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ON 2-DIFFEOMORPHISMS WITH ONE-DIMENSIONAL BASIC SETS AND A FINITE NUMBER OF MODULI

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ABSTRACT. This paper is a step towards the complete topological classification of Ω-stable diffeomorphisms on an orientable closed surface, aiming to give necessary and sufficient conditions for two such diffeomorphisms to be topologically conjugate without assuming that the diffeomorphisms are necessarily close to each other. In this paper we will establish such a classification within a certain class Ψ of Ω-stable diffeomorphisms defined below. To determine whether two diffeomorphisms from this class Ψ are topologically conjugate, we give (i) an algebraic description of the dynamics on their non-trivial basic sets, (ii) a geometric description of how invariant manifolds intersect, and (iii) define numerical invariants, called moduli, associated to orbits of tangency of stable and unstable manifolds of saddle periodic orbits. This description determines the *scheme* of a diffeomorphism, and we will show that two diffeomorphisms from Ψ are topologically conjugate if and only if their schemes agree.

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1. INTRODUCTION AND FORMULATION OF THE RESULTS

The topological classification of structurally stable diffeomorphisms on closed orientable surfaces has made tremendous progress in the last 25 years, due to the work of C. Bonatti, V. Grines, R. Langevin, A. Zhirov, R. Plykin et al. (see for example [1], [3], [2], [9] for the history of the subject and more information). Any such classification naturally includes a description of its basic sets and a non-trivial description of how invariant manifolds of periodic points intersect. If invariant manifolds of saddles have tangencies, then the topological classification also involves expressions, called *moduli*, related to eigenvalues at saddle points, as was discovered by J. Palis [14].

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The first important step in the direction of a topological classification of Ω-stable diffeomorphisms on orientable closed surfaces was made in the paper [11] by W. de Melo and S. J. van Strien, where they found necessary and sufficient conditions for Ω-stable diffeomorphisms to have a *finite number of moduli.* (A diffeomorphism f is said to have a finite number of moduli if one can parametrise topological conjugacy classes of a neighbourhood of f by a finite number of parameters). Their result is local in the sense that it only considers the topological conjugacy of two diffeomorphisms which are sufficiently close to each other. To deal with the global situation, T. Mitryakova and O. Pochinka [12] partly generalised the previous result by construction a complete invariant for Ω-stable diffeomorphisms for a certain class of Ω-stable diffeomorphisms (with at most a finite number of periodic points) which can in general be "far" from each other.

Here we present the topological classification considering a wider class than in [12], within this class the existence of one-dimensional attractors and repellers is allowed. This class Ψ will be defined formally below.

Let M^2 be an orientable closed surface and $f\colon M^2 \to M^2$ be an A-diffeomorphism, i.e., an Axiom A diffeomorphism. By S. Smale [17], the non-wandering set $NW(f)$ of f is represented as a finite union of disjoint closed invariant sets $\Lambda_1, \ldots, \Lambda_k$, called *basic sets*, each of which contains a dense orbit. A basic set which consists of a periodic orbit will be called *trivial* and otherwise it is called *non-trivial.*

Let Λ be a one-dimensional basic set of f. By R. Plykin [15], Λ is either an attractor or a repeller.

According to [5, Definition 3] a point p is called an *s-boundary* (*u-boundary*) *point* of attractor (repeller) Λ, if one of the connected components of the set $W^s_p \setminus p$ ($W^u_p \setminus p$) is disjoint from Λ; denote by ℓ_p such a component (see Figure 1, where the construction of a DA-diffeomorphism is represented and where p_1, p_2 are the s-boundary points).

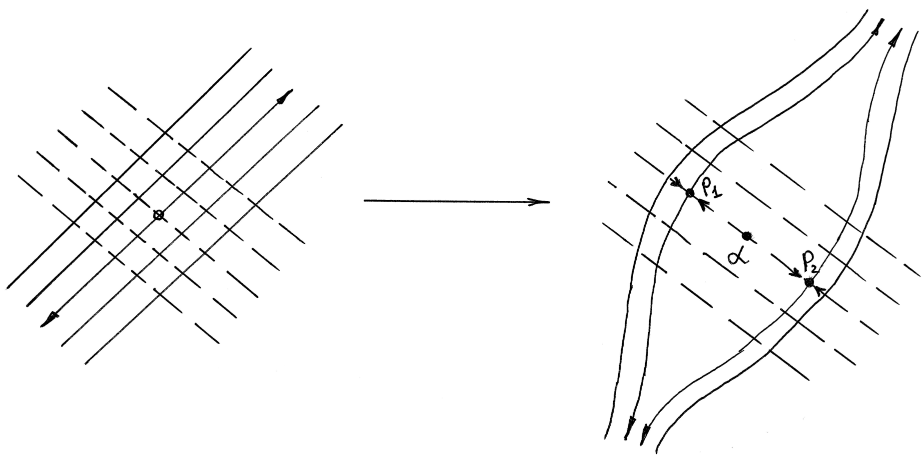

Figure 1. DA-diffeomorphism

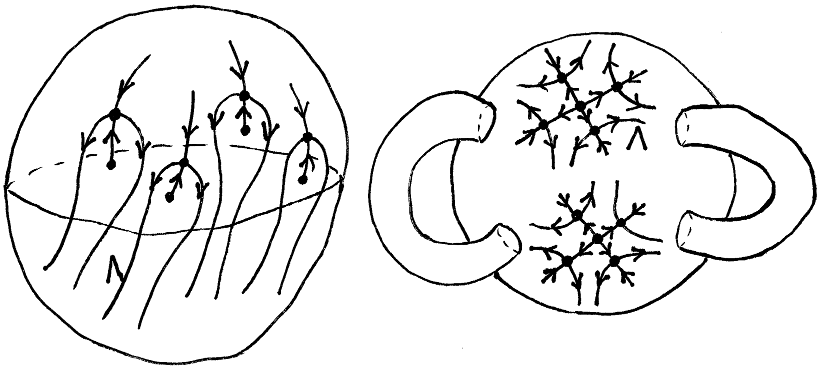

FIGURE 2. Other examples of 1-dimensional attractors

For definiteness we suppose that Λ is an attractor (all notions for repeller can be obtained by considering f^{-1}). By [6, Lemmas 2.4, 2.5], each s-boundary point is necessarily periodic and the set Λ has a non-empty and finite set of s-boundary points[1]. We denote this set by P_Λ.

Definition 1 (Separable one-dimensional attractors)**.** We say that a 1-dimensional attractor Λ of an A-diffeomorphism f is *separable* if a union Y_Λ of saddle and source trivial basic sets of the diffeomorphism f exists with the following properties:

1) $\operatorname{cl}(W^s_\Lambda) \setminus W^s_\Lambda = W^s_{Y_\Lambda}$;

2) $\operatorname{cl}(\ell_p) \setminus \ell_p = p \cup \alpha$ for every s-boundary point $p \in P_\Lambda$, where $\alpha \in Y_\Lambda$ is a source point;

3) for every saddle point $\sigma \in Y_\Lambda$ the manifold W^s_σ does not contain heteroclinic points.

This definition is illustrated in Figure 3.

It follows from [7, Lemma 1, Lemma 2] that any one-dimensional basic set of a structural stable diffeomorphism $f\colon M^2 \to M^2$ is separable. We prove the following stronger result.

Theorem 1. *If an Ω-stable diffeomorphism $f\colon M^2 \to M^2$ has a finite number of moduli then any of its one-dimensional basic set is separable.*

The proof of Theorem 1 is based on necessary and sufficient conditions, found in [11], under which a diffeomorphism of an orientable surface has a finite number of moduli of topological conjugacy, and described the structure of the neighborhood of such a diffeomorphism.

Statement 1 (Criteria of a finite number moduli, [11])**.** *Let $f\colon M^2 \to M^2$ be an Ω-stable C^2-diffeomorphism. Then f has a finite number moduli if and only if it satisfies the conditions below*:

[1]In fact the existence and finiteness of the set of boundary points without the term "boundary point" was proved by S. Newhouse and J. Palis in [13, Proposition 1].

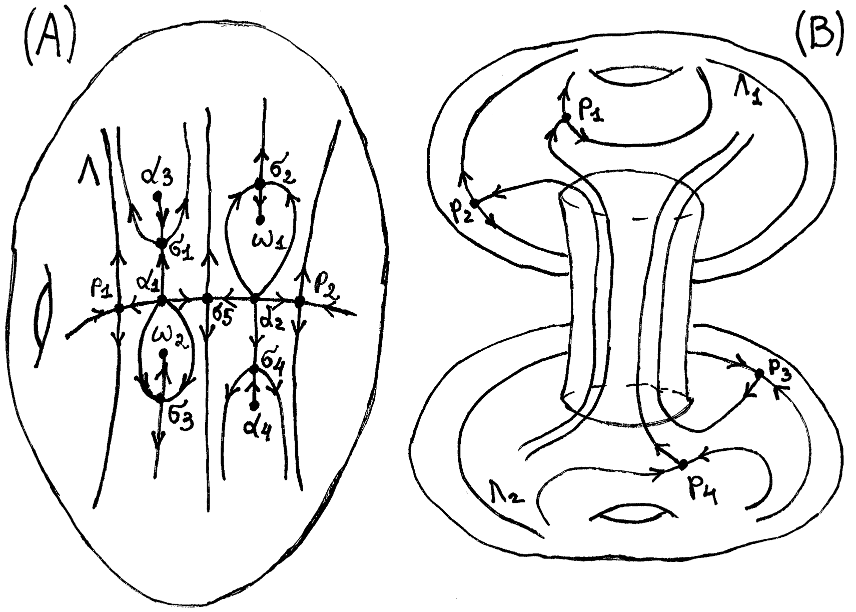

FIGURE 3. In these figures α, ω, σ denote periodic points of source, sink and saddle type. In Figure (A) a separable one-dimensional attractor on the torus is shown by the dark curves and ℓ_{p_i} are the curves connecting p_i to α_i; In Figure (B) a situation is shown with a non-seperable one-dimensional attractor on the pretzel by the dark curves; here the curves ℓ_{p_i} do not land on repelling fixed points, and so condition 2 in the definition of separable one-dimensional attractor is violated.

(1) *if $x, y \in NW(f)$ are such that W^u_x is not transverse to W^s_y then the basic sets containing x and y are trivial*;

(2) *there is only a finite number of orbits of non-transverse intersections between stable and unstable manifolds and the contact between these manifolds along each of these orbits is of finite order*;

(3) *if p, q are periodic points from trivial basic sets such that W^u_p has an orbit of non-transverse intersection with W^s_q then the number of orbits in W^s_p (resp. in W^u_q) belonging to some unstable (resp. stable) manifolds of periodic saddle points of f is finite*;

(4) *if x is a point of non-transverse intersection of W^u_p and W^s_q then there exists an arc Σ transverse to W^u_p at x such that no connected component of $\Sigma \setminus \{x\}$ contains points of both stable and unstable manifolds of saddles*;

(5) *if W^u_p has a point of non-transverse intersection with W^s_q, and W^u_q has a point of non-transverse intersection with W^s_r, then there is no saddle point of f whose unstable manifold (resp. stable manifold) intersects W^s_p (resp. W^u_r).*

Definition 2 (The class Ψ)**.** An orientation preserving Ω-stable C^2-diffeomorphism $f\colon M^2 \to M^2$ is called a diffeomorphism of class Ψ if it has a finite number of moduli and the following properties are satisfied:

1) each non-trivial basic set Λ of f is one-dimensional;

2) heteroclinic orbits can be contained in the stable or the unstable manifold of a periodic point of the trivial basic set of f, but not in both.

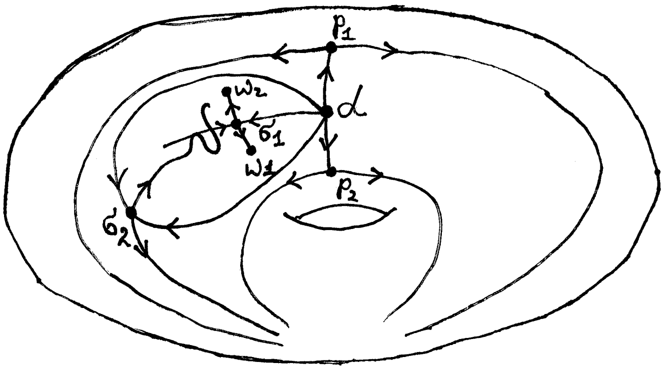

FIGURE 4. An example of a diffeomorphism f from Ψ

Let $f \in \Psi$ and W^u_x be not transverse to W^s_y for some saddle periodic points $x, y \in NW(f)$. Set $\Theta_{xy} = \frac{\ln|\lambda_x|}{\ln|\mu_y|}$, where λ_x is the eigenvalue of D_f at x which is less than one by absolute value and μ_y is the eigenvalue of Df at y which is greater than one by absolute value. Denote by Ψ^* the set of diffeomorphisms $f \in \Psi$ such that Θ_{xy} is an irrational for any such pair x, y.

