Conference on Topology, Embeddings, and Attractors
Confirmed speakers
http://www2.warwick.ac.uk/fac/sci/maths/people/staff/james_robinson/lf/venice

FINITE-DIMENSIONAL LIMITING DYNAMICS OF SEMILINEAR PARABOLIC EQUATIONS

A.V. Romanov (MIEM, Moscow)

My talk will be devoted the finite-dimensional description of limiting dynamics of semilinear parabolic equations. The idea of finite-dimensional nature of the dynamics in such equations for large time goes back to the work [Hopf1948] and historically relates with the problem of turbulence.

[Hopf1948] E. Hopf. A mathematical example displaying features of turbulence. Comm. Appl. Math., 1:4, 303–322.

1. PRELIMINARY

Let us consider the abstract dissipative semilinear parabolic equation (SPE)

$$u_t = -Au + F(u) \tag{1-1}$$

in a real separable Hilbert space X with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. Suppose A to be positive self-adjoint linear operator in X with compact inverse and $\{X^\theta\}_{\theta\geq 0}$ be the Hilbert scale determined by A. We assume also the estimate

$$|F(u)-F(v)| \leq K|u-v|_\theta \qquad (u,v\in X^\theta) \tag{1-2}$$

for the nonlinear term F with some $\theta\in[0,1)$ where $|u|_\theta = |A^\theta u|$. I shall call the number θ as exponent of nonlinearity of equation (1-1). The resolving phase semiflow Φ_t in X^θ is injective in the case $\theta\leq 1/2$ [Temam1997] and inherits the smoothness of function $F: X^\theta \to X$ [Henry1981] in a general case. In this situation there exists the compact global attractor $\mathcal{A}\subset X^\theta$ [Temam1997], i.e. the maximal bounded invariant set in X^θ (actually $\mathcal{A}$ attracts the bounded subsets in X^θ uniformly as $t\to+\infty$).

The finite-dimensional description of SPE (1-1) limiting dynamics means the existing of ODE

$$x_t = h(x) \quad (x\in R^n)$$

with (at least) continue vector field $h(x)$ and unique solutions being describe (as maximum) the behavior of all solutions $u(t)$ for a large time or (as minimum) behavior solutions $u(t) \in \mathcal{A}$ for $t \in (-\infty, \infty)$.

[Temam1997] R. Temam. Infinite-dimensional Dynamical Systems in Mechanics and Physics. Appl. Math. Sci. 68, Berlin: Springer, 2-nd ed.
[Henry1981] D. Henry. Geometric Theory of Semilinear Parabolic Equations. Lect. Notes in Math. 840, Berlin: Springer.

2. INERTIAL MANIFOLDS

The most radical approach to the problem of a description of the final phase dynamics SPE (1-1) by a suitable ODE in R^n is connected with the conception inertial manifold (IM): a Lipschitz or C^1-smooth finite-dimensional invariant surface $\mathcal{M}$ in X^θ containing the attractor and exponentially attracting all solutions $u(t)$ at a long time with asymptotic phase. Usually IM is building as a graph from low modes of linear part SPE to high ones. The restriction of equation (1-1) to $\mathcal{M}$ gives an ODE in R^n ($n = \dim \mathcal{M}$) which completely reproduces the final X^θ- phase dynamics of SPE as $t \to +\infty$. The most of known methods of IM's constructing demand the spectral gap condition

$$\lambda_{n+1} - \lambda_n > cK(\lambda_{n+1}^\theta + \lambda_n^\theta), \qquad (2\text{-}1)$$

where K is the number in inequality (1-2) and $\{\lambda_1 \le \lambda_2 \le ...\} = \sigma(A)$. In the case $\mathcal{M} \in \mathrm{Lip}$ the optimal constant c in this condition is 1 [Mik1991], [Rom1991-1993]; the simple and concise proofs with $c = 2$ has been obtained in [Rob1993-1995]. The best known value c in (2-1) for the case $\mathcal{M} \in C^1$ is equal $\sqrt{2}$ (see [Kok1997]).

Unfortunately, the spectral gap condition is highly restrictive and so the existing of IM may be proven for a narrow class SPEs only. This class contains for example scalar or vector 1d reaction-diffusion equations and scalar equations of the such type in some special domains $D \subset R^n$ $(n = 2,3)$ In the second case condition (2-1) usually is not executed and it needs to use so-called "Principle of spatial averaging" by [MP-Sell1988] (see [Kwean2001] too).

It needs to note the recent results due J. Vukadinovic (see [Vuk2009], [Vuk2011] and cites there). In particularly, the existing of IM to diffusive Burgers equations

$$u_t + (u \cdot \nabla)u = \Delta u + Tu + \nabla g(x) \qquad (u = \nabla\varphi) \qquad (2\text{-}2)$$

on the tori $[-\pi, \pi]^d$, $d = 1,2$ has been proved in the last paper. The functions $u(t,x)$ have vanishing spatial average and the operator T is assumed to be Fourier multiplier with arbitrary bounded symbol $m: Z^d \to R$, $m(0) = 0$ in (2-2). This author employs suitable transformations of original SPEs (the Cole-Hopf transformation for the Burgers equations) in order to satisfy the spectral gap condition. It is important that his transformations preserve the symmetry property of the linear term of underlying equation. It needs remind in this connection the numerous incorrect works published in 90-s years (M. Kwak and some other authors) in which this property were violated after a transformation of original equation.

