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Automorphism groups of finite dimensional simple algebras

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Abstract

We show that if a field k contains sufficiently many elements (for instance, if k is infinite), and K is an algebraically closed field containing k, then every linear algebraic k-group over K is k-isomorphic to $\mathrm{Aut}(A \otimes_k K)$, where A is a finite dimensional simple algebra over k.

1. Introduction

In this paper, 'algebra' over a field means 'nonassociative algebra', i.e., a vector space A over this field with multiplication given by a linear map $A \otimes A \to A$, $a_1 \otimes a_2 \mapsto a_1 a_2$, subject to no *a priori* conditions; cf. [Sc].

Fix a field k and an algebraically closed field extension K/k. Our point of view of algebraic groups is that of [Bor], [H], [Sp]; the underlying varieties of linear algebraic groups will be the affine algebraic varieties over K. As usual, algebraic group (resp., subgroup, homomorphism) defined over k is abbreviated to k-group (resp., k-subgroup, k-homomorphism). If E/F is a field extension and V is a vector space over F, we denote by V_E the vector space $V \otimes_F E$ over E.

Let A be a finite dimensional algebra over k and let V be its underlying vector space. The k-structure V on V_K defines the k-structure on the linear algebraic group $\mathrm{GL}(V_K)$. As $\mathrm{Aut}(A_K)$, the full automorphism group of A_K, is a closed subgroup of $\mathrm{GL}(V_K)$, it has the structure of a linear algebraic group. If $\mathrm{Aut}(A_K)$ is defined over k (that is always the case if k is perfect; cf. [Sp, 12.1.2]), then for each field extension F/k the full automorphism group $\mathrm{Aut}(A_F)$ of F-algebra A_F is the group $\mathrm{Aut}(A_K)(F)$ of F-rational points of the algebraic group $\mathrm{Aut}(A_K)$.

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