In Section 2 we introduce the notion of a scheme of diffeomorphism f containing

(i) an algebraic description of the dynamics on its non-trivial basic sets,
(ii) a geometric description of how invariant manifolds intersect,
(iii) numerical invariants, called moduli, associated to orbits of tangency of stable and unstable manifolds of saddle periodic orbits

and define an equivalence of two schemes.

The main result of this paper is the following theorem.

Theorem 2 (Classification with class Ψ)**.** 1. *If the schemes of diffeomorphisms f, $f' \in \Psi$ are equivalent, then the diffeomorphisms are topologically conjugate.*

2. *Diffeomorphisms f, $f' \in \Psi^*$ are topologically conjugate if and only if their schemes are equivalent.*

2. Descriptions of Diffeomorphisms from Ψ

2.1. An algebraic description of the dynamics on one-dimensional basic set. Now let Λ be a 1-dimensional attractor of an A-diffeomorphism $f\colon M^2 \to M^2$. From [4] Λ is represented as a finite union of disjoint compact sets $\Lambda_1, \ldots, \Lambda_k$, which are cyclically transformed into each other under the action of f. Moreover,

$\mathrm{cl}(W^s_x \cap \Lambda_i) = \Lambda_i$ and $\mathrm{cl}(W^u_x \cap \Lambda_i) = \Lambda_i$ for any point $x \in \Lambda_i$. Every Λ_i is called *periodic* (or C*-dense*) *component* of the basic set Λ. In this section we suppose that the attractor Λ consists of only one periodic component. We will now associate to Λ a closed neighbourhood N_Λ.

Definition 3 (The bunch of an attractor)**.** A *bunch* b of an attractor Λ is the union of the maximal number r_b of the unstable manifolds $W^u_{p_1}, \dots, W^u_{p_{r_b}}$ of the s-boundary points $p_1, \dots, p_{r_b}$ of the set Λ whose separatrices[2] $\ell_{p_1}, \dots, \ell_{p_{r_b}}$ belong to the same connected component of the set $W^s_\Lambda \setminus \Lambda$. The number r_b is said to be the *degree of the bunch*.

Let $\delta \in \{u, s\}$ and $x \in \Lambda$. For points $y, z \in W^\delta_x$ $(y \neq z)$,let

$$[y, z]^\delta,\ \ [y, z)^\delta,\ \ (y, z]^\delta,\ \ (y, z)^\delta$$

denote the connected arcs on the manifold W^δ_x with the boundary points y, z.

Denote by B_Λ the set of all bunches of Λ. From the definition of a bunch $b \in B_\Lambda$ of degree r_b it follows that there is a sequence of points $x_1, \dots, x_{2r_b}$ such that:

1) x_{2j-1}, x_{2j} belong to the different connected components of the set $W^u_{p_j} \setminus p_j$;

2) $x_{2j+1} \in W^s_{x_{2j}}$ (we set $x_{2r_b+1} = x_1$)

3) $(x_{2j}, x_{2j+1})^s \cap \Lambda = \varnothing$, $j = 1, \dots, r_b$.

For each $j \in \{1, \dots, r_b\}$ we pick a pair of points $\tilde{x}_{2j-1}, \tilde{x}_{2j}$ and a simple curve ℓ_j with boundary points $\tilde{x}_{2j-1}, \tilde{x}_{2j}$ such that:

1) $(\tilde{x}_{2j}, \tilde{x}_{2j+1})^s \subset (x_{2j}, x_{2j+1})^s$ $(x_{2r+1} = x_1)$;

2) the curve ℓ_j transversally intersects at a unique point the stable manifold of any point on the arc $(x_{2j-1}, x_{2j})^u$;

3) $L_b = \bigcup\limits_{j=1}^{r_b} [\ell_j \cup (\tilde{x}_{2j}, \tilde{x}_{2j+1})^s]$ is a simple closed smooth curve and the set $L_\Lambda = \bigcup\limits_{b \in B_\Lambda} L_b$ is such that:

a) $f(L_\Lambda) \cap L_\Lambda = \varnothing$;

b) for every curve L_b, $b \in B_\Lambda$, there is a curve in the set $f(L_\Lambda)$ such that these two curves are the boundaries of an annulus K_b;

c) the annuli $\{K_b,\, b \in B_\Lambda\}$ are pairwise disjoint (see Figure 5).

Let $N_\Lambda = \Lambda \cup \bigcup\limits_{n \geqslant 1} f^n(\bigcup\limits_{b \in B_\Lambda} K_b)$. By construction the annuli $\{K_b,\ b \in B_\Lambda\}$ consist of the wandering points of the diffeomorphism f, N_Λ is a surface with non-empty boundary and N_Λ is a neighbourhood of the attractor Λ, which we call the support of N_Λ.

Let $p_\Lambda \colon \mathbb{U}_{N_\Lambda} \to N_\Lambda$ be the universal covering where $\mathbb{U}_{N_\Lambda}$ is a subset of Lobachevsky plane and let $\mathbb{G}_{N_\Lambda}$ be the group of its covering transformations. Set $\mathbb{E}_{N_\Lambda} = \partial \mathbb{U}_{N_\Lambda}$. A lift $\bar{f}_\Lambda \colon \mathbb{U}_{N_\Lambda} \to \mathbb{U}_{N_\Lambda}$ of $f_\Lambda = f|_{N_\Lambda}$ with respect to p_Λ induces an automorphism $T_{\bar{f}_\Lambda}$ of the group $\mathbb{G}_{N_\Lambda}$ acting by the formula $T_{\bar{f}_\Lambda}(g) = \bar{f}_\Lambda g \bar{f}_\Lambda^{-1}$. Set $\bar{\Lambda} = p_\Lambda^{-1}(\Lambda)$. If $x \in \Lambda$ then let $\bar{x} \in \bar{\Lambda}$ denote a point in the preimage $p_\Lambda^{-1}(x)$ and let $w^\delta_{\bar{x}}$ be the connected component of $p^{-1}(W^\delta_x)$ containing $\bar{x}$. Let us choose a parametrisation $\mathbb{R} \ni t \to W^\delta_x(t)$ of W^δ_x such that $W^\delta_x(0) = x$. Then $w^\delta_{\bar{x}}(t)$ is a point

[2] *Stable* (*unstable*) *separatrix* of a hyperbolic periodic point p is a connected component of the set $W^s_p \setminus p$ $(W^u_p \setminus p)$.

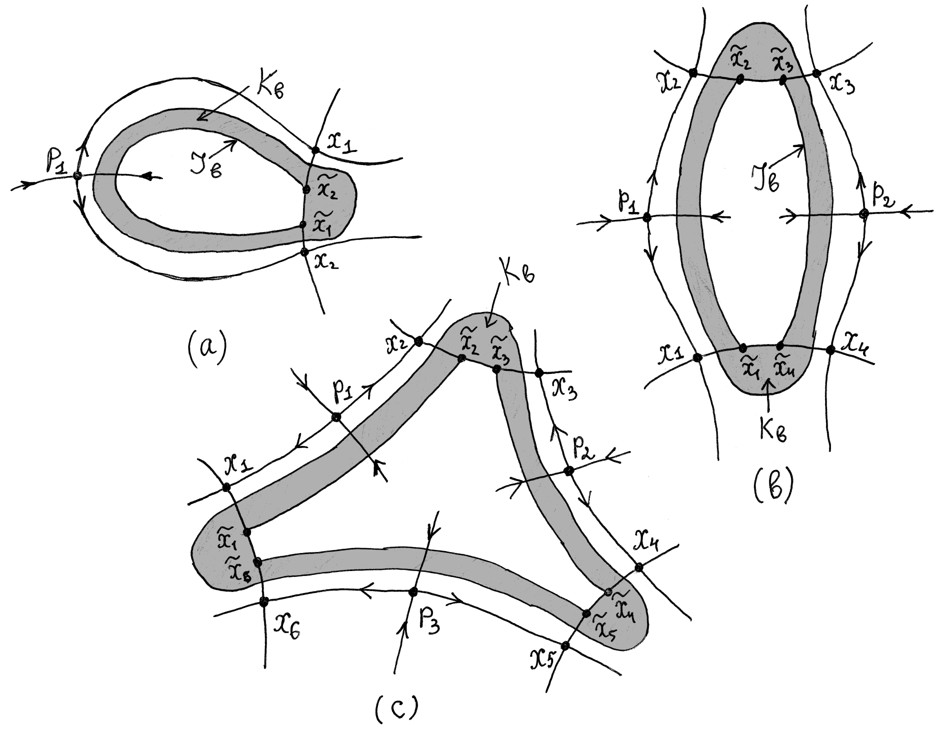

FIGURE 5. Construction of the surface N_Λ.

on $w^\delta_{\bar{x}}$ such that $p_\Lambda(w^\delta_{\bar{x}}(t)) = W^\delta_x(t)$ and $W^{\delta+}_x$, $W^{\delta-}_x$ ($w^{\delta+}_{\bar{x}}$, $w^{\delta-}_{\bar{x}}$) are the connected components of the curve $W^\delta_x \setminus x$ ($w^\delta_{\bar{x}} \setminus \bar{x}$) for $t > 0$, $t < 0$ respectively. For points $\bar{y}$, $\bar{z} \in w^\delta_{\bar{x}}$ ($\bar{y} \neq \bar{z}$), let $[\bar{y}, \bar{z}]^\delta$, $[\bar{y}, \bar{z})^\delta$, $(\bar{y}, \bar{z}]^\delta$, $(\bar{y}, \bar{z})^\delta$ denote the connected arcs on the manifold $w^\delta_{\bar{x}}$ with boundary points $\bar{y}$, $\bar{z}$.

Definition 4 (Asymptotic direction)**.** We say that a curve $w^{\delta\nu}_{\bar{x}}$ has the asymptotic direction $\delta^\nu_{\bar{x}}$ for $t \to \nu\infty$, $\nu \in \{-, +\}$ if $\mathrm{cl}(w^{\delta\nu}_{\bar{x}}) \setminus w^{\delta\nu}_{\bar{x}}$ is equal to $\bar{x}$ and $\delta^\nu_{\bar{x}} \in \mathbb{E}_{N_\Lambda} = \partial\mathbb{U}_{N_\Lambda}$.

As before, let P_Λ be the set of s-boundary points of Λ. For a boundary point $p \in P_\Lambda$ denote by ℓ^∞_p a connected component of $W^s_p \setminus p$ different from ℓ_p and by $\ell^\infty_{\bar{p}}$ the connected component of $w^s_{\bar{p}} \setminus \bar{p}$ for which $p_\Lambda(\ell^\infty_{\bar{p}}) = \ell^\infty_p$.

It was proved in [8] that for each point $x \in (\Lambda \setminus W^s_{P_\Lambda})$ the curve $w^s_{\bar{x}}$ has two distinct boundary points (asymptotic directions) $s^+_{\bar{x}}$, $s^-_{\bar{x}}$. Finally, for every point $p \in P_\Lambda$ the curve $w^{s\infty}_{\bar{p}}$ has an asymptotic direction $s^\infty_{\bar{p}}$.

Statement 2 (Conjugacy on one-dimensional attractors, [8])**.** *Let Λ, Λ' be attractors such that there is an automorphism $\psi_\Lambda \colon \mathbb{G}_{N_\Lambda} \to \mathbb{G}_{N_{\Lambda'}}$ with property $T_{\bar{f}'_{\Lambda'}} = \psi_\Lambda T_{\bar{f}_\Lambda} \psi_\Lambda^{-1}$. Then*

1) *ψ_Λ is uniquely induces a homeomorphism $\psi^*_\Lambda \colon \mathbb{E}_{N_\Lambda} \to \mathbb{E}_{N_{\Lambda'}}$;*

2) *for every point* $\bar{x} \in \bar{\Lambda}$ *there exists a unique point* $\bar{x}' \in \bar{\Lambda}'$ *such that*

$$\psi^*_\Lambda(\mathrm{cl}(w^\delta_{\bar{x}}) \cap \mathbb{E}_{N_\Lambda}) = \mathrm{cl}(w^\delta_{\bar{x}'}) \cap \mathbb{E}_{N_{\Lambda'}}$$

for $\delta \in \{u, s\}$ *and the map* $\bar{\varphi}_\Lambda \colon \bar{\Lambda} \to \bar{\Lambda}'$, *assigning* $\bar{x}'$ *to* $\bar{x}$, *is a homeomorphism*;

3) $\bar{\varphi}_\Lambda$ *induces the homeomorphism*

$$\varphi_\Lambda = p_{\Lambda'} \bar{\varphi}_\Lambda p^{-1}_\Lambda \colon \Lambda \to \Lambda'$$

conjugating $f|_\Lambda$ *with* $f'|_{\Lambda'}$ *and possesses the property*: *if* $a, b \in W^s_x$, $x \in \Lambda$, *then* $\varphi_\Lambda(a), \varphi_\Lambda(b) \in W^s_{\varphi_\Lambda(x)}$.

