Fermionic Formulas for Eigenfunctions of the Difference Toda Hamiltonian

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Abstract. We use the Whittaker vectors and the Drinfeld Casimir element to show that eigenfunctions of the difference Toda Hamiltonian can be expressed via fermionic formulas. Motivated by the combinatorics of the fermionic formulas we use the representation theory of the quantum groups to prove a number of identities for the coefficients of the eigenfunctions.

1. Introduction

The goal of this paper is to derive fermionic formulas for eigenfunctions of the finite difference Toda Hamiltonian H_{Toda} and to study these fermionic formulas. Eigenfunctions of H_{Toda} have been studied recently in connection with quantum cohomology of flag manifolds [3,12], Whittaker vectors [2,7,19], Macdonald polynomials and affine Demazure characters [11]. In particular, an important connection with the

representation theory of quantum groups was established. In our paper we show how fermionic formulas naturally appear in the representation-theoretical terms. These formulas provide explicit expressions for eigenfunctions of H_{Toda}, which can be studied from purely combinatorial point of view (note that existing expressions are based on the geometry of certain moduli spaces [2,3,12]). In the paper we combine these two approaches. We give some details below.

1.1. CENTRAL ELEMENTS AND WHITTAKER VECTORS

As we have already mentioned, the representation theory of quantum groups plays a very important role in the study of finite difference Toda Hamiltonian. In particular, one can construct eigenfunctions of H_{Toda} using Whittaker vectors in Verma modules [2,7,19]. In this paper we use pairing of Whittaker vectors with the dual ones.

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank l and let $U_v(\mathfrak{g})$ and $U_{v^{-1}}(\mathfrak{g})$ be two quantum groups with parameters v and v^{-1}. Let P, Q (resp. P_+, Q_+) be the weight and root lattices of $\mathfrak{g}$ (resp. their positive parts) and let $\mathcal{V}^\lambda = \sum_{\beta \in Q_+} (\mathcal{V}^\lambda)_\beta$ and $\overline{\mathcal{V}}^\lambda = \sum_{\beta \in Q_+} (\overline{\mathcal{V}}^\lambda)_\beta$ be Verma modules of $U_v(\mathfrak{g})$ and $U_{v^{-1}}(\mathfrak{g})$, respectively. In order to define a Whittaker vector θ^λ in the completion $\prod_{\beta \in Q_+} (\mathcal{V}^\lambda)_\beta$ of the Verma module $\mathcal{V}^\lambda$ one fixes elements $v_i \in P$ and scalars c_i $(1 \leq i \leq l)$. Then the Whittaker vector, associated with these data, is defined by the condition

$$E_i K_{v_i} \ \theta^\lambda = \frac{c_i}{1 - v^2} \ \theta^\lambda \tag{1.1}$$

(for simplicity, in Introduction, we assume that $\mathfrak{g}$ is simply-laced). Here $E_i \in U_v(\mathfrak{g})$ are the Chevalley generators (which act as annihilating operators) and K_{v_i} are certain elements from the Cartan subalgebra, associated with v_i. Similarly, one defines the dual Whittaker vector $\bar{\theta}^\lambda$ in the completion of $\overline{\mathcal{V}}^\lambda$ by the formula

$$\bar{E}_i \bar{K}_{v_i} \ \bar{\theta}^\lambda = \frac{c_i^{-1}}{1 - v^{-2}} \ \bar{\theta}^\lambda \tag{1.2}$$

The central object of our paper is the following function

$$J_\beta^\lambda = v^{-(\beta,\beta)/2 + (\lambda,\beta)} \ (\theta_\beta^\lambda, \bar{\theta}_\beta^\lambda),$$

where $\theta_\beta^\lambda \in (\mathcal{V}^\lambda)_\beta$ is the weight $\lambda - \beta$ component of the Whittaker vector and $(\ ,\ )$ is the natural non-degenerate pairing between $\mathcal{V}^\lambda$ and $\overline{\mathcal{V}}^\lambda$. It can be shown that J_β^λ is independent of possible choices of v_i and c_i.