[Mik1991] M. Miklavcic. A sharp condition for existence of an inertial manifold. J. Dyn. Differ. Eq., 3:3, 437-456.
[Rom1991] A.V. Romanov. Conditions for the asymptotic k - dimensionality of semilinear parabolic equations. Russ. Math. Surveys, 46:1, 255-256.
[Rom1993] A.V. Romanov. Sharp estimates of the dimension of inertial manifolds for nonlinear parabolic equations (in English, 1994). Izvestia: Mathematics, 43:1, 31–47.
[Rob1993] J. C. Robinson. Inertial manifolds and the cone condition. Dyn. Systems Appl., 2:3, 311–330.
[Rob1995] J. C. Robinson. A concise proof of the geometric construction of inertial manifolds. Phys. Lett. A, 200, 415–417.
[Kok1997] N. Koksch. Almost sharp conditions for the existence of smooth inertial manifolds. Conf. Diff. Equat. Appl. (Equadiff-9), 1997, Brno, 139-166.
[MP-Sell1988] J. Mallet-Paret and R. Sell. Inertial manifolds for reaction diffusion equations in higher space dimensions. J. Amer. Math. Soc. 1:4, 805–866.
[Kwean2001] H. Kwean. An inertial manifold and the principle of spatial averaging. Int. J. Math. Math. Sci. 28:5, 293–299.
[Vuk2009] J. Vukadinovic. Inertial manifolds for a Smoluchowski equation on a unit sphere. Comm. Math. Phys., 285:3, 975-990.
[Vuk2011] J. Vukadinovic. Global dissipativity and inertial manifolds for diffusive Burgers equations with low-wavenumber instability. Discr. Cont. Dyn. Syst., 29:1, 327-341.

3. NON-EXISTING OF INERTIAL MANIFOLD

At present it is known very little about non-existing of IM for SPE (1-1). In [Rom2000] has been constructed the system of two coupled 1d parabolic pseudo-differential equations which have no the smooth IM. This is rather artificial example, but the more natural one has been only mentioned in the short note [Rom2002]. Let us consider the integro-differential parabolic equation

$$u_t = ((I + B)u_x)_x + f(x,u,u_x) + Ku \tag{3-1}$$

on the unit circle Γ. The bounded linear operators K, $I = id$, $B = B^*$ act in $X = L^2(\Gamma)$ and the function $f(x,u,p)$ is smooth but non-analytic. The operator $I + B \geq 0$ plays a part of non-local diffusion coefficient and value Ku plays a part of non-local linear source. More exactly,

$$(Bh)(x) = \frac{1}{2\pi}\int_0^{2\pi} \ln\left|\sin\frac{x+y}{2}\right| h(y)\,dy$$

for $h \in X$. Remark that $\partial_x B$ is slightly modified Hilbert's operator with the kernel $\mathrm{ctg}\,\frac{x+y}{2}$ instead $\mathrm{ctg}\,\frac{x-y}{2}$.

THEOREM 3.1. *At the suitable choice of the function* f *and the compact integral operator* K *with* C^∞*- kernel the equation* (3-1) *generates the smooth* *dissipative semiflow in* X^θ, $\theta > 3/4$ *and there no exists the invariant finite-dimensional* C^1 *- manifold* $\mathcal{M} \subset X^\theta$ *containing the attractor of this equation.*

The both above-mentioned examples are based on properties of spectra

$$\sigma_u = \sigma(A - DF(u))$$

in stationary points of SPE (1-1). Let E is the set of hyperbolic stationary points $u \in X^\theta$ for which σ_u does not contain any real values $\lambda > 0$. Moreover let $l(u)$ be the number (with multiplicity) of the values $\lambda < 0$ in σ_u. It is clear that $l(u) < \infty$.

LEMMA 3.2 [Rom2000]. *If attractor* $\mathcal{A}$ *of SPE* (1-1) *is contained in some smooth invariant finite-dimensional manifold* $\mathcal{M} \subset X^\theta$ *then for any points* $u_1, u_2 \in E$ *the number* $l(u_1) - l(u_2)$ *is even.*

[Rom2000] A.V. Romanov. Three counterexamples in the theory of inertial manifolds. Math. Notes, 68:3, 378–385.
[Rom2002] A.V. Romanov. On the finite-dimensional dynamics of parabolic equations. Nonlinear dynamical analysis (NDA'2). Moscow, June 3-8, 195 (in Russian).