It immediately follows from Statement 2 that each isomorphism ψ_Λ with property $T_{\bar{f}'_{\Lambda'}} = \psi_\Lambda T_{\bar{f}_\Lambda} \psi^{-1}_\Lambda$ uniquely induces a one-to-one map

$$\hat{\psi}_\Lambda \colon P_\Lambda \to P_{\Lambda'}.$$

2.2. Moduli associated to diffeomorphisms from the class Ψ. For two diffeomorphisms from Ψ to be topologically conjugate certain moduli conditions have to be satisfied. Let us define these conditions now. For $f \in \Psi$ denote by Ω_f the set of trivial basic sets of f and by Ω^0, Ω^1, Ω^2 its subsets consisting of the sinks, saddles and sources, accordingly. For a saddle point $\sigma \in \Omega^1$ of a diffeomorphism $f \in \Psi$ denote by k_σ the period of σ and μ_σ, let λ_σ denote the eigenvalues of $Df^{k_\sigma}_\sigma$ which are greater and less than one by absolute value, accordingly ($|\mu_\sigma| > 1 > |\lambda_\sigma| > 0$).

For $0 < |\lambda| < 1 < |\mu|$ denote by $f_{\mu,\lambda} \colon \mathbb{R}^2 \to \mathbb{R}^2$ linear diffeomorphism given by the formula

$$f_{\mu,\lambda}(x, y) = (\mu x, \lambda y).$$

Set

$$U_{\mu,\lambda} = \{(x, y) \in \mathbb{R}^2 \colon |x||y|^{-\log_\lambda \mu} \leqslant 1\}.$$

Notice that the set $U_{\mu,\lambda}$ is $f_{\mu,\lambda}$-invariant and possesses two $f_{\mu,\lambda}$-invariant foliations $\mathcal{F}^s = \bigcup\limits_{c \in \mathbb{R}} \{(x, y) \in U_{\mu,\lambda} \colon x = c\}$ and $\mathcal{F}^u = \bigcup\limits_{c \in \mathbb{R}} \{(x, y) \in U_{\mu,\lambda} \colon y = c\}$.

Definition 5 (C^1 linearization)**.** A saddle point $\sigma \in \Omega^1$ and an f^{k_σ}-invariant neighbourhood U_σ of σ form a C^1-linearization (see Figure 6) if:

1) there is a C^1-diffeomorphism $\psi_\sigma \colon U_\sigma \to U_{\mu_\sigma,\lambda_\sigma}$ conjugating $f^{k_\sigma}|_{U_\sigma}$ with $f_{\mu_\sigma,\lambda_\sigma}|_{U_{\mu_\sigma,\lambda_\sigma}}$;

2) each leaf of the foliations $\mathcal{F}^s_\sigma = \psi^{-1}_\sigma(\mathcal{F}^s)$, $\mathcal{F}^u_\sigma = \psi^{-1}_\sigma(\mathcal{F}^u)$ is C^2-smooth.

The existence of a linearizable neighborhood for any saddle point of a diffeomorphism f from Ψ (or indeed any C^2 diffeomorphism) is well-known, see for example [16, Chapter 5].

Denote by $\mathcal{A}$ the set of points at which one-sided heteroclinic tangencies of invariant manifolds of saddle points of the diffeomorphism f there is. For each $a \in \mathcal{A}$ denote by $\sigma^u_a \in \Omega^1$, $\sigma^s_a \in \Omega^1$ the saddle points such that $a \in W^s_{\sigma^s_a} \cap W^u_{\sigma^u_a}$. Set $\mu_a = \mu_{\sigma^s_a}$ and $\lambda_a = \lambda_{\sigma^u_a}$.

Denote by U_a the connected component of $U_{\sigma^s_a} \cap U_{\sigma^u_a}$ containing the point a. For any point $z \in U_a$ let us set $z^s = \psi_{\sigma^s_a}(z) = (z^s_x, z^s_y)$ and $z^u = \psi_{\sigma^u_a}(z) = (z^u_x, z^u_y)$. Set

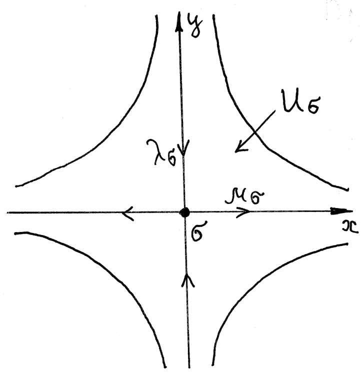

FIGURE 6. Linearizable neighborhood U_σ

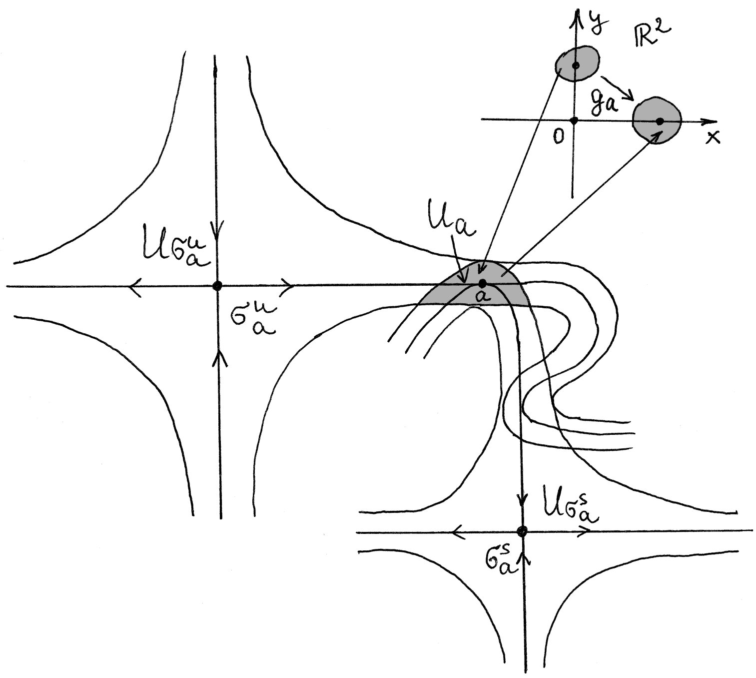

FIGURE 7. The transition map g_a

$g_a = \psi_{\sigma_a^u}(\psi_{\sigma_a^s}|_{U_a})^{-1} \colon \psi_{\sigma_a^s}(U_a) \to \psi_{\sigma_a^u}(U_a)$ (see Figure 7) and write the map g_a in the coordinate form

$$g_a(x,\, y) = (\xi_a(x,\, y),\, \eta_a(x,\, y)).$$

Set

$$\tau_a = \frac{\partial \eta_a}{\partial x}(a^s).$$

Set

$$H_a = \mathcal{A} \cap W^s_{\sigma^s_a} \cap W^u_{\sigma^u_a}.$$

Statement 3 (Moduli for $f \in \Psi$, [11], [12])**.**
(1) *Let $f \in \Psi$, $a \in \mathcal{A}$ and $k \in \mathbb{Z}$. Then*

$$\tau_{f^k(a)} = \left|\frac{\lambda_a}{\mu_a}\right|^k \cdot \tau_a.$$

(2) *If diffeomorphisms f, $f' \in \Psi$ are topologically conjugate by means of a homeomorphism h such that $h(a) = a'$ for a point $a \in \mathcal{A}$ and $h(\sigma^s_a) = \sigma^s_{a'}$, $h(\sigma^u_a) = \sigma^u_{a'}$ then*

$$\frac{\ln|\lambda_a|}{\ln|\mu_a|} = \frac{\ln|\lambda_{a'}|}{\ln|\mu_{a'}|}.$$

(3) *If diffeomorphisms f, $f' \in \Psi^*$ are topologically conjugate by means of a homeomorphism h such that $h(\sigma^s_a) = \sigma^s_{a'}$, $h(\sigma^u_a) = \sigma^u_{a'}$ for some points $a \in \mathcal{A}$, $a' \in \mathcal{A}'$ and $h(a_1) = a'_1$, $h(a_2) = a'_2$ for some points a_1, $a_2 \in H_a$, a'_1, $a'_2 \in H_{a'}$ then*

$$\left|\frac{\tau_{a_2}}{\tau_{a_1}}\right|^{\frac{1}{\ln|\mu_a|}} = \left|\frac{\tau_{a'_2}}{\tau_{a'_1}}\right|^{\frac{1}{\ln|\mu_{a'}|}}.$$

2.3. Geometric description of the intersection pattern of invariant manifolds. Denote by $\mathcal{L}^s$, $\mathcal{L}^u$ the sets of non-trivial attractors respectively repellers. Set $\mathcal{L} = \mathcal{L}^s \cup \mathcal{L}^u$. As before let Ω^0, Ω^1, Ω^2 be the sets of sinks, saddles and sources from the trivial basic set Ω_f. We let Ω^{1u} be the set of saddle points $p \in \Omega^1$ for which there is either a saddle point $q \in (\Omega^1 \setminus p)$ such that $W^u_p \cap W^s_q \neq \varnothing$ or a set $\Lambda \in \mathcal{L}^s$ such that $W^u_p \cap W^s_\Lambda \neq \varnothing$. Next define $\Omega^{1s} = \Omega^1 \setminus \Omega^{1u}$. Note that the definitions of the sets Ω^{1s} and Ω^{1u} are not symmetric, but, by the class Ψ assumptions, if $p \in \Omega^{1u}$ then there exists no saddle point q for which $W^s_p \cap W^u_q \neq \varnothing$ for some saddle point q and that there exists no set $\Lambda \in \mathcal{L}^u$ such that $W^s_p \cap W^u_\Lambda \neq \varnothing$.

Let us set

$$A_f = W^u_{\Omega^{1s}} \cup \Omega^0 \cup \mathcal{L}^s, \quad R_f = W^s_{\Omega^{1u}} \cup \Omega^2 \cup \mathcal{L}^u.$$

By construction the set A_f is an attractor and R_f is a repeller of f, see Figure 8.

Set

$$V_f = M^2 \setminus (A_f \cup R_f).$$

Let k_f be a minimal natural number such that each separatrix of saddle and boundary points is invariant with respect to f^{k_f}. By construction the orbit space $\hat{V}_f = V_f / f^{k_f}$ of the action of the diffeomorphism f^{k_f} on V_f consists of a finite number of copies of the two-dimensional torus, and the natural projection $p_f \colon V_f \to \hat{V}_f$ is a covering (see, for example [12, Lemma 2.1]). In Figure 9 this construction is illustrated for the diffeomorphism shown in Figure 8. Set

$$\hat{\phi}_f = p_f f p_f^{-1} \colon \hat{V}_f \to \hat{V}_f.$$

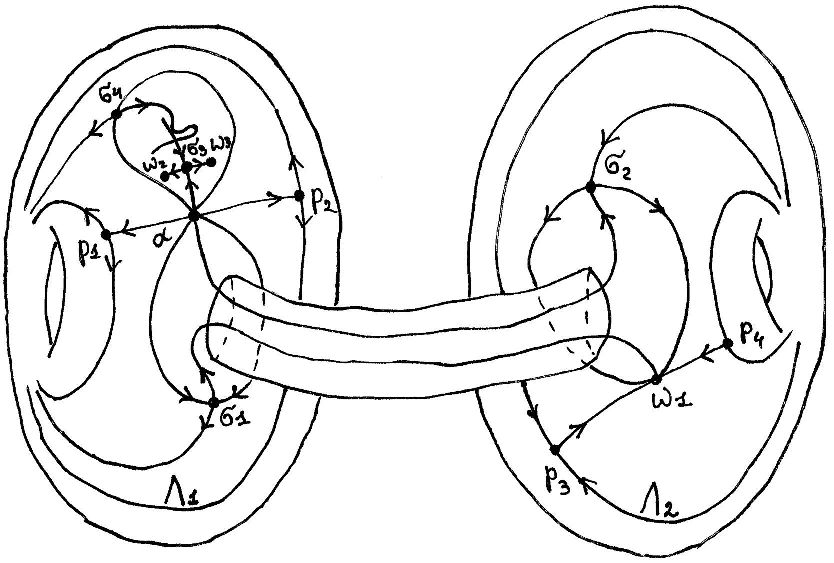

FIGURE 8. Attractor A_f and repeller R_f for a diffeomorphism $f \in \Psi$

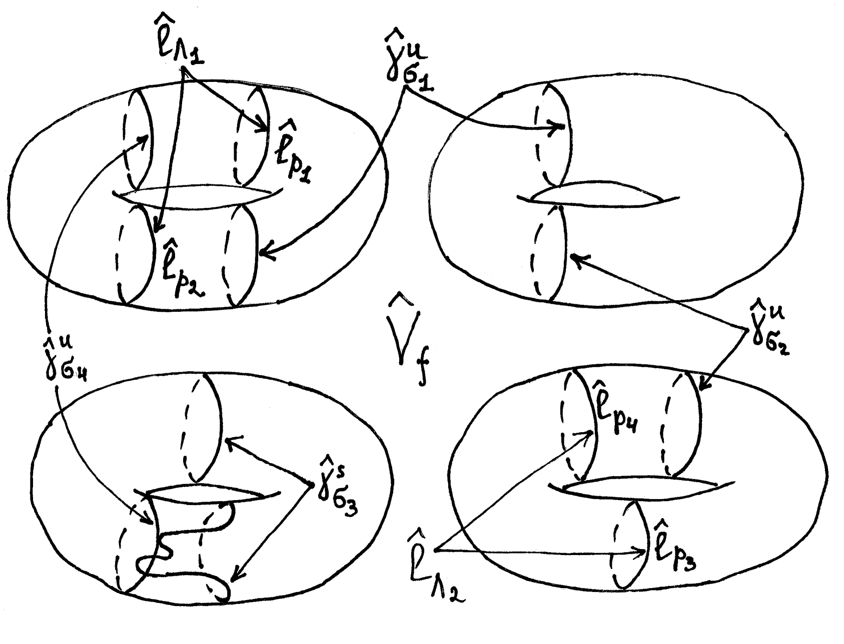

FIGURE 9. The orbit space $\hat{V}_f$ for the diffeomorphism f from Figure 8 with the projections of the separatrices