Consider the generating function

$$F(q, z_1, \ldots, z_l, y_1, \ldots, y_l) = \sum_\beta J_\beta^\lambda \prod_{i=1}^l y_i^{(\beta, \omega_i)},$$

where $z_i = q^{-(\lambda, \alpha_i)}$, $q = v^2$ and ω_i (resp. α_i) are fundamental weights (resp. simple roots). Then F is known to be an eigenfunction of the quantum difference Toda

operator [7,19]. In order to prove this statement one uses central elements of the quantum group. Roughly, the procedure works as follows. If u is a central element, then the scalar product

$$(u\theta_\beta^\lambda, \bar\theta_\beta^\lambda) \tag{1.3}$$

can be written in two ways. On the one hand, one can compute the action of u on $\mathcal{V}^\lambda$ (the corresponding scalar). On the other hand, if a precise formula for u is known then one can compute (1.3) using the relation

$$(F_i w, \bar w) = (w, \bar E_i \bar w)$$

and formulas (1.1), (1.2).

The Toda Hamiltonian appears when one uses the central element written as the trace of products of R matrices in finite-dimensional $U_v(\mathfrak{g})$ modules (see (3.24) and the end of Section 3 for the explicit form of H_{Toda}). Our key observation is that if the Drinfeld Casimir element is used instead then one obtains a recursion relation for F which leads to the fermionic formulas. In the next subsection we describe those formulas in more details.

1.2. FERMIONIC FORMULAS

Fermionic formulas appear in different problems of representation theory and mathematical physics (see for example [4,10,13,21]). Let us describe the class of formulas we treat in our paper.

Let $[r, s] = \{t \in \mathbb{Z} \mid r \le t \le s\}$ be a subset of $\mathbb{Z}$, where r, s are integers or $\pm\infty$. Let V be a vector space with a basis $e_{i,t}$ labeled by pairs $1 \le i \le l$, $t \in [r, s]$. Let $\Gamma_+ = \{\sum_{(i,t)} m_{i,t} e_{i,t} \mid m_{i,t} \in \mathbb{Z}_{\ge 0}\}$ be the positive part of the lattice generated by $\{e_{i,t}\}$. We fix a quadratic form $\langle \cdot, \cdot \rangle$ on V and a vector $\mu \in V$. Further, define maps w and d from V to the l-dimensional vector space with a basis $p_1, \ldots, p_l$ via the formulas

$$w\left(\sum_{(i,t)} m_{i,t} e_{i,t}\right) = \sum_{i=1}^l p_i \sum_{t \in [r,s]} m_{i,t}, \quad d\left(\sum_{(i,t)} m_{i,t} e_{i,t}\right) = \sum_{i=1}^l p_i \sum_{t \in [r,s]} t m_{i,t}.$$

Define functions I_m depending on q, $z = (z_1, \ldots, z_l)$ and $m = (m_1, \ldots, m_l)$ as follows

$$I_m(q, z) = \sum_{w(\gamma) = m} z^{d(\gamma)} \frac{q^{\langle \gamma, \gamma \rangle + \langle \mu, \gamma \rangle}}{(q)_\gamma}, \tag{1.4}$$

where the summands are labeled by $\gamma = \sum_{(i,t)} m_{i,t} e_{i,t} \in \Gamma_+$ and $(q)_\gamma = \prod_{(i,t)} (q)_{m_{i,t}}$, $z^{d(\gamma)} = \prod_{i=1}^l z_i^{d(\gamma)_i}$. We call the right hand side of (1.4) a fermionic formula. The generating function $F(q, z, y) = F(q, z_1, \ldots, z_l, y_1, \ldots, y_l)$ is given by the formula

$$F(q, z, y) = \sum_m y^m I_m(q, z), \quad y^m = y_1^{m_1} \ldots y_l^{m_l}. \tag{1.5}$$

Let the matrix of the quadratic form $\langle \cdot, \cdot \cdot \rangle$ be a tensor product $D = C \otimes G(r, s)$, where C is the Cartan matrix of $\mathfrak{g}$ (we assume here that C is symmetric) and $G = (G_{t,t'})_{i,j \in [r,s]}$, $G_{t,t'} = \min(t, t')$. Such matrices appear in [5,20] in the fermionic formulas for the Kostka polynomials. Let $[r, s] = [0, \infty)$. Then functions $I_m(q, z)$ satisfy the following recursion relation:

$$I_m(q, z) = \sum_{0 \leq a \leq m} \frac{z^a q^{W(a)}}{(q)_{m-a}} I_a(q, z), \tag{1.6}$$

where $W(a) = \frac{1}{2}(Ca \cdot a - \mathrm{diag} C \cdot a)$, $\cdot$ denotes the standard scalar product and $0 \leq a \leq m$ abbreviates the set of inequalities $0 \leq a_i \leq m_i$. The relation (1.6) shows that $I_m(q, z)$ are determined by $I_0(q, z)$.