4. THE LIPSCHITZ FINITE-DIMENSIONAL DYNAMICS ON ATTRACTOR

Since there are above-mentioned problems with the IM existing it has been suggested in the works [EFNT1994] and [Rob1999] to consider ODEs reproducing the phase dynamics of evolutionary equation (1-1) on the attractor only. Let us say in this connection (following [Rom2000]) that phase dynamics on the attractor is Lipschitz finite-dimensional (it takes place the property LFDA) if for some ODE

$$x_t = h(x) \qquad\qquad (x \in R^n,\ h \in \mathrm{Lip})$$

with phase flow S_t and S_t - invariant compact $V \subset R^n$ the dynamical systems Φ_t on $\mathcal{A}$ and S_t on V are Lipschitz adjoined for $t > 0$. The property LFDA is formally weaker than the property of existing of inertial manifold because in this case we consider the dynamics on attractor only.

I shall not discuss now very important question whether here V be the global attractor of the flow S_t and I hope to learn many interesting in this connection on our meeting.

Further let $G(u) = F(u) - Au$ and P_n be orthogonal spectral projection in X^θ (Fourier projection) corresponding the first n (with multiplicity) low modes of the operator A. Let more over $\mathcal{A}_0$ be the set of points $w = u - v$ for different $u, v \in \mathcal{A}$.

In the papers [Rom2000-2006] were proven the following criteria of the property LFDA.

(Vf) The vector field of equation (1-1) is Lipschitz on attractor, i.e.

$$\left|G(u)-G(v)\right|_{\theta} \le C_1\left|u-v\right|_{\theta}$$

for $u,v \in \mathcal{A}$.

(Fl) The semiflow Φ_t on attractor is injective and extends to Lipschitz flow. It means that

$$\left|\Phi_t u-\Phi_t v\right|_{\theta} \le C_2 e^{k|t|}\left|u-v\right|_{\theta}$$

on $\mathcal{A}$ for any $t \in R$ and $k = k(\mathcal{A})$.

(Fl0) It takes place the estimate

$$\left|\Phi_\tau u-\Phi_\tau v\right|_{\theta} \ge c(\tau)\left|u-v\right|_{\theta}$$

on $\mathcal{A}$ for some fixed $\tau > 0$.

(Em) The metrics X^{θ} and X^{β} are equivalent on $\mathcal{A}$ for some (for any) $\beta \in [0,1]$, $\beta \ne \theta$.

(Le) There exists a linear bi-Lipschitz embedding of attractor $\mathcal{A}$ into R^n. This means $\mathcal{A}$ to be a Lipschitz graph.

(Lle) For any point $u \in \mathcal{A}$ some its X^{θ}- neighbourhood on $\mathcal{A}$ may be linear bi-Lipschitz embedded into R^n. The rank of corresponding embedding can depend from u. This property means that $\mathcal{A}$ is of local Lipschitz-Descartes structure.

(GrF) It takes place the estimate

$$\left|u-v\right|_{\theta} \le C_3\left|P_n(u-v)\right|_{\theta}$$

on $\mathcal{A}$ for some $n \ge 1$. It means the possibility of linear bi-Lipschitz embedding of attractor $\mathcal{A}$ into R^n by some Fourier projection.

There are $C_i = C_i(\mathcal{A})$ in the formulas above. Some (but not all) of these criteria suggest the additional smooth condition $F \in C^2(X^{\theta}, X)$.

The criterion (Em) have been obtained independently by J.C. Robinson and E. Pinto de Moura [Rob2003], [PM-Rob2010]) for $1 > \beta > \theta$. It is known [CFKM1997] that Fourier bi-Lipschitz embedding property (GrF) follows from the property (Em) with $\beta = 0,\ \theta = 1/2$ what in essentiality is the boundness of Dirichlet quotient $\mu = \dfrac{(w, Aw)}{(w,w)}$ on $\mathcal{A}_0$.

If $\mathcal{A}^s$ is the set of points $\omega = \dfrac{w}{|w|}$ ($w \in \mathcal{A}_0$), we can formulate some more following three criteria of LFDA.

(Sk) The set $\mathcal{A}^s$ is relatively compact in X^{θ} ([Rom2001]).

(Sk0) The Hausdorff measure of non-compactness of the set $\mathcal{A}^S$ is less then 1, i.e. $\mathcal{A}^S$ lies on the ε-neighbourhood of some compact set $H \subset X^\theta$ with $\varepsilon < 1$ ([Rom2006]).

(Skw) The weak closure of the set $\mathcal{A}^S$ in X^θ does not contain zero ([ML-W2005]).

H. Movahedi-Lankarani has called the properties (Sk) and (Skw) of the set $\mathcal{A}$ by “spherically compactness ” and “weak spherically compactness ” respectively. More exactly, it has been proved in the paper [ML-W2005]) that (Sk) $\Rightarrow$ (Le) and (Skw) $\Leftrightarrow$ (Le) *for any compact set* $\mathcal{A}$ *in any Banach space.*

Let finally formulate the useful sufficient condition [Rom2000-2001]: if attractor $\mathcal{A}$ is contained in a finite-dimensional C^1-submanifold $\mathcal{M} \subset X^\theta$ then LFDA holds. *We do not suggest here* $\mathcal{M}$ *to be invariant.* Remark that it follows from properties (Le), (GrF) that attractor is the part of a finite-dimensional Lipschitz submanifold $\mathcal{M} \subset X^\theta$.