For each point $\sigma \in \Omega^{1s}$ set $\hat\gamma_\sigma = p_f(W^s_\sigma \setminus \sigma)$ and for each point $\sigma \in \Omega^{1u}$ set $\hat\gamma_\sigma = p_f(W^u_\sigma \setminus \sigma)$. By the construction $\hat\gamma_\sigma$ is a pair of circles. Set

$$\hat\Gamma^s_f = \bigcup_{\sigma\in\Omega^{1s}} \hat\gamma_\sigma, \quad \hat\Gamma^u_f = \bigcup_{\sigma\in\Omega^{1u}} \hat\gamma_\sigma.$$

For each non-trivial basic set Λ denote by k_Λ the number of periodic components of Λ and by g_Λ the restriction of f^{k_Λ} on a periodic component of N_Λ. Set $\hat\ell_p = p_f(\ell_p)$, $p \in P_\Lambda$, $\hat\ell_\Lambda = \bigcup\limits_{p\in\Lambda} \hat\ell_p$ and

$$\hat L^s_f = \bigcup_{\Lambda\in\mathcal{L}^s} \hat\ell_\Lambda, \quad \hat L^u_f = \bigcup_{\Lambda\in\mathcal{L}^u} \hat\ell_\Lambda.$$

Set $\hat H_a = p_f(H_a)$, $\lambda_{\hat H_a} = \lambda_a$ and $\mu_{\hat H_a} = \mu_a$ for $a \in \mathcal{A}$. Denote by $\hat H_f$ the union of all sets $\hat H_a$. For a connected component $\hat T$ of $\hat V_f$ let us set $\hat H_{\hat T} = \hat H_f \cap \hat T$. If the set $\hat H_{\hat T}$ is not empty let us choose a simple closed curve $\hat\beta_{\hat T}$ which intersects each curve from $\hat T \cap \hat\Gamma^s_f$ at exactly one point not being from $\hat H_f$ (such curve exists as $\hat T \cap \hat\Gamma^s_f$ is a set of disjoint non-contractible curves). Denote by $\beta_{\hat T}$ a connected component of the preimage $p_f^{-1}(\hat\beta_{\hat T})$ and by $K_{\hat T}$ an annulus on V_f situated between $\beta_{\hat T}$ and $f^{k_f}(\beta_{\hat T})$. For an oriented path $\hat\nu \subset \hat T$ from a point $\hat x$ to a point $\hat y$ there is a unique lift $\nu \subset V_f$ with the start point $x = p_f^{-1}(\hat x) \cap K_{\hat T}$ (see, for example, [10]). Then the end point of ν is situated in $f^{k_f\cdot k_{\hat\nu}}(K_{\hat T})$ for some integer $k_{\hat\nu}$.

Let $\hat a_1, \hat a_2 \in \hat H_{\hat T}$. If $\hat a_1$ and $\hat a_2$ belong to the same connected component of $\hat\Gamma^s_f$ then denote by $\hat\nu_{\hat a_1,\hat a_2}$ a directed curve connecting the points $\hat a_1$ with $\hat a_2$ which is the part of curve from $\hat\Gamma^s_f$ oriented along the stable manifold. If $\hat a_1$ and $\hat a_2$ belong to different connected components $\hat\gamma^s_1$, $\hat\gamma^s_2$ of $\hat\Gamma^s_f$ then set $\hat z_1 = \hat\gamma^s_1 \cap \hat\beta_{\hat T}$, $\hat z_1 = \hat\gamma^s_1 \cap \hat\beta_{\hat T}$ and denote by $\hat\nu_{\hat a_1,\hat a_2}$ a directed curve connecting the points $\hat a_1$ with $\hat a_2$ consisting of a part of $\hat\gamma^s_1$ oriented opposite the stable manifold, a part of curve $\hat\beta_{\hat T}$ connecting $\hat z_1$ with $\hat z_2$ and a part of $\hat\gamma^s_2$ oriented along the stable manifold.

For each point $b \in (H_f \cap K_{\hat T})$ let us calculate τ_b and set $\tau_{\hat b} = \tau_b$ for $\hat b = p_f(b)$. For $\hat H_a$ is from $\hat H_f$ let us set

$$\tau_{\hat H_a} = \{\tau_{\hat b},\, \hat b \in \hat H_a\} \quad \text{and} \quad \hat C_{\hat H_a} = \{\lambda_{\hat H_a},\, \mu_{\hat H_a},\, \tau_{\hat H_a}\}.$$

Set

$$\hat C_f = \{\hat C_{\hat H_a},\, \hat H_a \subset \hat H_f\}.$$

Definition 6 (The scheme of a diffeomorphism)**.** We call the set

$$S_f = (\hat V_f,\, \phi_f,\, \hat\Gamma^s_f,\, \hat\Gamma^u_f,\, \hat C_f,\, \hat L^s_f,\, \hat L^u_f)$$

a scheme of the diffeomorphism $f \in \Psi$.

Definition 7 (Equivalence of schemes)**.** The schemes

$$S_f = (\hat V_f,\, \phi_f,\, \hat\Gamma^s_f,\, \hat\Gamma^u_f,\, \hat C_f,\, \hat L^s_f,\, \hat L^u_f) \quad \text{and} \quad S_{f'} = (\hat V_{f'},\, \phi_{f'}\hat\Gamma^s_{f'},\, \hat\Gamma^u_{f'},\, \hat C_{f'},\, \hat L^s_{f'},\, \hat L^u_{f'})$$

of diffeomorphisms $f, f' \in \Psi$, respectively, are said to be equivalent if there exists an orientation-preserving homeomorphism $\hat\varphi\colon \hat V_f \to \hat V_{f'}$ such that:

1) $\hat{\varphi}\hat{\phi}_f = \hat{\phi}_{f'}\hat{\varphi}$;

2) $\hat{\varphi}(\hat{\Gamma}^s_f) = \hat{\Gamma}^s_{f'}$, $\hat{\varphi}(\hat{\Gamma}^u_f) = \hat{\Gamma}^u_{f'}$ and for each $\sigma \in \Omega^1$ there is a unique $\sigma' \in \Omega'^1$ such that $\hat{\varphi}(\hat{\gamma}_\sigma) = \hat{\gamma}_{\sigma'}$;

3) if $\hat{H}_{a'} = \hat{\varphi}(\hat{H}_a)$ then $\frac{\ln|\lambda_{\hat{H}_a}|}{\ln|\mu_{\hat{H}_a}|} = \frac{\ln|\lambda_{\hat{H}_{a'}}|}{\ln|\mu_{\hat{H}_{a'}}|}$;

4) if $\hat{H}_{a'} = \hat{\varphi}(\hat{H}_a)$ for $\hat{H}_a$ from $\hat{H}_f$ then

4a) for any points $\hat{a}_1$, $\hat{a}_2 \in \hat{H}_a$ belonging to the same connected component $\hat{T}$ of $\hat{V}_f$ we have $\left|\frac{\tau_{\hat{a}_2}}{\tau_{\hat{a}_1}}\right|^{\frac{1}{\ln|\mu_{\hat{H}_a}|}} = \left(\left|\frac{\lambda_{\hat{H}_{a'}}}{\mu_{\hat{H}_{a'}}}\right|^{k_{\hat{\varphi}(\hat{\nu}_{\hat{a}_1,\hat{a}_2})}} \cdot \left|\frac{\tau_{\hat{\varphi}(\hat{a}_2)}}{\tau_{\hat{\varphi}(\hat{a}_1)}}\right|\right)^{\frac{1}{\ln|\mu_{\hat{H}_{a'}}|}}$;

4b) for any points $\hat{a}_1$, $\hat{a}_2 \in \hat{H}_a$ from different connected components $\hat{T}_1$, $\hat{T}_2$ of $\hat{V}_f$ there is a number $m_{\hat{a}_1,\hat{a}_2}$ such that $\left(\frac{\tau_{\hat{a}_2}}{\tau_{\hat{a}_1}}\right)^{\frac{1}{\ln|\mu_{\hat{H}_a}|}} = \left(\left|\frac{\lambda_{\hat{H}_{a'}}}{\mu_{\hat{H}_{a'}}}\right|^{m_{\hat{a}_1,\hat{a}_2}} \cdot \frac{\tau_{\hat{\varphi}(\hat{a}_2)}}{\tau_{\hat{\varphi}(\hat{a}_1)}}\right)^{\frac{1}{\ln|\mu_{\hat{H}_{a'}}|}}$;

5) if H_a, H_b are from H_f and $\hat{a}_1$, $\hat{a}_2 \in \hat{H}_a$, $\hat{b}_1$, $\hat{b}_2 \in \hat{H}_b$ such that $\hat{a}_1$, $\hat{b}_1 \in \hat{T}_1$, $\hat{a}_2$, $\hat{b}_2 \in \hat{T}_2$ then the numbers $m_{\hat{a}_1,\hat{a}_2}$, $m_{\hat{b}_1,\hat{b}_2}$ satisfy the equality $m_{\hat{b}_1,\hat{b}_2} = -k_{\hat{\varphi}(\hat{\nu}_{\hat{a}_1,\hat{b}_1})} + m_{\hat{a}_1,\hat{a}_2} + k_{\hat{\varphi}(\hat{\nu}_{\hat{a}_2,\hat{b}_2})}$;

6) $\hat{\varphi}(\hat{L}^s_f) = \hat{L}^s_{f'}$, $\hat{\varphi}(\hat{L}^u_f) = \hat{L}^u_{f'}$ and for each $\Lambda \in \mathcal{L}$ there is a unique $\Lambda' \in \mathcal{L}'$ such that $\hat{\varphi}(\hat{\ell}_\Lambda) = \hat{\ell}_{\Lambda'}$;

7) if $\hat{\varphi}(\hat{\ell}_\Lambda) = \hat{\ell}_{\Lambda'}$ then there is an isomorphism ψ_Λ conjugating $T_{\bar{g}_\Lambda}$ with $T_{\bar{g}'_{\Lambda'}}$ for some $\bar{g}_\Lambda$, $\bar{g}'_{\Lambda'}$ and such that $\hat{\varphi}(\hat{\ell}_p) = \hat{\ell}_{\hat{\psi}_\Lambda(p)}$.

3. Separability of a 1-dimensional Attractor (Repeller) of a Diffeomorphism of a Surface with a Finite Number Moduli

Proof of Theorem 1. We now prove that a 1-dimensional attractor of an A-diffeomorphism $f\colon M^2 \to M^2$ with a finite number moduli is separable.

Let Λ be an attractor of an A-diffeomorphism $f\colon M^2 \to M^2$ with a finite number of moduli. Let us prove that the three conditions of Definition 1 hold.

1) To prove item 1 of Definition 1 it suffices to prove that $W^u_{\Lambda'} \cap W^s_\Lambda = \varnothing$ holds for every non-trivial basic set Λ' distinct from Λ. Suppose the contrary: there are points $x \in \Lambda$, $x' \in \Lambda'$ such that $W^s_x \cap W^u_{x'} \neq \varnothing$. Since stable manifolds of the points of Λ (unstable manifolds of the points of Λ') are C^1-close on compact sets, without loss of generality one can assume that the manifold W^s_x contains no s-boundary periodic points of the basic set Λ and the manifold $W^u_{x'}$ contains no u-boundary periodic points of the basic set Λ'. By Statement 1, the intersection $W^u_{\Lambda'} \cap W^s_\Lambda$ is transverse.

Let $y \in (W^s_x \cap W^u_{x'})$. As Λ and Λ' have local structure of the product on interval by Cantor set then the point y belongs to an adjacent interval $(a, b)^s \subset W^s_x$ which consists of the wandering points of the diffeomorphism f and such that $a, b \in \Lambda$ and W^u_a, W^u_b contain one s-boundary point each p_a, p_b ($p_a = p_b$ if $W^u_a = W^u_b$). Denote by L_a (L_b) the connected component of the set $W^u_a \setminus a$ ($W^u_b \setminus b$) disjoint from the point p_a (p_b). Then the curve $l_{ab} = L_a \cup L_b \cup [a, b]^s$ bounds a domain D_{ab}. This domain is a continuous immersion of the open disk into the manifold M^2, all of its points are the wandering points of the diffeomorphism f and the curve l_{ab} is the boundary of D_{ab} which is accessible from inside.

