Trajectory Classification of Random Motion with Stochastic Self-Acceleration and its Anomalous Properties

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Abstract. An original classification of random trajectories formed by a Brownian particle whose motion is governed by stochastic self-acceleration is constructed. In particular, it enables us to elucidate the mechanism endowing the analyzed continuous Markovian random walks with the characteristic properties of Lévy flights or Lévy walks. Lévy flights appear in the case when the particle velocity is governed by stochastic self-acceleration, Lévy walks are the case when the particle acceleration undergoes stochastic self-acceleration whereas the particle velocity is approximately fixed in magnitude. Besides, the constructed trajectory classification can be regarded as a generalized continuous time random walks model.

DISCRETE AND CONTINUOUS MODELS OF LÉVY TYPE PROCESSES

Lévy type stochastic processes are met in many systems of various natures (e.g., Ref. [1, 2]). One of the widely used approaches to describing such processes is the so-called continuous time random walks (CTRW) [3, 4]. This model implements, in particular, a general class of Lévy type stochastic processes represented as random motion $x(t)$ of a wandering particle in the space $\mathbb{R}^N$ and described by the fractional Fokker-Planck equation [5]. It imitates random motion of a particle by assigning to each its jump a length δx and a waiting time δt elapsing between two successive jumps, drawn from the probability density $\psi(\delta x, \delta t)$. When it is possible to write this probability density as the product of the individual probability densities of the jump length, $\psi_x(\delta x)$, and the waiting time, $\psi_t(\delta t)$, i.e., $\psi(\delta x, \delta t) = \psi_x(\delta x)\psi_t(\delta t)$, the two quantities can be regarded as independent random variables. In this case the model is called decoupled CTRW. In the opposite case called coupled CTRW the probability density $\psi(\delta x, \delta t)$ is represented usually as the product of two functions $\psi(\delta x, \delta t) = \psi_x(\delta x)\psi_v(\delta x/\delta t)$ or $\psi(\delta x, \delta t) = \psi_t(\delta t)\psi_v(\delta x/\delta t)$, so the jump characteristics δx and δt are no longer independent random variables but the independent variables are the jump length δx (or the waiting time δt) and the mean particle velocity $v = \delta x/\delta t$.

In particular, Lévy flights are one of the special types of CTRW. A Lévy flight's waiting time distribution is narrow, for instance, Poissonian with $\psi_t(\delta t) = \tau^{-1}\exp\{-\delta t/\tau\}$, where τ is a certain "microscopic" time scale and, thus, the system dynamics becomes Markovian on time scales $t \gg \tau$. The jump length distribution of Lévy flights is long-tailed and the resulting distribution function of the particle displacement x during a time interval $t \gg \tau$, for example, in one-dimensional case ($N = 1$) exhibits the asymptotic behavior $\Psi(x,t) \sim [x(t)]^\alpha / x^{1+\alpha}$ for $x \gg x(t)$, where $x(t)$ is the characteristic particle displacement during the time interval t and the exponent α meets the inequality $0 < \alpha < 2$. The time dependence of the value $\bar{x}(t)$ obeys the scaling law $\bar{x}(t) \propto t^{1/\alpha}$. As a result the distinctive feature of Lévy flights is the divergence of the second moment $\langle x^2 \rangle \to \infty$.

Lévy walks are another characteristic example of these processes and observed widely in animal movements and human traveling patterns (for a review see Refs. [6] and [7], respectively). In contrast to Lévy flights, Lévy walks possess a finite mean squared displacement, albeit having a broad jump length distribution. It becomes possible due to coupling waiting times and jump lengths, $\psi_t(\delta t)\psi_v(\delta x/\delta t)$, such that a long jump involves a long waiting time.

The given discrete implementation of CTRW admits a continuous representation of particle motion assuming that within one jump a wandering particle moves with a fixed velocity v along the straight line connecting its initial and terminal points. A more detailed description of particle motion lies beyond the CTRW model. Unfortunately, for Lévy flights sophisticated details of the particle motion within one step could be substantial especially in heterogeneous media or systems with boundaries. For example, the fact that Lévy flights do exhibit nontrivial properties on scales of single steps was found in studying the first passage time problem for Lévy flights based on the leapover statistics [8].