[EFNT1994] A. Eden, C. Foias, B. Nicolaenko, R. Temam. Exponential Attractors for Dissipative Evolution Equations. Wiley: New York.
[Rob1999] J.C. Robinson. Global attractors: topology and finite-dimensional dynamics. J. Dyn. Differ. Eq., 11:3, 557–581.
[Rom 2000] A.V. Romanov. Finite-dimensional limiting dynamics for dissipative parabolic equations. Sbornik: Mathematics, 191:3, 415-429.
[Rom2001] A.V. Romanov. Finite-dimensional dynamics on attractors of non-linear parabolic equations. Izvestia: Mathematics, 65:5, 977-1001.
[Rom2006] A.V. Romanov. Effective finite parametrization in phase spaces of parabolic equations. Izvestia: Mathematics, 70:5, 1015-1029.
[Rob2003] J. C. Robinson. Attractors and finite-dimensional behavior in the Navier- Stokes equations. Instructional Conf. Math. Anal. of Hydrodynamics.
[PM-Rob2010] E. Pinto de Moura, J.C. Robinson. Log-Lipschitz continuity of the vector field on the attractor of certain parabolic equations, arXiv:1008.4949v1 [math.AP], 29 Aug.
[CFKM1997] P. Constantin, C. Foias, I. Kukavica, A. Majda. Dirichlet quotients for periodic 2 dimensional Navier–Stokes equations. J. Math Pure Appl., 76:2, 125–153.
[ML-W2005] H. Movahedi-Lankarani, R. Wells. On bi-Lipschitz embeddings. Portugaliae Mathematica, 62:3, 247–268.

5. THE EXAMPLES OF SPEs WITH PROPERTY LFDA

The property LFDA looks as successful replacement of the IM’s conception but at present there are no much examples equations (1-1) for which the existing of inertial manifold is not known but which are demonstrating LFDA. There are for example [Rom2001] PDEs

$$u_t = d\,u_{xx} + f(x,u,u_x), \quad x \in (0,1), \quad d > 0 \qquad \text{(5-1)}$$

in the suitable phase space with the smooth function f and standard (Sturm or periodic) boundary conditions. The considerations are based on nonlinear version cone condition with using the criterion of (Lle) and the Liouville transformation of the linearized equation.

Independently I. Kukavica has obtained [Kuk2003] the Fourier embedding property (GrF) to the equation (5-1) in the periodic case. Moreover he has obtained this property for dissipative equations of the form

$$u_t + (-1)^m u^{(2m)} = f(x,u,u_x,...u^{(2m-2)}) \quad (m \ge 1)$$

on the circle. The arguments in [Kuk2003] connected with boundness of Dirichlet quotient and a version of Liouville transformation are simpler that one's in [Rom2001] but give more particular result for equation (5-1). Also it was noted in [Rom2001] that property LFDA takes place for systems 1d equations on form

$$u_t^j \;=\; (d(x)u_x^j)_x \;+\; f_j(x,u,u_x), \quad 1 \le j \le n$$

with Dirichlet boundary condition and any smooth function $d(x) > 0$. Some modification of the Liouville transformation [Kam1992] must be used in this case.

[Rom2001] A.V. Romanov. Finite-dimensional dynamics on attractors of non-linear parabolic equations. Izvestia: Mathematics, 65:5, 977-1001.
[Kuk2003] I. Kukavica. Fourier parametrization of attractors for dissipative equations in one space dimension. J. Dyn. Differ. Eq., 15:2/3, 473-484.
[Kam1992] D.A. Kamaev. Families of stable manifolds of invariant sets of systems of parabolic equations. Russ. Math. Surveys, 47:5, 185–186.

6. THE LOG-LIPSCHITZ FINITE-DIMENSIONAL DYNAMICS ON ATTRACTOR

The following weakening of the property LFDA due E. Pinto de Moura and J. Robinson [PM-Rob2010] seems very perspective. Suppose that the vector field $G(u)$ is η-log-Lipschitz on attractor in X^θ-norm, i.e.

$$\left|G(u)-G(v)\right|_\theta \le C_1\left|u-v\right|_\theta \ln^\eta \frac{M}{\left|u-v\right|^\theta} \qquad (u,v \in \mathcal{A}),$$

and exists the linear embedding $L: X^\theta \to R^n$ with γ-log-Lipschitz inverse on the image $L(\mathcal{A})$. Recently the possibility of such embedding has been proved ([Ol-Rob2010], [Rob2010]) with any $\gamma > 1/2$ in assumption that the set $\mathcal{A} - \mathcal{A}$ is homogeneously, i.e. has the finite Bouligand-Assouad dimension. If here the inequality $\eta + \gamma \le 1$ holds then the phase dynamics of SPE (1-1) on attractor is described by suitable ODE in R^n with unique solutions. One can speak in this case about log-Lipshcitz finite-dimensional dynamics on attractor (the property log-LFDA) of the equation (1-1).