Denote by W^{u*}_y the connected component of the set $W^u_y \setminus y$ disjoint from the point x'. The transversality condition implies $W^{u*}_y \cap D_{ab} \neq \varnothing$. On the other hand the component W^{u*}_y contains a set which is dense in the periodic component of the set Λ'. Therefore there are points in W^{u*}_y disjoint from the domain D_{ab}. Then there is a point $y' \in (a, b)^s$ distinct from the point y and such that the arc $(y, y')^u \subset W^u_{x'}$ belongs to the domain D_{xy}. Since for any point $\tilde{a} \in L_{p_a}$ there is a unique point $\tilde{b} \in L_{p_b}$ such that $\tilde{a} \in W^s_{\tilde{x}}$, $\tilde{x} \in \Lambda$ and $(\tilde{a}, \tilde{b})^s \subset D_{ab}$ it follows that there is a point $\tilde{x}$ for which the arc $(\tilde{a}, \tilde{b})^s$ is tangent to the arc $(y, y')^u$ and this contradicts the transversality condition.

2) To prove item 2 of Definition 1 it suffices to show that for every s-boundary point p of the basic set Λ there is no saddle point σ from the trivial basic set of the diffeomorphism f such that $W^u_\sigma \cap \ell_p \neq \varnothing$. If we assume the contrary then similarly to the proof of item 1 we come to a contradiction to the transversality condition.

3) Assuming the contrary in this case we come to a contradiction to the transversality condition as well. □

4. Proof of the Classification Theorem

It follows from the geometrical construction of the schemes and Statement 3 that diffeomorphisms $f, f' \in \Psi^*$ are topologically conjugate then their schemes are equivalent. Let us show that if the schemes of diffeomorphisms $f, f' \in \Psi$ are equivalent, then the diffeomorphisms are topologically conjugate.

Proof of Theorem 2. Let

$$S_f = (\hat{V}_f, \phi_f, \hat{\Gamma}^s_f, \hat{\Gamma}^u_f, \hat{C}_f, \hat{L}^s_f, \hat{L}^u_f) \quad \text{and} \quad S_{f'} = (\hat{V}_{f'}, \phi_{f'}\hat{\Gamma}^s_{f'}, \hat{\Gamma}^u_{f'}, \hat{C}_{f'}, \hat{L}^s_{f'}, \hat{L}^u_{f'})$$

be schemes of diffeomorphisms $f, f' \in \Psi$, respectively, for which there exists an orientation-preserving homeomorphism $\hat{\varphi}\colon \hat{V}_f \to \hat{V}_{f'}$ with the properties 1–7 of Definition 7. We divide the construction of a conjugating homeomorphism $h\colon M^2 \to M^2$ such that $hf = f'h$ in to steps.

Step 1. The existence of the homeomorphism $\varphi\colon \hat{V}_f \to \hat{V}_{f'}$ with property $\hat{\varphi}\hat{\phi}_f = \hat{\phi}_{f'}\hat{\varphi}$ implies that there exists a homeomorphism $\varphi\colon V_f \to V_{f'}$ that conjugates the restriction of the diffeomorphism f to V_f with the restriction of the diffeomorphism f' to $V_{f'}$ and is such that $\hat{\varphi} = p_{f'}\varphi p_f^{-1}$. For each point $b \in (H_a \cap K_{\hat{T}})$ let us denote by $n(\hat{b})$ an integer such that $\varphi(b) \in f'^{k_{f'}\cdot n(\hat{b})}(K_{\hat{\varphi}(\hat{T})})$. Due to condition 5 in Definition 7, we can suppose that φ is chosen such that if $a_1 \in (H_a \cap K_{\hat{T}_1})$ and $a_2 \in (H_a \cap K_{\hat{T}_2})$ then $n(\hat{a}_2) - n(\hat{a}_1) = m_{\hat{a}_1,\hat{a}_2}$. So we have a conjugating homeomorphism on the set $M^2 \setminus (A_f \cup R_f)$.

Due to condition 2 in Definition 7, for any point $\sigma \in \Omega^{1\delta}$, $\delta \in \{u, s\}$ there exists a point $\sigma' \in \Omega'^{1\delta}$ such that $\varphi(W^\delta_\sigma \setminus \sigma) = W^\delta_{\sigma'} \setminus \sigma'$. Let us extend φ on the set Ω^1 by setting $\varphi(\sigma) = \sigma'$. Due to condition 1 in Definition 7, φ conjugates $f|_{W^\delta_{\Omega^{1\delta}}}$ with $f'|_{W'^\delta_{\Omega'^{1\delta}}}$.

Due to condition 6 in Definition 7, for any basic set $\Lambda \in \mathcal{L}^\delta$ there exists a basic set $\Lambda' \in \mathcal{L}'^\delta$ such that $\varphi(\ell_\Lambda) = \ell_{\Lambda'}$. Let us extend φ by continuity on the set $\bigcup\limits_{\Lambda \in \mathcal{L}} P_\Lambda$

of boundary points of non-trivial basic sets. Due to condition 1 in Definition 7, φ conjugates $f|_{\bigcup\limits_{\Lambda\in\mathcal{L}}\ell_\Lambda}$ with $f'|_{\bigcup\limits_{\Lambda'\in\mathcal{L}'}\ell_{\Lambda'}}$.

STEP 2. In this step we define homeomorphisms $\varphi^s_{\Omega^{1u}}\colon W^s_{\Omega^{1u}}\to W^s_{\Omega'^{1u}}$ and $\varphi^u_{\Omega^{1s}}\colon W^u_{\Omega^{1s}}\to W^u_{\Omega'^{1s}}$ conjugating $f|_{W^s_{\Omega^{1u}}}$ with $f'|_{W^s_{\Omega'^{1u}}}$ and $f|_{W^u_{\Omega^{1s}}}$ with $f'|_{W^u_{\Omega'^{1s}}}$.

Let $\sigma\in\Omega^{1u}$ and $\sigma'=\varphi(\sigma)$. Set

$$\rho^s_\sigma=\frac{\ln|\lambda_{\sigma'}|}{\ln|\lambda_\sigma|}.$$

Let ℓ^s_σ be a stable separatrix of σ. Denote by $\ell^s_{\sigma'}$ the stable separatrix of $\sigma'=\varphi(\sigma)$ such that for a connected component E of $U_\sigma\setminus W^u_\sigma$ containing ℓ^s_σ and a connected component E' of $U_{\sigma'}\setminus W^u_{\sigma'}$ containing $\ell^s_{\sigma'}$ the following condition holds $\varphi(E\setminus W^u_\sigma)\cap E'\neq\varnothing$. Let us define a homeomorphism $\varphi_{\ell^s_\sigma}\colon \ell^s_\sigma\to\ell^s_{\sigma'}$ by the following way. For a point $t\in\ell^s_\sigma$ such that $t^u=(0,\,t^u_y)$ let us set $\varphi_{\ell^s_\sigma}(t)=t'$ where ${t'}^u=(0,\,{t'}^u_y)$,

$$|{t'}^u_y|=|t^u_y|^{\rho^s_\sigma}.$$

It is easy to verify that $\varphi_{\ell^s_\sigma}$ conjugates the diffeomorphisms $f^{k_\sigma}|_{\ell^s_\sigma}$ and ${f'}^{k_\sigma}|_{\ell^s_{\sigma'}}$.

Due to property 2 of Definition 7 we get $k_\sigma=k_{\sigma'}$. Then for each $k=0,\,\dots,\,k_\sigma$ we can define a homeomorphism $\varphi_{\ell^s_{f^k(\sigma)}}\colon \ell^s_{f^k(\sigma)}\to\ell^s_{f'^k(\sigma')}$ by the formula

$$\varphi_{\ell^s_{f^k(\sigma)}}(x)=f'^k(\varphi_{\ell^s_\sigma}(f^{-k}(x)))$$

for each $x\in\ell^s_{f^k(\sigma)}$. Doing a similar construction for all saddle periodic orbits of the set Ω^{1u} we get the sought conjugating homeomorphism $\varphi^s_{\Omega^{1u}}$.

Now let $\sigma\in\Omega^{1s}$ and $\sigma'=\varphi(\sigma)$. Set

$$\rho^u_\sigma=\frac{\ln|\mu_{\sigma'}|}{\ln|\mu_\sigma|}.$$

Similar to the construction above for corresponding separatrices ℓ^u_σ, $\ell^u_{\sigma'}$ we define a homeomorphism $\varphi_{\ell^u_\sigma}\colon \ell^u_\sigma\to\ell^u_{\sigma'}$ by the following way. For a point $t\in\ell^u_\sigma$ such that $t^s=(t^s_x,\,0)$ let us set $\varphi_{\ell^u_\sigma}(t)=t'$ where ${t'}^s=({t'}^s_x,\,0)$,

$$|{t'}^s_x|=c_{\ell^u_\sigma}|t^s_x|^{\rho^u_\sigma},$$

where $c_{\ell^u_\sigma}=\dfrac{\left|\frac{\lambda_{\hat H_{a'}}}{\mu_{\hat H_{a'}}}\right|^{n(\hat a)}\cdot|\tau_{\hat a'}|}{|\tau_{\hat a}|^{\rho^u_\sigma}}$ if there is a point $\hat a\in\hat\ell^u_\sigma\cap\hat H_f$ and equals 1 in the opposite case. As above it is possible to verify that φ^u_σ conjugates the diffeomorphisms $f^{k_\sigma}|_{\ell^u_\sigma}$ and $f'^{k_\sigma}|_{\ell^u_{\sigma'}}$. For each $k=0,\,\dots,\,k_\sigma$ we can define a homeomorphism $\varphi_{\ell^u_{f^k(\sigma)}}\colon \ell^u_{f^k(\sigma)}\to\ell^u_{f'^k(\sigma')}$ by the formula $\varphi_{\ell^u_{f^k(\sigma)}}(x)=f'^k(\varphi_{\ell^u_\sigma}(f^{-k}(x)))$ for each $x\in\ell^u_{f^k(\sigma)}$. Doing a similar construction for all saddle periodic orbits of the set Ω^{1s} we get the sought conjugating homeomorphism $\varphi^u_{\Omega^{1s}}$.

STEP 3. In this step we construct a homeomorphism $\varphi_{\mathcal{L}^s}\colon\mathcal{L}^s\to\mathcal{L}'^s$ ($\varphi_{\mathcal{L}^u}\colon\mathcal{L}^u\to\mathcal{L}'^u$) which conjugates $f|_{\mathcal{L}^s}$ with $f'|_{\mathcal{L}'^s}$ ($f|_{\mathcal{L}^u}$ with $f'|_{\mathcal{L}'^u}$). Let us construct $\varphi_{\mathcal{L}^s}$, the construction of $\varphi_{\mathcal{L}^u}$ is similar.

Let Λ be a one-dimensional attractor of f and L be one from k_Λ periodic components of Λ. Then L is a one-dimensional attractor of the diffeomorphism $g=f^{k_\Lambda}$

with the unique periodic component. Denote by L' a periodic component of Λ' such that φ sends the boundary points of L to the boundary points of L'. Then L' is a one-dimensional attractor of the diffeomorphism $g' = f'^{k_{\Lambda'}}$ with the unique periodic component. Due to conditions 1) and 6) in Definition 7, $k_\Lambda = k_{\Lambda'}$. Due to condition 7) in Definition 7, there is an isomorphism ψ_Λ conjugating $T_{\bar g_L}$ with $T_{\bar g'_{L'}}$ for some $\bar g_L$, $\bar g'_{L'}$ and such that $\varphi(p) = \hat\psi_\Lambda(p)$ for each point p from the set P_L of the boundary points of L. Statement 2 implies that there is a homeomorphism $\bar\varphi_L\colon \bar L \to \bar L'$ such that $\bar g'_{L'}\bar\varphi_L|_{\bar L} = \bar\varphi_L \bar g_L|_{\bar L}$. Set

$$\varphi_L = p_{N_{L'}}\bar\varphi_L p_{N_L}^{-1}\colon L \to L'.$$

Then $\varphi_L g_L|_L = g'_{L'}\varphi_L|_L$ and $\varphi_L(p) = \varphi(p)$ for each $p \in P_L$. Define $\varphi_\Lambda\colon \Lambda \to \Lambda'$ by the formula $\varphi_\Lambda(v) = f'^k(\varphi_L(f^{-k}(v)))$ where $f^k(v) \in L$ for $v \in \Lambda$. Doing a similar construction for all attractors of the set $\mathcal{L}^s$ we get the sought conjugating homeomorphism $\varphi_{\mathcal{L}^s}$.