Recently, a new approach to modeling Lévy type random motion based on continuous nonlinear Markovian processes has been proposed [9, 10, 11, 12]. The pivot point of this approach is stochastic differential equations

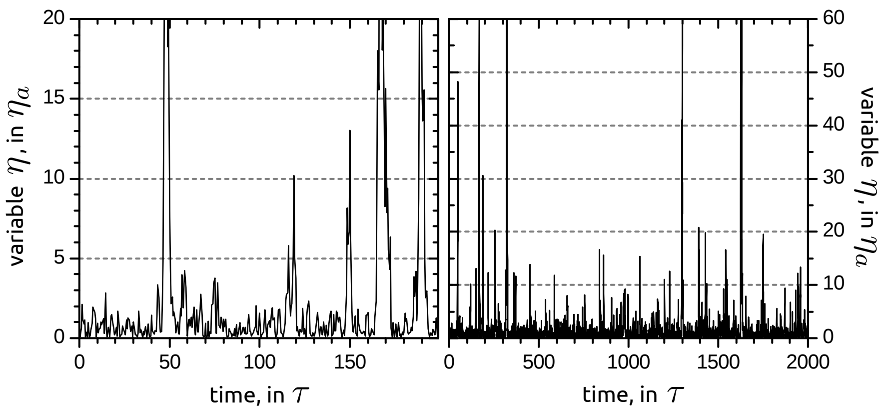

FIGURE 1. The characteristic form of the time pattern $\{\eta(t)\}$ generated by model (1). The two frames depict the same patten on different time scales. Based on the results presented in Ref. [9], the value $\alpha = 1.6$ was used.

with nonlinear noise governing the particle motion that generate the desired Lévy type random walks on scales $t \gg \tau$ exceeding some microscopic time τ. It enables one to cope with continuous trajectories of Brownian motion whose properties can be analyzed using the standard techniques and then to pass to the limit $t \gg \tau$. To clarify the basic features of this approach let us consider a rather simple case when the phase variable $\eta \in \mathbb{R}$ determining the particle motion in the space $\mathbb{R}$ is governed by the equation

$$\frac{d\eta}{dt} = -\frac{1+\alpha}{\tau} \cdot \eta + \sqrt{\frac{2}{\tau}} \sqrt{\eta_a^2 + \eta^2} * \xi(t)\,, \tag{1}$$

where, α is some coefficient coinciding, e.g., for Lévy flights with the exponent α noted above, $\xi(t)$ is white noise of unit amplitude, and the parameter η_a quantifies the relative weight of additive and multiplicative components of the random Langevin force [11]. The multiplication sign '$*$' in equation (1) indicates that it is written in the Hänggi-Klimontovich form. The η-dependence of the noise intensity $g(v) := \sqrt{2/\tau} \cdot \sqrt{\eta_a^2 + \eta^2}$ growing linearly with η, i.e., $g(v) \propto \eta$ for $\eta \gg \eta_a$ is the implementation of stochastic self-acceleration, real or effective one. As illustrated in Fig. 1, stochastic self-acceleration causes a complex multi-scale structure of the time pattern $\{\eta(t)\}$. Its extreme η-fluctuations are distributed according to a power-law and their characteristic amplitude grows also according to a power-low as the observation time interval t increases. In particular, the former property gives rise to the Lévy distribution of the particle displacement, whereas the latter one is responsible for the Lévy time scaling [9, 10].

These features make it attractive to partition the pattern $\{\eta(t)\}$ in such a way that its fragments $\{\eta(t)\}_i^{i+1}$ could be classified as random walks inside a certain neighborhood $\mathscr{L}$ of the origin $\eta = 0$ or outside it. Then, the latter ones may be treated as some peaks in the time pattern $\{\eta(t)\}$. A direct implementation of this idea, however, faces a serious obstacle. The matter is that a random trajectory is not a smooth curve. Therefore, although for a wandering particle it is possible to calculate the probability of getting the boundary of $\mathscr{L}$ for the first time, the question about it crossing this boundary for the second time is meaningless, the particle will cross the boundary immediately after the first one. The trajectory classification to be constructed below enables us to overcome this problem and, thus, to implement the desired partition. In this way the two approaches, discrete and continuous ones, to modeling Lévy type random motion become interrelated. The CTRW model admits rather efficient numerical implementation whereas its basic characteristics can be found appealing to physical properties of the system.

