k are linear orders.

In a general case, both social choices and social preferences are functions of the *profile* of individual preferences $\mathbf{\Pi} = \{\Pi_k \subseteq X \times X \mid k \in G\}$.

That is, a social choice correspondence is a mapping SC: $2^X \backslash \varnothing \times (2^{X \times X})^n \to 2^A$ with arguments A and $\mathbf{\Pi}$ and values in the set of subsets of A.

We consider only those SC that depends on A and $\mathbf{\Pi}$ only through restriction of $\mathbf{\Pi}$ on A.

Social preferences is a mapping P: $(2^{X \times X})^n \to 2^{X \times X}$ with argument $\mathbf{\Pi}$ and values in the set of all binary relations on X.

A special case of P – ***the majority rule*:**

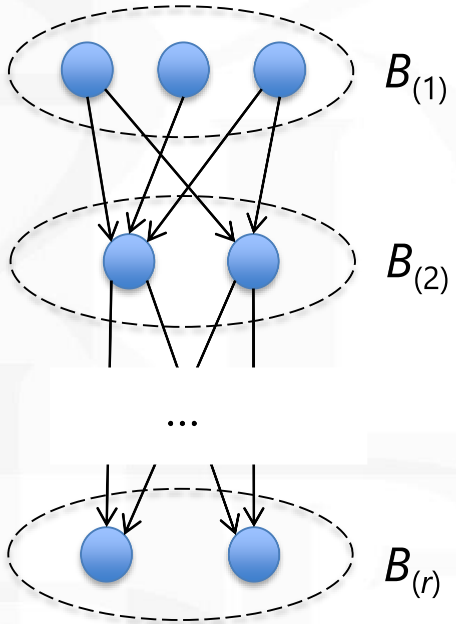
$xPy \Leftrightarrow |G_1|>|G_2|$, where $G_1=\{k \in G \mid x\, \Pi_k\, y\}$, $G_2=\{k \in G \mid y\, \Pi_k\, x\}$;

Ranking based on a tournament solution

Suppose, we are interested in ranking alternatives from A.

Then we may use the following procedure:

- Tournament solution $S(P, A)$ choses the set $B_{(1)}$ of the best alternatives in A, $B_{(1)}=S(P, A)$.
- Exclude these alternatives from A and apply S to the rest. $B_{(2)}=S(P, A\backslash B_{(1)})=S(P, A\backslash S(P, A))$ will be the set of the second-best alternatives in A.
- By repeated exclusion of the best alternatives determined at each step of the procedure the set A is separated into groups $B_{(r)}=S(P, A\backslash(B_{(r-1)}\cup B_{(r-2)}\cup\ldots\cup B_{(2)}\cup B_{(1)}))$, and that is the ranking.
- Let $r = r\,(x, P)$ denote the rank of x in this ranking.



The properties of the ranking rule based on sorting either by *ES* or by *RES*

- ***Weak Pareto principle***: if x Pareto dominates y, then $xQ\ (P)y$

  The Pareto principle is violated.

- ***Weak monotonicity w.r.t the individual preferences*** Π_i (*Smith's monotonicity*):

$(\mathbf{\Pi}|_{A\setminus\{x\}} = \mathbf{\Pi}'|_{A\setminus\{x\}} \wedge \forall i \in G, \forall y \in A, x\ \Pi_i y \Rightarrow x\ \Pi_i' y) \Rightarrow$

$\Rightarrow (\forall y \in A, xQ\ (P)y \Rightarrow xQ\ (P')y\ )$

Independence of irrelevant classes of alternatives

Implementation of a tournament solution

A social choice correspondence $S(\mathbf{\Pi}|_A)$ is *Nash implementable* if for any feasible choice set A there is a non-cooperative game form Γ with a set of players G and set of outcomes A such that for any admissible profile $\mathbf{\Pi}$ the set of social choices coincides with the set of outcomes corresponding to Nash equilibria of the game $(\Gamma, \mathbf{\Pi}|_A)$.

A social choice correspondence $S(\mathbf{\Pi}|_A)$ is *Maskin monotonic* if for any feasible choice set A and any two admissible profiles $\mathbf{\Pi}$ and $\mathbf{\Pi}'$ the following holds:

$$\forall x \in S(\mathbf{\Pi}|_A), (\forall y \in A, \forall k \in G, x\Pi_k y \Rightarrow x\Pi_k' y) \Rightarrow x \in S(\mathbf{\Pi}'|_A)$$

Maskin's theorem. $SC(\mathbf{\Pi}|_A)$ is Nash implementable only if it is Maskin monotonic.

If social preferences P are based on majority rule and if any set of n linear orders on X is admissible as a profile then no tournament solution $S(P|_A)$ is Maskin monotonic and, consequently, Nash implementable.

Implementation of a tournament solution

I.Özkal-Sanver and R.Sanver (2009) demonstrate that it is possible to Nash implement some tournaments solutions if individual preferences are extended, that is, if voters have preferences not only for alternatives but also for sets of alternatives.

