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Modern Developments in Invariant Theory

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Invariant theory, which is almost one and a half centuries old, is now going through a new period of vigorous development: a satisfactory comprehension of a general picture within the classical situation has been attained and the new important directions, methods, and applications are being discovered. Now, as a matter of fact, invariant theory is identified with the theory of algebraic group actions on algebraic varieties (schemes) whose local aspect (concerned with the actions on affine varieties) is, on the one hand, the most immediately connected with the classical understanding of invariant theory and, on the other, the most elaborated.

I have to restrict myself here mostly to some results of principal importance that form a group around the classical theme of description of invariants and covariants, and I hope that the addresses by C. de Concini and W. Borho will give an idea of some other up-to-date aspects and applications of invariant theory.

1. Classical problems. Let k be an algebraically closed field, V a finite-dimensional vector space over k, $k[V]$ the algebra of regular functions (polynomials) on V, and G an algebraic subgroup of $\mathrm{GL}(V)$. G acts on $k[V]$.

The main problem of the classical theory is to give "the explicit description" of the algebra of invariants $k[V]^G$. This presupposes (a) solving the question of whether $k[V]^G$ is finitely generated over k and, if yes, (b) giving a constructive way to find explicitly a minimal system of homogeneous generators of $k[V]^G$ (or, what is ideally desirable, even to exhibit such a system actually).

The problem of describing all G such that $k[V]^G$ is finitely generated is known as the original Hilbert's fourteenth problem. (In fact Hilbert's formulation at the 1900 Paris Congress was somewhat different.) In his address at the 1958 Edinburgh Congress, M. Nagata gave an example of such G that $k[V]^G$ is not finitely generated. Actually it is more natural to consider a general situation when G rationally acts on an arbitrary affine (i.e., finitely generated and reduced) k-algebra A or, geometrically, when G algebraically acts on an affine algebraic variety X with the coordinate algebra $k[X] = A$, see [1, 2]. M. Nagata [1]