However, before constructing the desired trajectory classification let us present two particular models using this approach. The first one generates Lévy flights and describes random motion $\mathbf{x}(t) \in \mathbb{R}^M$ of a wandering particle in $\mathbb{R}^N$ whose velocity $\mathbf{v} = d\mathbf{x}/dt$ is governed by an equation similar to Eq. (1), namely, [9, 10, 11]

$$\frac{d\mathbf{v}}{dt} = -\frac{N+\alpha}{\tau} \cdot \mathbf{v} + \sqrt{\frac{2}{\tau}} \sqrt{v_a^2 + v^2} * \xi(t)\,, \text{ where } \xi = \{\xi_1, \xi_2, \ldots, \xi_N\} \text{ and } \left\langle \xi_i(t)\xi_{i'}(t') \right\rangle = \delta(t-t')\delta_{ii'}\,. \tag{2}$$

The second one generates Lévy walks and imitates stochastic search during animal foraging [13]. It appeals to the concept of a self-propelled particle moving in $\mathbb{R}^2$ mainly with a velocity of a preferable magnitude v_0 and the stochastic variable $\eta \in \mathbb{R}^2$ describes its active behavior in intentional change of the motion direction. In the simplest form this

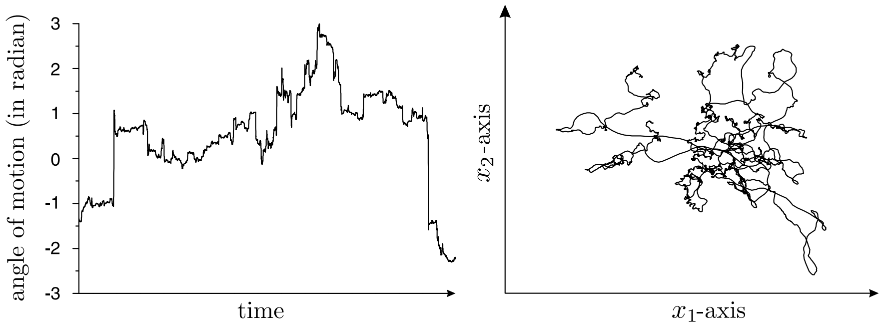

FIGURE 2. The characteristic form of the time pattern representing the motion direction of a wandering particle (left frame) and the corresponding trajectory of its motion (right frame) generated by model (3). Based on the results of Ref. [13].

model reads

$$\frac{d\mathbf{v}}{dt} = \lambda_v \left(1 - \frac{v^2}{v_0^2}\right)\mathbf{v} + \eta\,, \qquad\qquad \frac{d\eta}{dt} = -\frac{\lambda_\eta}{\tau}\,\eta + \frac{\sqrt{2}\,\eta_s}{\sqrt{\tau}}\sqrt{\frac{\Delta^2\eta_s^2 + \eta^2}{\eta_s^2 + \eta^2}} * \xi(t)\,. \tag{3}$$

Here λ_v and λ_η are some kinetic coefficients and the value $\Delta\eta_s$ quantifying the relative contribution of the additive components of the Langevin forces is related to the value η_s characterizing the limit capacity of the animal active behavior via the proportionality coefficient $\Delta \ll 1$. Figure 2 exemplifies the characteristic properties of random walks generated by model (3). As seen, the motion trajectory does contain a sequence of fragments matching gradual variations of the motion direction followed by step-wise jumps. These fragments can be approximated by the particle motion along straight lines whereas the jumps may be treated as turning points in the particle motion.

TRAJECTORY CLASSIFICATION

Figure 3 (left frame) exhibits the desired classification of random walks in $\mathbb{R}$. Let us introduce two boundaries η_l and η_u for the layer $\mathscr{L}$ such that $\eta_a \lesssim \eta_l < \eta_u$. Then, without loss of generality, we assume that the first fragment of a given trajectory represents the motion of the wandering particle outside the interval $[0,\eta_l)$ until it gets the point $\eta = \eta_l$ for the first time at a time moment $t_1 > 0$. Such particle motion is referred to as random walks outside the neighborhood $\mathscr{L}$ of the origin $\eta = 0$. The next fragment matches the particle wandering inside the interval $[0,\eta_u)$ until it gets the point η_u for the first time at a certain moment $t_2 > t_1$. The particle motion of this type is referred to as random walks inside the neighborhood $\mathscr{L}$. The following fragment is similar to the first one and so on. A sequence of such fragments alternating one another makes up the desired trajectory partition. The collection of quantities $\{\Theta_i\}$ bounding from above the allowed η-variations in the corresponding fragments of random walks outside $\mathscr{L}$ enables us to analyze the statistics of the extreme η-fluctuations of the pattern $\{\eta(t)\}$. Exactly this representation of particle trajectories generalizes the CTRW model because it does not assume the particle to move uniformly withing time steps composed of two succeeding fragments of random walks inside and outside $\mathscr{L}$.

