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Is the function field of a reductive Lie algebra purely transcendental over the field of invariants for the adjoint action?

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Abstract

Let k be a field of characteristic zero, let G be a connected reductive algebraic group over k and let $\mathfrak{g}$ be its Lie algebra. Let $k(G)$, respectively, $k(\mathfrak{g})$, be the field of k-rational functions on G, respectively, $\mathfrak{g}$. The conjugation action of G on itself induces the adjoint action of G on $\mathfrak{g}$. We investigate the question whether or not the field extensions $k(G)/k(G)^G$ and $k(\mathfrak{g})/k(\mathfrak{g})^G$ are purely transcendental. We show that the answer is the same for $k(G)/k(G)^G$ and $k(\mathfrak{g})/k(\mathfrak{g})^G$, and reduce the problem to the case where G is simple. For simple groups we show that the answer is positive if G is split of type A_n or C_n, and negative for groups of other types, except possibly G_2. A key ingredient in the proof of the negative result is a recent formula for the unramified Brauer group of a homogeneous space with connected stabilizers. As a byproduct of our investigation we give an affirmative answer to a question of Grothendieck about the existence of a rational section of the categorical quotient morphism for the conjugating action of G on itself.

Contents

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