There are a few facts which testify that the LFDA or log-LFDA may be more general properties of SPEs then the existing of inertial manifold. I shall enumerate its.

1). In fact yet in the paper [Lad1972] (see [Rom2000] too) for a class SPEs containing 2d N-S on torus was obtained the estimate

$$\left|\Phi_\tau u - \Phi_\tau v\right|_\theta^\kappa \ge c(\tau)\left|u-v\right|_\theta \;\; (\kappa = e^{-k\tau}, \;\; \tau > 0)$$

on $\mathcal{A}$ with some $k = k(\mathcal{A})$ (the «almost» property (Fl0)).

2) Accordingly to the resent result [Kuk2007] (see [PM-Rob2010] too) if $\theta \le 1/2$ then

$$\left| A^{1/2} w \right| \le C_1 |w| \log^{1/2} \frac{M}{|w|} \qquad (w \in \mathcal{A}_0). \qquad (6\text{-}1)$$

If attractor $\mathcal{A}$ is bounded in X^1, $\theta = 1/2$ and $F \in \mathrm{Lip}(X^{1/2}, X^1)$ then ([PM-Rob2010]) the estimate

$$|Aw| \le C_2 |w| \log \frac{M}{|w|} \qquad (w \in \mathcal{A}_0) \qquad (6\text{-}2)$$

follows from (6-1). The inequality

$$|G(u) - G(v)| \le C_3 |u - v| \log \frac{M}{|u - v|} \qquad (u, v \in \mathcal{A})$$

follows from (6-2) easily (the constants M, C_i depending from $\mathcal{A}$ only). Three last estimates are the «almost» properties (Em) and (Vf).

3) Accordingly to results of works [PM-Rob2010], [Ol-Rob2010], if the inequality (6-2) is valid then for any positive $\kappa < 1$ exists a linear embedding $L : X^\theta \to R^n$ with $n = n(\kappa)$ such that

$$|u - v|_\theta \le C_\kappa |Lu - Lv|_{R^n}^\kappa \qquad (u, v \in \mathcal{A}).$$

This is the «almost» property (Le).

[PM-Rob2010] E. Pinto de Moura, J.C. Robinson. Log-Lipschitz continuity of the vector field on the attractor of certain parabolic equations, arXiv:1008.4949v1 [math.AP] 29 Aug.
[Ol-Rob2010] E.J. Olson, J.C. Robinson. Almost bi-Lipschitz embeddings and almost homogeneous sets. Trans. Amer. Math. Soc., 362:1, 45-168.
[Rob2010] J.C. Robinson. Log-Lipschitz embeddings of homogeneous sets with sharp logarithmic exponents and slicing the unit cube, arXiv:1007.4570v [math.MG] 26 Jul.
[Lad1972] O. Ladyzhenskaya. On the dynamical system generated by the Navier–Stokes equations (in English, 1975). J. Soviet Math., 3:4, 458-479.
[Rom 2000] A.V. Romanov. Finite-dimensional limiting dynamics for dissipative parabolic equations. Sbornik: Mathematics, 191:3, 415-429.
[Kuk2007] I. Kukavica. Log-log convexity and backward uniqueness. Proc. Amer. Math. Soc., 35:8, 2415-2421.

7. THE IMPROVEMENT OF THE KUKAVICA ESTIMATE

I formulate the following statement improving the Kukavica estimate (6-1) if $\theta < 1/2$ and actually repeating it for $\theta = 1/2$. Such improvement is interesting in connection the property log-LFDA.

THEOREM 7.1. *If* $\theta \in [0, 1/2]$ *and* $\alpha = \dfrac{1}{2(1-\theta)}$ *then*

$$\left| A^{1/2} w \right| \le \sqrt{d_\alpha}\, |w| \, (\ln \frac{M^2}{|w|^2})^{\alpha/2} \tag{7-1}$$

for $w \in \mathcal{A}_0$ *with* $d_\alpha = \sqrt{\dfrac{1-\theta}{\alpha-\theta}} (2K^2)^{\frac{1}{2(1-\theta)}}$ *and* $M = m\sqrt{e}$, $m = \operatorname{diam} \mathcal{A}$ *in* X.

In particular:

$$\alpha = \frac{1}{2} \text{ for } \theta = 0, \quad \alpha = \frac{2}{3} \text{ for } \theta = \frac{1}{4}, \quad \alpha = 1 \text{ for } \theta = \frac{1}{2}$$

and the estimate

$$\left| A^{1/2} w \right| \le 2K |w| \ln^{1/4} \frac{M^2}{|w|^2} \qquad (w \in \mathcal{A}_0)$$

holds in the case $\theta = 0$. Acting now as in [PM-Rob2010] one can get the following statement.

COROLLARY 7.2. *If* $\theta = 0$ *and* $F \in \mathrm{Lip}(X^{1/2}, X^{1/2})$ *then*

$$|Aw| \le C |w| \log^{1/2} \frac{M}{|w|} \qquad (w \in \mathcal{A}_0)$$

with $C = C(\mathcal{A})$.