Analyzing the statistics of the constructed trajectory fragments we demonstrate the possibility of converting to a new variable $\mathfrak{u}$ and time $\mathfrak{t}$ via the formula

$$\eta(t) = \sinh\left\{\frac{\mathfrak{u}(\mathfrak{t})}{\alpha} + \Delta_\alpha \operatorname{sign}[\mathfrak{u}(\mathfrak{t})]\right\}, \text{ where } t = \frac{1}{\alpha^2}\,\mathfrak{t} \text{ and } \Delta_\alpha = \frac{1}{\alpha}\ln\left[\frac{4\Gamma(\alpha)}{\alpha\Gamma^2(\alpha/2)}\right] < 0.35 \text{ for } \alpha \in (0,2)\,, \tag{4}$$

such that the stochastic process $\mathfrak{u}(\mathfrak{t})$ be governed by the equation

$$\frac{d\mathfrak{u}}{d\mathfrak{t}} = -\operatorname{sign}(\mathfrak{u}) + \sqrt{2}\xi(\mathfrak{t}) \tag{5}$$

containing no parameters at all. It is called the core stochastic process. All the system parameters such as the exponent of the Lévy scaling law enter only the transformation formulas. This, in particular, explains why the main results rigorously obtained for Lévy flights of the superdiffusive regime [9, 10] hold for the other possible regimes as

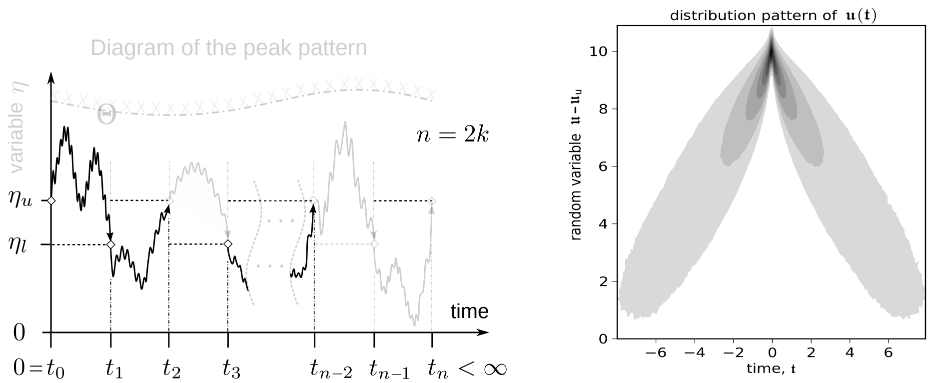

FIGURE 3. The classification diagram of random walks in the space $\mathbb{R}_\eta$ and their basic fragments (left frame). Without loss of generality, we may consider trajectories returning to the initial point located at the upper boundary of the layer $\mathscr{L}$. The right frame visualizes the time-space structure of trajectories formed by the random motion $\mathfrak{u}(t)$ in the vicinity of the attained maximal value.

demonstrated numerically [11]. In addition it is worthwhile to note that in studying the extreme characteristics of particle motion outside the neighborhood $\mathscr{L}$ the shape of the time pattern $\{\eta(t)\}$ can be approximated using the most probable trajectory $\{\eta_{\text{opt}}(t)\}$. It opens a gate to modeling such processes in heterogeneous media constructing the most optimal trajectories of particle motion in heterogeneous environment. Figure 3 (right frame) justifies this by showing how the points of trajectories are scatted around $\{\eta_{\text{opt}}(t)\}$.

As far as Lévy flights are concerned, for the random walks $x(t) \in \mathbb{R}$ described by the relationship $dx/dt = \eta$ it is demonstrated that the power-law tails in the distribution function $\mathscr{P}(\delta_t x)$ of the particle spatial displacements during the time interval t are mainly due to the extreme fluctuations in the particle velocity within individual fragments of random motion outside the neighborhood $\mathscr{L}$; all the other details of these random walks are of minor importance. It is called the one-peak approximation in order to underline the fact that in calculating the asymptotics of the distribution function $\mathscr{P}(\delta_t x)$ it is sufficient to take into account only one fragment containing the extreme value of the particle velocity. Naturally, the other fragments contribute to the magnitude of this distribution function taken at $\delta_t x = 0$ and to describe this effect one has to go beyond the one-peak approximation, which is worthy of individual analysis.

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