Step 4. In this step we modify the homeomorphism $\varphi|_{W^s_{\mathcal{L}^s}\setminus\mathcal{L}^s}$ ($\varphi|_{W^u_{\mathcal{L}^u}\setminus\mathcal{L}^u}$) by replacing it with $\tilde h_{\mathcal{L}^s}\colon W^s_{\mathcal{L}^s}\setminus\mathcal{L}^s \to W^s_{\mathcal{L}'^s}\setminus\mathcal{L}'^s$ ($\tilde h_{\mathcal{L}^u}\colon W^s_{\mathcal{L}^u}\setminus\mathcal{L}^u \to W^s_{\mathcal{L}'^u}\setminus\mathcal{L}'^u$), which extends continuously to the set $\mathcal{L}^s$ ($\mathcal{L}^u$) by the mapping $\varphi_{\mathcal{L}^s}$ ($\varphi_{\mathcal{L}^u}$). We construct $\tilde h_{\mathcal{L}^s}$ (construction of $\tilde h_{\mathcal{L}^u}$ is similar).

Let Λ be a one-dimensional attractor of f. We modify the homeomorphism $\varphi|_{W^s_\Lambda\setminus\Lambda}$ by replacing it with the homeomorphism $\tilde h_\Lambda\colon W^s_\Lambda\setminus\Lambda \to W^s_{\Lambda'}\setminus\Lambda'$, which extends continuously to Λ by the mapping φ_Λ, and which conjugates $f|_{W^s_\Lambda\setminus\Lambda}$ with $f'|_{W^s_{\Lambda'}\setminus\Lambda'}$.

Let L be a periodic component of Λ as in Step above. Denote by B_L the set of all bunches of L and will use further the denotations of Section 2.1. Let us consider the closed curve L_b of the bunch $b \in B_L$ and enumerate the separatrices $l_1, \ldots, l_m$ of all saddle points and boundary points which intersect L_b due to some orientation on L_b. Set $b' = \varphi_L(b)$. Due to items 2 and 6 of Definition 7 we have that the separatrices $\varphi(l_1), \ldots, \varphi(l_m)$ intersect $L_{b'}$ in order. If $\varphi_L|_{P_L} = \varphi|_{P_L}$ then for each $j \in \{1, \ldots, r_b\}$ there is a homeomorphism $h^s_j\colon [x_{2j}, x_{2j+1}]^s \to [\varphi_L(x_{2j}), \varphi_L(x_{2j+1})]^s$ such that $h^s_j([x_{2j}, x_{2j+1}]^s \cap l_\mu) = [\varphi_L(x_{2j}), \varphi_L(x_{2j+1})]^s \cap \varphi(l_\mu)$ for each $\mu \in \{1, \ldots, m\}$ and $h^s_j(x_{2j}) = \varphi_L(x_{2j})$. Set $I^s_b = \bigcup\limits_{j=1}^{r_b}[x_{2j}, x_{2j+1}]^s$, $I^s_{b'} = \bigcup\limits_{j=1}^{r_{b'}}[\varphi_L(x_{2j}), \varphi_L(x_{2j+1})]^s$ and denote by $h^s_b\colon I^s_b \to I^s_{b'}$ a homeomorphism which composed from h^s_j, $j \in \{1, \ldots, r_b\}$. Set $I^s_L = \bigcup\limits_{b\in B_L} I^s_b$, $I^s_{L'} = \bigcup\limits_{b'\in B_{L'}} I^s_{b'}$ and denote by $h^s_L\colon I^s_L \to I^s_{L'}$ a homeomorphism which composed from h^s_b, $b \in B_L$.

Denote by $y_{2j-1}, y_{2j} \in W^u_{p_j}$ the intersection points of $f(I^s_b)$ with $W^u_{p_j}$ such that $p_j \notin [x_{2j}, y_{2j}]^u$ (see Figure 10). Set $I^u_b = \bigcup\limits_{i=j}^{r_b}[x_{2j}, y_{2j}]^u$, $I^u_L = \bigcup\limits_{b\in B_L} I^u_b$, and $I^u_{L'} = h_L(I^u_L)$. Set $h^u_L = \varphi_L|_{I^u_L}\colon I^u_L \to I^u_{L'}$. Let Π_L ($\Pi_{L'}$) be the closure of the set $W^s_{I^u_L}\setminus L$ ($W^s_{I^u_{L'}}\setminus L'$). Let us construct on Π_L a pair of transverse one-dimensional foliation F^s_L, F^u_L with the following properties:

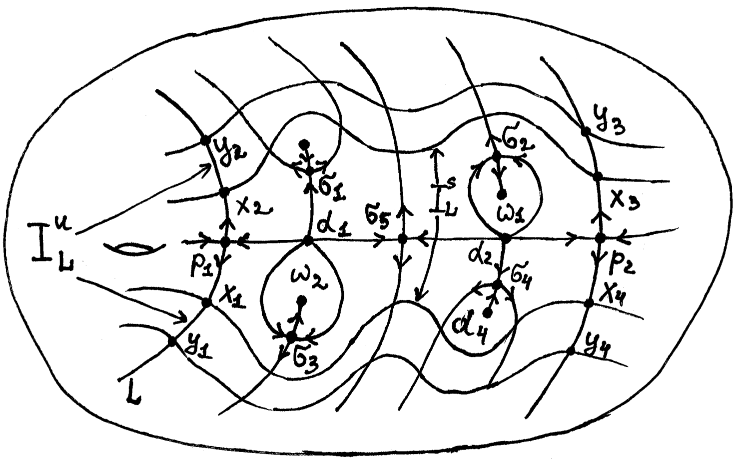

FIGURE 10. Illustration to the Step 4

a) each leaf of F^s_L is a connected component of the intersection stable manifold of a point from L with Π_L;
b) each leaf of F^u_L is a segment $[x,\, y]$ with the boundary points $x,\ y$ such that $x \in [x_{2j},\, x_{2j+1}]^s,\ y \in [y_{2j},\, y_{2j+1}]^s$;
c) if $[x,\, y]$ belongs to F^u_L then $[g^{-1}(y),\, g(x)]$ also belongs to ot;
d) each connected component of intersection of the separatrices of the saddle points with Π_L is a leaf of F^u_L.

For each point $z \in I^u_L$ denote by $F^s_{L,z}$ a leaf of the foliation F^s_L passing through the point z. For each point $x \in I^s_L$ denote by $F^u_{L,x}$ a leaf of the foliation F^u_L passing through the point x. Let us construct similar foliations $F^s_{L'},\ F^u_{L'}$ on $\Pi_{L'}$ and define a homeomorphism $h_{\Pi_L} : \Pi_L \to \Pi_{L'}$ by the formula

$$h_{\Pi_L}(F^s_{L,z} \cap F^u_{L,x}) = F^s_{L,h^u_L(z)} \cap F^u_{L,h^s_L(x)}.$$

Notice that $\Pi_L \setminus B_L$ is a fundamental domain of f restriction on $W^s_\Lambda \setminus \Lambda$. Then for each point $w \in (W^s_\Lambda \setminus \Lambda)$ there is $k \in \mathbb{Z}$ such that $f^k(w) \in \Pi_L$. As h_{Π_L} conjugates $g|_{\Pi_L}$ with $g'|_{\Pi_{L'}}$ then we can extend h_{Π_L} up to $\tilde{h}_\Lambda \colon W^s_\Lambda \setminus \Lambda \to W^s_{\Lambda'} \setminus \Lambda'$ conjugating f and f' by the formula

$$\tilde{h}_\Lambda(w) = f'^k(h_{\Pi_L}(f^{-k}(w))).$$

Doing a similar construction for all attractors of the set $\mathcal{L}^s$ we get the sought conjugating homeomorphism $\tilde{h}_{\mathcal{L}^s}$.

Denote by $\varphi_1 \colon V_f \to V_{f'}$ a homeomorphism given by the formula

$$\varphi_1(z) = \begin{cases} \tilde{h}_{\mathcal{L}^s}(z), & z \in (W^s_{\mathcal{L}^s} \setminus \mathcal{L}^s); \\ \tilde{h}_{\mathcal{L}^u}(z), & z \in (W^u_{\mathcal{L}^u} \setminus \mathcal{L}^u); \\ \varphi(z), & z \in V_f \setminus (W^s_{\mathcal{L}^s} \cap W^u_{\mathcal{L}^u}). \end{cases}$$

Set $\hat{\varphi}_1 = p_{f'}\varphi_1 p_f^{-1} \colon \hat{V}_f \to \hat{V}_{f'}$.

Step 5. In this step we modify the homeomorphism $\varphi_1|_{W^s_{\mathcal{L}^s}\setminus\mathcal{L}^s}$ ($\varphi_1|_{W^u_{\mathcal{L}^u}\setminus\mathcal{L}^u}$) by replacing it with $h_{\mathcal{L}^s}\colon W^s_{\mathcal{L}^s}\setminus\mathcal{L}^s\to W^s_{\mathcal{L}'^s}\setminus\mathcal{L}'^s$ ($h_{\mathcal{L}^u}\colon W^s_{\mathcal{L}^u}\setminus\mathcal{L}^u\to W^s_{\mathcal{L}'^u}\setminus\mathcal{L}'^u$), which extends to the set $\mathcal{L}^s$ ($\mathcal{L}^u$) by $\varphi_{\mathcal{L}^s}$ ($\varphi_{\mathcal{L}^u}$) by the mapping $\varphi_{\mathcal{L}^s}$ ($\varphi_{\mathcal{L}^u}$), and which extends to the set $\mathrm{cl}(W^s_{\mathcal{L}^s})\setminus(\mathcal{L}^s\cup\Omega^2)$ ($\mathrm{cl}(W^u_{\mathcal{L}^u})\setminus(\mathcal{L}^u\cup\Omega^0)$) by the mapping $\varphi^s_{\Omega^{1u}}$ ($\varphi^u_{\Omega^{1s}}$).

By Theorem 1, each non-trivial attractor of the diffeomorphism f is separable. Then there is a set $\Sigma^u\subset\Omega^{1u}$ such that $\mathrm{cl}\,(W^s_{\mathcal{L}^s})\setminus(\mathcal{L}^s\cup\Omega^2)=W^s_{\Sigma^u}$. Let $\sigma\in\Sigma^u$. Set $h^s_\sigma=\varphi^s_{\Omega^{1u}}|_{W^s_\sigma}\colon W^s_\sigma\to W^s_{\sigma'}$ and $h^u_\sigma=\varphi_1|_{W^u_\sigma}\colon W^u_\sigma\to W^u_{\sigma'}$. In an f^{k_σ}-invariant neighbourhood N_σ of σ let us construct a pair of transverse f^{k_σ}-invariant foliations G^s_σ, G^u_σ with the following properties:

a) $W^s_\sigma\in G^s_\sigma$, $W^u_\sigma\in G^u_\sigma$;
b) if $W^u_\sigma\cap W^s_L\neq\varnothing$ for some periodic component L of a non-trivial attractor then each connected component of $W^s_x\cap N_\sigma$, $x\in L$ is a leaf of G^s_σ.

For each point $z_u\in W^u_\sigma$ denote by G^s_{σ,z_u} a leaf of the foliation G^s_σ passing through the point z_u. For each point $z_s\in W^s_\sigma$ denote by G^u_{σ,z_s} a leaf of the foliation G^u_σ passing through the point z_s. Let us construct similar foliations $G^s_{\sigma'}$, $G^u_{\sigma'}$ on $N_{\sigma'}$ and define a homeomorphism $h_{N_\sigma}\colon N_\sigma\to N_{\sigma'}$ by the formula

$$h_{N_\sigma}(G^s_{\sigma,z_u}\cap G^u_{\sigma,z_s})=G^s_{\sigma',h^u_\sigma(z_u)}\cap G^u_{\sigma',h^s_\sigma(z_s)}.$$

Then in some tubular neighbourhood $N(\hat\gamma_\sigma)$ of $\hat\gamma_\sigma$ a map $\hat h_{N_\sigma}$ is well-defined by the formula $\hat h_{N_\sigma}=p_{f'}h_{N_\sigma}p_f^{-1}$. Chose a tubular neighbourhood $\tilde N(\hat\gamma_\sigma)$ of $\hat\gamma_\sigma$ such that $N(\hat\gamma_\sigma)\subset\tilde N(\hat\gamma_\sigma)$, $\hat h_{N_\sigma}(N(\hat\gamma_\sigma))\subset\hat\varphi_1(\tilde N(\hat\gamma_\sigma))$ and the set $Q=\mathrm{cl}(\tilde N(\hat\gamma_\sigma)\setminus N(\hat\gamma_\sigma))$, $Q'=\mathrm{cl}(\hat\varphi_1(\tilde N(\hat\gamma_\sigma))\setminus\hat h_{N_\sigma}(N(\hat\gamma_\sigma)))$ are two-dimensional annulus. Then there is a homeomorphism $\hat\varphi_{\hat Q}\colon\hat Q\to\hat Q'$ such that $\hat\varphi_{\hat Q}|_{\partial N(\hat\gamma_\sigma)}=\hat h_{N_\sigma}$ and $\hat\varphi_{\hat Q}|_{\partial\tilde N(\hat\gamma_\sigma)}=\hat\varphi_1$. As the homeomorphisms φ_1 and h_{N_σ} send leaves of the foliation W^s_x, $x\in\mathcal{L}^s$, to leaves of the foliation $W^s_{x'}$, $x'\in\mathcal{L}'^s$, and are coincide on W^u_σ then we can construct $\hat\varphi_{\hat Q}$ such that its lift sends leaves of the foliation W^s_x, $x\in\mathcal{L}^s$, to leaves of the foliation $W^s_{x'}$, $x'\in\mathcal{L}'^s$.