A tournament solution $S(P|_A)$ is *Sanver monotonic* if for any feasible choice set A and any two strict social preference relations P and P' the following statement holds:

$$(\forall x \in S(P|_A), \forall y \in A, xPy \Rightarrow xP'y) \Rightarrow S(P|_A) \subseteq S(P'|_A)$$

Sanvers' theorem. Suppose 1) social preferences are based on the majority rule:

2) P is a *tournament*, i.e. P is complete: $\forall\ x \neq y, xPy \vee yPx$;

3) individual preferences P_k are (coherently) extended on sets of alternatives,

then a tournament solution $S(P|_A)$ is Nash implementable if it is Sanver monotonic.

Implementation of a tournament solution

When *P* is a tournament, solutions *RES* and *WS* coincide with *ES*. *WTC* coincide with *STC* and is called simply *the top cycle TC*. *D* is distinct from *ES* and *TC*.

TC is Sanver monotonic (Özkal-Sanver and Sanver 2010).

Theorem 3. *ES* is Sanver monotonic, and *D* is not.

The covering relations and the uncovered sets

The covering relations (Fishburn, 1977; Miller, 1980)

The covering relation $C(P|_A) \subseteq A^2$, is a strengthening of the strict social preferences P:

1. The Miller covering relation C_M : $x\, C_M\, y \Leftrightarrow x P y \wedge P|_A^{-1}(y) \subset P|_A^{-1}(x)$.
2. The weak Miller covering C_{WM}: $x\, C_{WM}\, y \Leftrightarrow P|_A^{-1}(y) \subset P|_A^{-1}(x)$.
3. The Fishburn covering C_F : $x\, C_F\, y \Leftrightarrow x P y \wedge P|_A(x) \subset P|_A(y)$.
4. The weak Fishburn covering C_{WF} : $x\, C_{WF}\, y \Leftrightarrow P|_A(x) \subset P|_A(y)$.

Note that $C(P|_A)$ is not a restriction of $C(P)$ on A: $C(P|_A) \not\equiv C(P) \cap A^2$.

The set of all alternatives that are not covered in A by any alternative is called ***the uncovered set*** of a feasible set A.
The set of all alternatives that are not weakly covered in A will be called ***the inner uncovered set*** of a feasible set A.

The Miller and Fishburn uncovered sets and their inner versions will be denoted UC_M and UC_F, UC_{IM} and UC_{IF}, correspondingly.

The uncovered sets and the externally stable sets

Theorem A. Suppose $|A|<\infty$. $x \in ES \Leftrightarrow \exists\, y \in UC_F: x\, P\, y \vee x \in UC_F$.

Corollary to Theorem A. *ES* is a union of UC_F and all $P(x)$ such that $x \in UC_F$

Theorem B. Suppose $|A|<\infty$. $x \in RES \Leftrightarrow \exists\, y \in UC_{IM}: x\, R\, y.$

Corollary to Theorem B. *RES* is a union of all $R(x)$ such that $x \in UC_{IM}$

Theorem C. Suppose $|A|<\infty$. $x \in D \Leftrightarrow \exists\, y \in UC_{IF}: x\, P\, y.$

Corollary to Theorem C. *D* is a union of all $P(x)$ such that $x \in UC_{IF}$

$$UC_F \subseteq ES \qquad\qquad UC_M \subseteq RES$$

D is not nested with the *UC* even when *P* is a tournament.

The uncovered sets and the externally stable sets

Proposition: Assume $R(x)$ is compact for all $x \in A$ then $UC_{IF} \neq \varnothing$

(Banks, Duggan & Le Breton 2006)

Let $\Omega = (A, \{\omega\})$ be the topology generated by $\{P^{-1}(x) \mid x \in A\}$.

Theorem Z: Suppose A is compact in Ω. Then $x \in D \Leftrightarrow \exists\, y \in UC_{IF}: x\, P\, y$.

Corollary. Suppose A is compact in Ω. Then $D \neq \varnothing$, $WS \neq \varnothing$ and $RES \neq \varnothing$. Additionally suppose that the core is either empty or P-externally stable then $ES \neq \varnothing$.

Publications

1. Subochev A. 2017. Axiomatic analysis of the union of minimal externally stable sets concerning application and implementation of this novel social choice correspondence. *Proceedings of the XVII April International Academic Conference on Economic and Social Development: in 4 volumes*. *Volume 1* (pp. 111-120). Moscow. National Research University Higher School of Economics. (in Russian)

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3. Aleskerov F., Subochev A. 2009. On Stable Solutions to the Ordinal Social Choice Problem. *Doklady Mathematics*, V. 73, N. 3. P. 437-439.

4. Subochev, A. 2008. Dominant, Weakly Stable, Uncovered Sets: Properties and Extensions. Working paper WP7/2008/03. Moscow: SU – Higher School of Economics.

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