Let as denote $R = F(u) - F(v)$ for $u, v \in X^\theta$.

LEMMA 7.3. *It takes place the estimate*

$$|w(t)| \le |w(\tau)| e^{k(t-\tau)} \qquad (t > \tau, \quad k = \frac{K^2}{4\lambda_1^{1-2\theta}})$$

for the difference $w(t) = u(t) - v(t)$ *of any solutions of equation* (1-1).

Proof. Multiplying (1-1) scalarly on w and taking into account relation (1-2) we can write

$$\frac{1}{2}\frac{d}{dt}|w|^2 = -(A^{1-2\theta} A^\theta w, A^\theta w) + (R, w) \le -\lambda_1^{1-2\theta} \left| A^\theta w \right|^2 + K \left| A^\theta w \right| |w|$$

where

$$K\left|A^{\theta}w\right|\left|w\right| \le \lambda_1^{1-2\theta}\left|A^{\theta}w\right|^2 + \frac{K^2}{4\lambda_1^{1-2\theta}}\left|w\right|^2.$$

Also we obtain the inequality

$$\frac{1}{2}\frac{d}{dt}\left|w\right|^2 \le \frac{K^2}{4\lambda_1^{1-2\theta}}\left|w\right|^2$$

and lemma 7.3 follows.

Proof the theorem 7.1. Let $\mu = \dfrac{(w, Aw)}{(w, w)}$ be classical Dirichlet quotient for $w \in \mathcal{A}_0$ and $L = \ln\dfrac{M^2}{\left|w\right|^2}$. Note that $L \ge 1$. If to consider for any $\alpha \in [0,1]$ the log-Dirichlet quotient $Q(t) = \dfrac{\mu(t)}{L^{\alpha}(t)}$ (really $Q(t) = Q(t; w_0)$, $w_0 \in \mathcal{A}_0$, $w(0) = w_0$) then [Kuk2007, p.2417] the inequality

$$Q_t + \alpha\frac{Q^2}{L^{1-\alpha}} \le \frac{2\left|R\right|^2}{\left|w\right|^2 L^{\alpha}} \tag{7-2}$$

holds. Accordingly (1-2) we have the estimate $\left|R\right| \le K\left|A^{\theta}w\right|$. There follows from interpolation inequality

$$\left|A^{\theta}w\right| \le \left|w\right|^{1-2\theta}\left|A^{1/2}w\right|^{2\theta}$$

that

$$\left|R\right|^2 \le K^2\left|A^{\theta}w\right|^2 \le K^2\left|w\right|^{2-4\theta}\left|A^{1/2}w\right|^{4\theta}$$

and

$$\frac{\left|R\right|^2}{\left|w\right|^2} \le K^2\frac{\left|A^{1/2}w\right|^{4\theta}}{\left|w\right|^{4\theta}} = K^2\mu^{2\theta}.$$

Taking into account the last relation one can obtain from (7-2) the estimate

$$Q_t + \alpha\frac{Q^2}{L^{1-\alpha}} \le \frac{2K^2}{L^{\alpha-2\alpha\theta}}Q^{2\theta} .$$

Postulating equality $1-\alpha = \alpha - 2\alpha\theta$ we have $\alpha(1-\theta) = \dfrac{1}{2}$ for values $\theta \in [0, 1/2]$ and $\alpha \in [1/2, 1]$ (remark that $\alpha > \theta$). Thus,

$$Q_t + \alpha b(t)Q^2 \le 2K^2 b(t)Q^{2\theta}$$

for every $\theta \in [0, 1/2]$ with $\alpha = \dfrac{1}{2(1-\theta)}$ and $b(t) = \dfrac{1}{L^{1-\alpha}(t)}$. Using the Young inequality

$$2K^2 Q^{2\theta} \le \frac{Q^2}{p} + \frac{(2K^2)^q}{q}$$

with $p = \dfrac{1}{\theta}$ and $q = \dfrac{1}{1-\theta}$ we get the estimates

$$Q_t + (\alpha - \theta) b(t) Q^2 \le (1-\theta) b(t) (2K^2)^{\frac{1}{1-\theta}} \tag{7-3}$$

for the family functions $Q(t) = Q(t; w_0)$. Suppose for the clarity $d_\alpha = d$ and apply to the analysis of inequality (7-3) the typical arguments (see [Temam1997] for example). As $Q(t) \equiv d$ is the solution of the differential equation corresponding to (7-3) then $Q(t) \le d$ on $[0, +\infty)$ when $Q(0) \le d$. If $Q(0) > d$ then $Q(t) > d$ on $(-\infty, 0)$ and we have

$$\frac{dz}{dt} \le -(\alpha - \theta) b(t) z^2$$

for the positive value $z(t) = Q(t) - d$. Integrating this inequality and denoting $a(s) = (\alpha - \theta) \int_s^0 b(t)\, dt$ we found:

$$\int_{z(s)}^{z(0)} \frac{dt}{t^2} \le -a(s), \quad \frac{1}{z(0)} - \frac{1}{z(s)} \ge a(s), \quad z(0) \le \frac{z(s)}{1 + z(s) a(s)} \le \frac{1}{a(s)}.$$