Denote by $\hat\varphi_{\hat\gamma_\sigma}\colon\hat V_f\to\hat V_{f'}$ a homeomorphism given by the formula

$$\hat\varphi_{\hat\gamma_\sigma}(\hat z)=\begin{cases}\hat\phi^k_{f'}(\hat h_{N_\sigma}(\phi_f^{-k}(\hat z))), & \hat z\in\hat\phi^k_f(N(\hat\gamma_\sigma));\\ \hat\phi^k_{f'}(\hat\varphi_{\hat Q}(\phi_f^{-k}(\hat z))), & \hat z\in\hat\phi^k_f(\hat Q);\\ \hat\varphi_1(\hat z), & \hat z\in(\hat V_f\setminus\tilde N(\hat\gamma_\sigma)).\end{cases}$$

Denote by $\varphi_{\hat\gamma_\sigma}$ a lift of $\hat\varphi_{\hat\gamma_\sigma}$ coinciding with φ_1 on $V_f\setminus p_f^{-1}(\tilde N(\hat\gamma_\sigma))$. Doing in series a similar construction for all saddle periodic orbits of the set Σ^u we get a homeomorphism $\varphi_{\Sigma^u}\colon V_f\to V_{f'}$. Also we construct a homeomorphism $\varphi_{\Sigma^s}\colon V_f\to V_{f'}$.

Denote by $\varphi_2\colon V_f\to V_{f'}$ a homeomorphism given by the formula

$$\varphi_2(z)=\begin{cases}\varphi_{\Sigma^u}(z), & z\in(W^s_{\mathcal{L}^s}\setminus\mathcal{L}^s);\\ \varphi_{\Sigma^s}(z), & z\in(W^u_{\mathcal{L}^u}\setminus\mathcal{L}^u);\\ \hat\varphi_1(z), & z\in V_f\setminus(W^s_{\mathcal{L}^s}\cap W^u_{\mathcal{L}^u}).\end{cases}$$

Set $\hat\varphi_2=p_{f'}\varphi_2p_f^{-1}\colon\hat V_f\to\hat V_{f'}$.

STEP 6. Let $\sigma \in \Omega^1$. For a point $x \in U_\sigma$ denote by $\mathcal{F}^u_{\sigma,x}$ ($\mathcal{F}^s_{\sigma,x}$) the unique leaf of $\mathcal{F}^u_\sigma$ ($\mathcal{F}^s_\sigma$) that passes through the point x. Define projections $\pi^u_\sigma\colon U_\sigma \to W^s_\sigma$ ($\pi^s_\sigma\colon U_\sigma \to W^u_\sigma$) along the leaves of the foliation $\mathcal{F}^u_\sigma$ ($\mathcal{F}^s_\sigma$) as follows: $\pi^u_\sigma(x) = \mathcal{F}^u_{\sigma,x} \cap W^s_\sigma$ ($\pi^s_\sigma(x) = \mathcal{F}^s_{\sigma,x} \cap W^u_\sigma$).

Let $a \in W^s_{\sigma^s_a} \cap W^u_{\sigma^u_a}$ be a point of one-sided tangency and $a' = \varphi(a)$. Set $l_a = \psi^{-1}_{\sigma^u_a}(\{(x, y) \in U_{\mu_{\sigma^u_a}, \lambda_{\sigma^u_a}} : x = a^u_x\}) \cap U_a$, $l_{a'} = \psi^{-1}_{\sigma^u_{a'}}(\{(x, y) \in U_{\mu_{\sigma^u_{a'}}, \lambda_{\sigma^u_{a'}}} : x = {a'}^u_x\}) \cap U_{a'}$. Set $L_\mathcal{A} = \bigcup\limits_{a \in \mathcal{A}} l_a$ and $L_{\mathcal{A}'} = \bigcup\limits_{a' \in \mathcal{A}'} l_{a'}$. In this step we construct a homeomorphism $\varphi_{L_\mathcal{A}}\colon L_\mathcal{A} \to L_{\mathcal{A}'}$ which conjugates $f|_{L_\mathcal{A}}$ with $f'|_{L_{\mathcal{A}'}}$. This homeomorphism extends continuously to the set $W^s_{\Omega^{1u}}$ by the mapping $\varphi^s_{\Omega^{1u}}$, and to the set $W^u_{\Omega^{1s}}$ by the mapping $\varphi^u_{\Omega^{1s}}$.

Define a homeomorphism $\varphi_{l_a}\colon l_a \to l_{a'}$ by the formula

$$\varphi_{l_a}(z) = z' = ((\pi^u_{\sigma^u_{a'}})^{-1}(\varphi^s_{\sigma^u}(\pi^u_{\sigma^u_a}(z)))) \cap l_{a'}.$$

Set $L_a = \bigcup\limits_{n \in \mathbb{Z}} f^{kn}(l_a)$ and $L_{a'} = \bigcup\limits_{n \in \mathbb{Z}} f'^{kn}(l_{a'})$, where k is the period of unstable separatrix containing a. Define a homeomorphism $\varphi_{L_a}\colon L_a \to L_{a'}$ by the formula $\varphi_{L_a}(z) = z' = f'^{kn}(\varphi_{l_a}(f^{-kn}(z)))$ for each point $z \in f^{kn}(l_a)$. Set $\mathcal{E}_a = W^u_{\sigma^s_a} \cup W^s_{\sigma^u_a} \cup L_a$ and $\mathcal{E}_{a'} = W^u_{\sigma^s_{a'}} \cup W^s_{\sigma^u_{a'}} \cup L_{a'}$. Denote by $\varphi_a\colon \mathcal{E}_a \to \mathcal{E}_{a'}$ a map, coinciding with $\varphi^u_{\sigma^s_a}$ on $W^u_{\sigma^s_a}$, with $\varphi^s_{\sigma^u_a}$ on $W^s_{\sigma^u_a}$ and with φ_{L_a} on L_a. Using condition 2 of Definition 7 it is possible to verify that φ_a is a homeomorphism (see [12] for details).

Denote by $A \subset \mathcal{A}$ a set of such points that any two from their are not belonging to the same orbit of f and $\bigcup\limits_{n \in \mathbb{Z}} f^n(A) = \mathcal{A}$. Set $A' = \varphi(A)$, $\mathcal{E}_\mathcal{A} = \bigcup\limits_{a \in A} \mathcal{E}_a$ and $\mathcal{E}_{\mathcal{A}'} = \bigcup\limits_{a' \in A'} \mathcal{E}_{a'}$. Let us define a map $\varphi_\mathcal{A}\colon \mathcal{E}_\mathcal{A} \to \mathcal{E}_{\mathcal{A}'}$ as coinciding with φ_a on each set $\mathcal{E}_a$. Using condition 4 of Definition 7 it is possible to verify that $\varphi_\mathcal{A}$ is a homeomorphism (see [12] for details).

STEP 7. In the neighborhood U_a of a point $a \in A$ define foliations $\mathcal{F}^u_a$ and $\mathcal{F}^s_a$ by the following way. The leaves of $\mathcal{F}^u_a$ are coincide with the leaves of $\mathcal{F}^u_{\sigma^u_a} \cap U_a$. In the neighborhood $\psi_{\sigma^u_a}(U_a)$ the curve $\psi_{\sigma^u_a}(W^s_{\sigma^s_a})$ has the equation $q(x) = Q(x - a^u_x)^n + o((x - a^u_x)^n)$, where $\frac{o((x-a^u_x)^n)}{(x-a^u_x)^n} \to 0$ for $x \to a^u_x$. Set $\mathcal{F}^s_a = \psi^{-1}_{\sigma^u_a}(\bigcup\limits_{c \in \mathbb{R}} \{(x, y) \in U_{\mu_\sigma, \lambda_\sigma} : y = q(x) + c\}) \cap U_a$. Hence, in a neighborhood U_a the leaves of $\mathcal{F}^u_a$ are transverse to the leaves of $\mathcal{F}^s_a$ on the set $U_a \setminus l_a$ and have tangency along the curve l_a. Set $U_A = \bigcup\limits_{a \in A} U_a$, $U_\mathcal{A} = \bigcup\limits_{n \in \mathbb{Z}} f^n(U_A)$, $\mathcal{F}^u_A = \bigcup\limits_{a \in A} \mathcal{F}^u_a$, $\mathcal{F}^u_\mathcal{A} = \bigcup\limits_{n \in \mathbb{Z}} f^n(\mathcal{F}^u_A)$, $\mathcal{F}^s_A = \bigcup\limits_{a \in A} \mathcal{F}^s_a$ and $\mathcal{F}^s_\mathcal{A} = \bigcup\limits_{n \in \mathbb{Z}} f^n(\mathcal{F}^s_A)$. The similar foliations $\mathcal{F}^u_{\mathcal{A}'}$ and $\mathcal{F}^s_{\mathcal{A}'}$ let us construct in the neighbourhood $U_{\mathcal{A}'}$ of the set $\mathcal{A}'$.

Let d be a point of the heteroclinic intersection of the manifolds $W^s_{\sigma^s_d} \cap W^u_{\sigma^u_d}$ be not belonging to the set $\mathcal{A}$. Denote U_d a connected component of the set $U_{\sigma^s_d} \cap U_{\sigma^u_d}$ which contains d. Define foliations $\mathcal{F}^u_d$ a $\mathcal{F}^s_d$ by the following way: $\mathcal{F}^u_d = \mathcal{F}^u_{\sigma^u_d} \cap U_d$ and $\mathcal{F}^s_d = \mathcal{F}^s_{\sigma^s_d} \cap U_d$. Let $\mathcal{D}$ ($\mathcal{D}'$) be the set of all heteroclinic points of f (f') not belonging to $\mathcal{A}$, $D \subset \mathcal{D}$ the set of points such that any two of them do not belong to the same orbit of the diffeomorphism f and $\bigcup\limits_{n \in \mathbb{Z}} f^n(D) = \mathcal{D}$. Set $U_D = \bigcup\limits_{d \in D} U_d$,

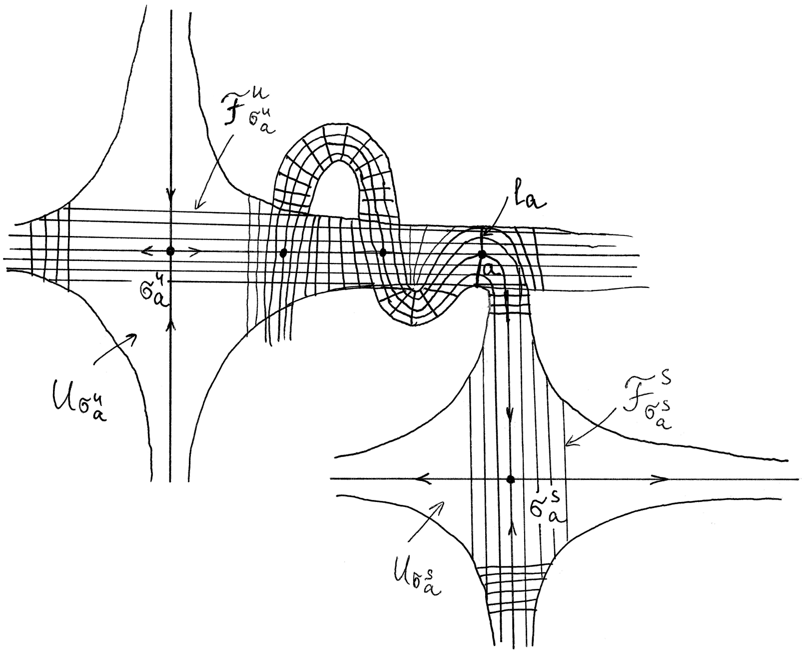

FIGURE 11. Construction of foliations

$U_{\mathcal{D}} = \bigcup\limits_{n\in\mathbb{Z}} f^n(U_D)$, $\mathcal{F}^u_D = \bigcup\limits_{d\in D} \mathcal{F}^u_d$, $\mathcal{F}^u_{\mathcal{D}} = \bigcup\limits_{n\in\mathbb{Z}} f^n(\mathcal{F}^u_D)$, $\mathcal{F}^s_D = \bigcup\limits_{d\in D} \mathcal{F}^s_d$ and $\mathcal{F}^s_{\mathcal{D}} = \bigcup\limits_{n\in\mathbb{Z}} f^n(\mathcal{F}^s_D)$. Construct the similar foliation $\mathcal{F}^u_{\mathcal{D}'}$ and $\mathcal{F}^s_{\mathcal{D}'}$ in the neighbourhood $U_{\mathcal{D}'}$ of the set $D' = \varphi(D)$.