Returning to the variable Q we obtain the estimate

$$Q(0) \le d + \frac{1}{a(s)}. \tag{7-4}$$

for $s < 0$. Further accordingly lemma 7.3 we have

$$|w(0)| \le |w(s)| e^{-ks}, \quad |w(s)|^{-1} \le |w(0)|^{-1} e^{-ks}$$

for $s < 0$. Consequently,

$$L(s) \le \ln(M^2 |w(0)|^{-2} e^{-2ks}) = L(0) - 2ks$$

and

$$b(s) \ge \frac{1}{(L(0) - 2ks)^{1-\alpha}}.$$

As $\alpha \in [1/2,1]$ then independently from the choice of initial points $w_0 \in \mathcal{A}_0$ the integrals $\int_{-\infty}^{0} b(s)\,ds$ diverge hence $a(s) \to +\infty$ as $s \to -\infty$ and $Q(0) \le d_\alpha$ from (7-4). After extracting a square root we obtain the founded estimate (7-1) and the theorem 7.1 is proved.

[Kuk2007] I. Kukavica. Log-log convexity and backward uniqueness.
Proc. Amer. Math. Soc., 35:8, 2415-2421.
[Temam1997] R. Temam. Infinite-dimensional Dynamical Systems in Mechanics and Physics.
Appl. Math. Sci. 68, Berlin: Springer, 2-nd ed.

8. THE FINITE-DIMENSIONAL HYPERBOLIC DYNAMICS

I want to remind here one old and undeservedly forgotten result of D. Kamaev [Kam1980] which seems interesting in the context my talk's theme. It needs note in this connection that all results about the finite-dimensional dynamics on attractor being formulate above are valid for any compact invariant set (CI-set) $\mathcal{K} \subset \mathcal{A}$.

Let P_n be the spectral projector of linear part A of SPE (1-1) corresponding its n (with multiplicity) low modes. Assuming that $p = P_n u$ for $u \in X^\theta$ consider Galerkin's approximations

$$p_t = -Ap + P_n F(p). \tag{8-1}$$

For any fixed n it is ODE in R^n with the Lipschitz vector field and the flow $S_n(t)$.

THEOREM 8.1 [Kam1980]. *If* $\mathcal{K}$ *be the hyperbolic CI-set of SPE* (1-1) *then for* $n \ge n_0$ *exist the hyperbolic* CI-*sets* $\mathcal{K}_n$ *of* ODE (8-1) *and the homeomorphisms* $h_n : \mathcal{K} \to \mathcal{K}_n$ *satisfying* (*for* $t > 0$) *the following conditions*:

$$1)\ S_n(t)\circ h_n = h_n \circ \Phi_t \ \ on\ \ \mathcal{K};\quad 2)\ \left|h_n(u) - u\right|_\theta \le c_n \ \ on\ \ \mathcal{K}\ \ and\ \ c_n \to 0.$$

It follows from this statement that the resolving semiflow Φ_t is injective on hyperbolic CI-set $\mathcal{K}$ and the semiflows Φ_t on $\mathcal{K}$ and $S_n(t)$ on $\mathcal{K}_n$ are topologically adjoined.

The cited paper do not contain the proof but the one contained in PhD dissertation of D. Kamaev (1980, in Russian).

Of course the most interesting case here is $\mathcal{K} = \mathcal{A}$. It is known that the moving on the (nontrivial) hyperbolic attractor must be chaotically. It is striking that hyperbolic limiting dynamics of SPE (when it takes place) is equivalent the one of ODE. The nontrivial hyperbolic attractor is clearly the large rarity in SPE's dynamics, but one can hope that the strong (classical) hyperbolicity in the theorem 8.1 may be replaced to some weaker one.

[Kam1980] D.A. Kamaev. Hyperbolic limit sets of evolutionary equations and the Galerkin
method. Russ. Math. Surveys, 35:3, 239-243.

9. WHAT MAY BE DONE IN FURTHER

1. The main target must be the case of zero exponent of nonlinearity ($\theta = 0$) being typical for scalar or vector reaction-diffusion equations. Corresponding reasons are the following.

(A) Accordingly corollary 7.2 the estimate

$$|A(u-v)| \le C|u-v| \log^{1/2} \frac{M}{|u-v|}$$

holds on attractor $\mathcal{A}$. If somebody will establish the estimate

$$\left| L^{-1}(x-y) \right| \le C|x-y|_{R^n} \log^{\gamma} \frac{M}{|x-y|_{R^n}} \qquad (\gamma \le 1/2) \qquad \text{(9-1)}$$

on the image $L\mathcal{A}$ for some linear (injective on $\mathcal{A}$) embedding $L: X \to R^n$ then the property log-LFDA be hold. At present one can obtain any $\gamma > 1/2$ in (9-1) if to prove the set $\mathcal{A} - \mathcal{A}$ be homogeneous, i.e. having the finite Bouligand-Assouad dimension (see [Ol-Rob2010], [Rob2010]). The last would be the «almost» property log-LFDA.

(B) The scalar reaction-diffusion equations

$$u_t = d\Delta u + f(x,u) \qquad \text{(9-2)}$$

in bounded domains $D \subset R^N$ ($N \ge 2$) with smooth f are gradient-similar and admit the strict Lyapounov function

$$G(u) = \int_D \left(\frac{1}{2}|\nabla u(x)|^2 - F(x,u)\right) dx,$$

where $F_u(x,u) = f(x,u)$. It is well known [Henry1981], [Temam1997] that in generic case the attractor of equation (9-2) consists of the finite number hyperbolic equilibrium points and its (smooth) unstable manifolds. Moreover in generic case the stable (smooth too!) and unstable manifolds of equilibrium points intersect transversally (see recent review [J-R2011] by R. Joly and G. Raugel). Lastly, the property LFDA holds [Rom2000-2001] if the attractor is contained in a finite-dimensional C^1- submanifold in phase space. All this arguments seems be useful to proving the properties LFDA or log-LFDA for scalar reaction-diffusion equations in arbitrary dimension.

2. It would be interesting to construct the example of SPE without property LFDA or log-LFDA. It needs note in this connection that even more simple examples of the smooth IM absence are constructing with the hard. A SPE do not possess properties LFDA and log-LFDA if its resolving semiflow is not injective on attractor but now does not known the similar equation for which backward uniqueness be false. It is strikingly, but at present still unknown the example of SPE without Lipschitz IM!

3. I seem the properties LFDA or log-LFDA must be valid (may be in some additional conditions) for SPEs (1-1) with analytic nonlinear term (2d N-S, for example). In this case [Henry1981] the resolving semiflow is jointly analytic on time and on phase variable. To certain PDEs the solutions $u(t)$ are analytic on space variable too [Foi-Tem1989]. It may be showed (doing as in [Prom1991], for example) that the solutions $u(t), v(t)$ lying on attractor $\mathcal{A}$ continue analytically and uniformly boundedly to the strip $D_\delta : |\operatorname{Im} z| < \delta,\ z = t + i\tau$, $\delta = \delta(\mathcal{A})$. The complex analysis arguments may be apply now to the difference $w(z) = u(z) - v(z)$ and the following statement may be got.

PROPOSITION 9.1. *Suppose may be found values* $\sigma > 1,\ 0 < t_0 < \delta/\sigma$ *and* $M = M(\mathcal{A})$ *for which the estimates*

$$\left\| \frac{d^k w}{dt^k}(t_0) \right\|_\theta \le \frac{M\,k!}{(\sigma t_0)^k} \| w(0) \|_\theta \tag{9-3}$$

take place for all $k \ge 1,\ w = u - v\ (u, v \in \mathcal{A})$. *Then the property* LFDA *holds for* SPE (1-1).

Really the estimate $\| \Phi_{-\tau} u - \Phi_{-\tau} v \|_\theta \le M_1 \| u - v \|_\theta$ with $\tau = (\sigma - 1) t_0 / 2$ is obtaining here and criterion (Fl0) is using. We emphasize that at least in the case $\theta = 0$ the inequalities of a type (9-3) with $\sigma = 1$ follow automatically from the Cauchy formula and the estimate $| w(z) |_\theta \le \mathrm{const} \cdot | \mathrm{w}(0) |_\theta$ being valid in the rectangle $0 < \operatorname{Re} z < 2t_0$, $|\operatorname{Im} z| < \delta$.

[Ol-Rob2010] E.J. Olson, J.C. Robinson. Almost bi-Lipschitz embeddings and almost homogeneous sets. Trans. Amer. Math. Soc., 362:1, 45-168.

[Rob2010] J.C. Robinson. Log-Lipschitz embeddings of homogeneous sets with sharp logarithmic exponents and slicing the unit cube, arXiv:1007.4570v [math.MG] 26 Jul.

[Henry1981] D. Henry. Geometric Theory of Semilinear Parabolic Equations. Lect. Notes in Math. 840, Berlin: Springer.

[Temam1997] R. Temam. Infinite-dimensional Dynamical Systems in Mechanics and Physics.

[Rom 2000] A.V. Romanov. Finite-dimensional limiting dynamics for dissipative parabolic equations. Sbornik: Mathematics, 191:3, 415-429.

[Rom2001] A.V. Romanov. Finite-dimensional dynamics on attractors of non-linear parabolic equations. Izvestia: Mathematics, 65:5, 977-1001.

[J-R2011] R. Joly, G. Raugel. A striking correspondence between the dynamics generated by the vector fields and by the scalar parabolic equations, arXiv:1005.1727v2 [math.AP], 14 Jan.

[Foi-Tem1989] C. Foias, R. Temam. Gevrey class regularity for the solutions of the Navier- Stokes equations. J. Funct. Anal., 87:2, 359-369.

[Prom1991] K. Promislow. Time analyticity and Gevrey regularity for solutions of a class of dissipative partial differential equations. Nonl. Anal., TMA, 16:11, 959-980.