Set $L_{\mathcal{A}} = \bigcup\limits_{a\in A} L_a$. For a point $\sigma \in \Omega_f$ define foliations $\tilde{\mathcal{F}}^s_\sigma$ and $\tilde{\mathcal{F}}^u_\sigma$ transverse to each other everywhere except $L_{\mathcal{A}}$ by the following way. The foliation $\tilde{\mathcal{F}}^s_\sigma$ ($\tilde{\mathcal{F}}^u_\sigma$) coincides with $\mathcal{F}^s_{\mathcal{A}}$ ($\mathcal{F}^u_{\mathcal{A}}$) on $U_\sigma \cap U_{\mathcal{A}}$, coincides with $\mathcal{F}^s_{\mathcal{D}}$ ($\mathcal{F}^u_{\mathcal{D}}$) on $U_\sigma \cap U_{\mathcal{D}}$ and coincides with $\mathcal{F}^s_\sigma$ ($\mathcal{F}^u_\sigma$) out of the set $U_{\mathcal{A}} \cup U_{\mathcal{D}}$ (see figure 11). Denote by $\tilde{\pi}^s_\sigma\colon U_\sigma \to W^u_\sigma$ ($\tilde{\pi}^u_\sigma\colon U_\sigma \to W^s_\sigma$) a projection along the leaves of the foliation $\tilde{\mathcal{F}}^s_\sigma$ ($\tilde{\mathcal{F}}^u_\sigma$). Construct similarly the foliation $\tilde{\mathcal{F}}^s_{\sigma'}$ ($\tilde{\mathcal{F}}^u_{\sigma'}$) and define the projection $\tilde{\pi}^s_{\sigma'}\colon U_{\sigma'} \to W^u_{\sigma'}$ ($\tilde{\pi}^u_{\sigma'}\colon U_{\sigma'} \to W^s_{\sigma'}$) in the neighbourhood $U_{\sigma'}$. Denote by $\tilde{\pi}^s_{\Omega^1}$, $\tilde{\pi}^u_{\Omega^1}$, $\tilde{\pi}^s_{\Omega'^1}$, $\tilde{\pi}^s_{\Omega'^1}$ maps consisting of $\tilde{\pi}^s_\sigma$, $\tilde{\pi}^u_\sigma$, $\tilde{\pi}^s_{\sigma'}$, $\tilde{\pi}^s_{\sigma'}$, $\sigma \in \Omega^1$, accordingly.

STEP 8. For each point $a \in \mathcal{A}$ let us define a homeomorphism $\varphi_{U_a}\colon U_a \to U_{a'}$ by the following way. Denote by U^+_a and U^-_a the connected components of $U_a \setminus l_a$ following a rule that any point $z = (z^u_x, 0) \in U_a$ belongs to U^+_a if $z^u_x > a^u_x$ and belongs to U^-_a if $z^u_x < a^u_x$. Similarly denote the connected components of $U_{a'} \setminus l_{a'}$. Define a homeomorphism $\varphi_{U^+_a}\colon U^+_a \to U^+_{a'}$ by the following way: for a point $z \in U^+_a$ set $\varphi_{U^+_a}(z) = z'$, where $z' \in U^+_{a'}$ is the intersection point of the leaves $(\tilde{\pi}^s_{\sigma^s_{a'}})^{-1}(\varphi^u_{\sigma^s_a}(\tilde{\pi}^s_{\sigma^s_a}(z)))$ and $(\tilde{\pi}^u_{\sigma^u_{a'}})^{-1}(\varphi^s_{\sigma^u_a}(\tilde{\pi}^u_{\sigma^u_a}(z)))$. In the similar way let us define a homeomorphism $\varphi_{U_a^-}: U_a^- \to U_{a'}^-$. Set

$$\varphi_{U_a}(z) = \begin{cases} \varphi_{U_a^+}(z), & z \in U_a^+; \\ \varphi_{U_a^-}(z), & z \in U_a^-; \\ \varphi_{l_a}, & z \in l_a. \end{cases}$$

Define a homeomorphism $\varphi_{U_\mathcal{A}}: U_\mathcal{A} \to U_{\mathcal{A}'}$ as coinciding with φ_{U_a} for each $a \in \mathcal{A}$.

For each point $d \in \mathcal{D}$ define a homeomorphism $\varphi_{U_d}: U_d \to U_{d'}$ by the following way: $\varphi_{U_d}(z)$ is the intersection point of the leaves $(\tilde{\pi}^s_{\sigma^u_{d'}})^{-1}(\varphi^u_{\sigma^u_d}(\tilde{\pi}^s_{\sigma^u_d}(z)))$ and $(\tilde{\pi}^u_{\sigma^u_{d'}})^{-1}(\varphi^s_{\sigma^u_d}(\tilde{\pi}^u_{\sigma^u_d}(z)))$ belonging to $U_{d'}$. Let us define a homeomorphism $\varphi_{U_\mathcal{D}}: U_\mathcal{D} \to U_{\mathcal{D}'}$ as a homeomorphism coinciding with φ_{U_d} for each $d \in \mathcal{D}$.

For $\delta \in \{u, s\}$ denote by $\varphi^\delta_{\Omega^{1\delta}}: W^\delta_{\Omega^{1\delta}} \to W^\delta_{\Omega'^{1\delta}}$ a homeomorphism conjugating the diffeomorphisms $f|_{W^\delta_{\Omega^{1\delta}}}$, $f'|_{W^\delta_{\Omega'^{1\delta}}}$, coinciding with the homeomorphism $\varphi_{U_\mathcal{A}}$ on $W^\delta_{\Omega^{1\delta}} \cap U_\mathcal{A}$, coinciding with the homeomorphism $\varphi_{U_\mathcal{D}}$ on $W^\delta_{\Omega^{1\delta}} \cap U_\mathcal{D}$ and coinciding with the homeomorphism φ out of some neighborhood of the set $W^\delta_{\Omega^{1\delta}} \cap (U_\mathcal{A} \cup U_\mathcal{D})$. Denote by $\varphi^u_{\Omega^1}: W^u_{\Omega^1} \to W^u_{\Omega^1}$ a homeomorphism composed from $\varphi^u_{\Omega^{1u}}$ and $\varphi^u_{\Omega^{1s}}$. Denote by $\varphi^s_{\Omega^1}: W^s_{\Omega^1} \to W^s_{\Omega^1}$ a homeomorphism composed from $\varphi^s_{\Omega^{1u}}$ and $\varphi^s_{\Omega^{1s}}$.

Set $U_{\Omega^1} = \bigcup\limits_{\sigma \in \Omega^1} U_\sigma$ and $U_{\Omega'^1} = \bigcup\limits_{\sigma' \in \Omega'^1} U_{\sigma'}$. Define a homeomorphism $\varphi_{U_{\Omega^1}}: U_{\Omega^1} \to U_{\Omega'^1}$ as a homeomorphism conjugating the diffeomorphisms $f|_{U_{\Omega^1}}$ and $f'|_{U_{\Omega'^1}}$, coinciding with the homeomorphism $\varphi_{U_\mathcal{A}}$ on $U_{\Omega^1} \cap U_\mathcal{A}$, coinciding with the homeomorphism $\varphi_{U_\mathcal{D}}$ on $U_{\Omega^1} \cap U_\mathcal{D}$ and such that for a $z \in (U_\sigma \setminus (U_\mathcal{A} \cup U_\mathcal{D}))$, $\varphi_{U_{\Omega^1}}(z)$ is the intersection point of the leaves $(\tilde{\pi}^s_{\Omega^1})^{-1}(\varphi^u_{\Omega^1}(\tilde{\pi}^s_{\Omega^1}(z)))$ and $(\tilde{\pi}^u_{\Omega^1})^{-1}(\varphi^s_{\Omega^1}(\tilde{\pi}^u_{\Omega^1}(z)))$.

Step 9. For any $t \in (0, 1)$ set $U^t_{\mu,\lambda} = \{(x, y) \in \mathbb{R}^2 : |x||y|^{-\log_\lambda \mu} \leqslant t\}$. For any $\sigma \in \Omega^1$ set $U^t_\sigma = \psi_\sigma^{-1}(U^t_{\mu_\sigma,\lambda_\sigma})$ and $U^t_{\Omega^1} = \bigcup\limits_{\sigma \in \Omega^1} U^t_\sigma$.

Let us choose a value $t_0 \in (0, 1)$ such that $\varphi_{U_{\Omega^1}}(U^{t_0}_{\Omega^1}) \subset (\varphi(U_{\Omega^1}) \cup W^s_{\Omega'^{1u}} \cup W^u_{\Omega'^{1s}})$. Set $Q = U_{\Omega^1} \setminus \operatorname{int} U^{t_0}_{\Omega^1}$, $R = \partial U_{\Omega^1}$, $R_0 = \partial U^{t_0}_{\Omega^1}$, $Q' = \varphi(U_{\Omega^1}) \setminus \operatorname{int} \varphi_{\Omega^1}(U^{t_0}_{\Omega^1})$, $R' = \varphi(\partial U_{\Omega^1})$, $R'_0 = \varphi_{U_{\Omega^1}}(\partial U^{t_0}_{\Omega^1})$, $\hat{Q} = p_f(Q)$, $\hat{Q}' = p_{f'}(Q')$ and

$$\hat{\varphi}_{U_{\Omega^1}} = p_{f'} \varphi_{U_{\Omega^1}} (p_f|_{R_0})^{-1}: \hat{R}_0 \to \hat{R}'_0.$$

By the construction the sets $\hat{Q}$, $\hat{Q}'$ have the same number of the connected components each of them is homeomorphic to the standard two-dimensional annulus (see Figure 12, where the set $\hat{Q}$ is coloured). Then there is a homeomorphism $\hat{\varphi}_{\hat{Q}}: \hat{Q} \to \hat{Q}'$ such that $\hat{\varphi}_{\hat{Q}}|_{\hat{R}} = \hat{\varphi}$ and $\hat{\varphi}_{\hat{Q}}|_{\hat{R}_0} = \hat{\varphi}_{U_{\Omega^1}}$.

Denote by $\varphi_Q: Q \to Q'$ a lift of the homeomorphism $\hat{\varphi}_{\hat{Q}}$ coinciding with φ on ∂U_{Ω^1}. Define a homeomorphism $\varphi_3: V_f \to V_{f'}$ by the formula:

$$\varphi_3(x) = \begin{cases} \varphi_{U_{\Omega^1}}(x), & x \in U^{t_0}_{\Omega^1}; \\ \varphi_Q(x), & x \in Q; \\ \varphi(x), & x \in M^2 \setminus U_{\Omega^1}. \end{cases}$$

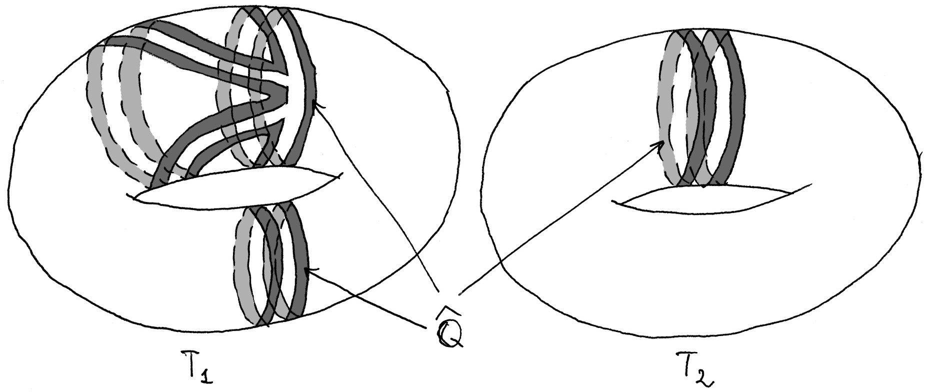

FIGURE 12. Illustration to Step 9

Let us define a homeomorphism $h\colon M^2 \setminus (\Omega^0 \cup \Omega^2) \to M'^2 \setminus (\Omega'^0 \cup \Omega'^2)$ by the formula:

$$h(x) = \begin{cases} \varphi_2(x), & x \in (W^s_{\mathcal{L}^s} \setminus \mathcal{L}^s \cup W^u_{\mathcal{L}^u} \setminus \mathcal{L}^u); \\ \varphi_3(x), & x \in (V_f \setminus (W^s_{\mathcal{L}^s} \cup W^u_{\mathcal{L}^u})); \\ \varphi_{\mathcal{L}^s}(x), & x \in \mathcal{L}^s; \\ \varphi_{\mathcal{L}^u}(x), & x \in \mathcal{L}^u; \\ \varphi^u_{\Omega^{1s}}(x), & x \in W^u_{\Omega^{1s}}; \\ \varphi^s_{\Omega^{1u}}(x), & x \in W^s_{\Omega^{1u}}. \end{cases}$$

Then, to obtain a desired homeomorphism, it suffices to extend the homeomorphism h continuously to the set $\Omega^0 \cup \Omega^2$. $\square$

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