Recall the functions J_β^λ. Using the Drinfeld Casimir element and the procedure described in the end of subsection 1.1, we show that J_β^λ satisfy the relation

$$J_\beta^\lambda = \sum_{\beta'} \frac{1}{(q)_{\beta-\beta'}} q^{(\beta',\beta')/2 - (\lambda+\rho,\beta')} J_{\beta'}^\lambda.$$

This leads to the following identification

$$J_\beta^\lambda = I_m(q, z), \quad \beta = \sum_i m_i \alpha_i, \quad z = q^{-(\lambda,\alpha_i)}.$$

In particular, this gives a fermionic formula for eigenfunctions of H_{Toda}.

Fermionic sums can be considered as a sort of statistical sum for some "models". The models depend on parameters r and s and enjoy many "physical" combinatorial properties. For example, we look into what happens with the fermionic sums when the parameters, e.g., r and s, go to infinity. We sort the terms by the dependence on the parameters which go to infinity and call the result the quasi-classical decomposition. Then we expect that the quasi-classical decompositions are exact, which means that the coefficients in a decomposition are summed up to rational functions and the result gives correct formulas for finite values of parameters. That expectation predicts recursion relations for the fermionic sums I_m.

We then prove the recursion relations using the Whittaker vectors and the representation theory of the quantum group $U_v(\mathfrak{g})$, (see Theorems 4.11–4.13). In some cases the relations become finite. From the point of view of fermionic sums it means the vanishing property: some fermionic expressions must be zero. The quantum group approach explains the vanishing property as well. Namely, some terms in the recursions are zero because the corresponding weight space is zero in the irreducible representation of the quantum group.

It is well known [7] that the eigenfunctions of the difference Toda Hamiltonian can be obtained as a certain limit of the Macdonald polynomials. In the case of $\mathfrak{sl}_n$, one of the recursions we prove [see (4.20)] is the corresponding limit of the Pierri rule for the Macdonald polynomials. It is interesting to study the other identities in relation with the Macdonald polynomials. We hope to address this problem in future publications.

1.3. AFFINE LIE ALGEBRAS: MOTIVATIONS AND FURTHER DIRECTIONS

In this subsection we discuss various connections between eigenfunctions of H_{Toda} and representation theory of affine Kac–Moody algebras. The fermionic formulas provide a very useful tool for computation of various "affine" characters. Though we do not treat this subject in the main body of the paper, it was our original motivation for studying the "fermionic part" of the quantum difference Toda story. We are not providing any proofs here. We hope to return to this subject in more details elsewhere.

1.3.1. Refined characters Let $\mathfrak{a}$ be a Lie algebra and W its representation. The character of W is an expression

$$\chi(z_1, \ldots, z_n) = \mathrm{tr}_W\left(z_1^{a_1} \cdots z_n^{a_n}\right),$$

where a_i are some commuting elements of $\mathfrak{a}$. Usually, $\mathfrak{a}$ is semi-simple and $(a_1, \ldots, a_n)$ is a basis of its Cartan subalgebra. There is a simple way to "refine" the character. To do it, suppose that W is a cyclic representation with a cyclic vector v, and choose a subspace $S \subset U(\mathfrak{a})$ such that $1 \in S$. We define subspaces $F_j \hookrightarrow W$, $j = 0, 1, \ldots$: $F_0 = \mathbb{C} \cdot v$ and $F_{j+1} = S \cdot F_j$. Assuming that F_j converge to W we obtain a filtration in W. Suppose also that the space S is invariant with respect to the adjoint action of a_i for all i:

$$[a_i, S] \subset S.$$

In this case, if F_0 is $\{a_i\}$ invariant, i.e., $a_i F_0 \subset F_0$, then all spaces F_j in the filtration are $\{a_i\}$-invariant. Consider now the associated graded space

$$\overline{W} = F_0 \oplus \left(\bigoplus_{j>0} F_j/F_{j-1}\right).$$

On $\overline{W}$ we have an action of $\{a_i\}$ and also the action of an additional operator b, which acts as the constant j on F_j/F_{j-1}. Now, we define the refined character

$$\chi(z_1, \ldots, z_n; y) = \mathrm{tr}_{\overline{W}}\left(z_1^{a_1} \cdots z_n^{a_n} y^b\right),$$
$$\chi(z_1, \ldots, z_n; 1) = \chi(z_1, \ldots, z_n).$$

We consider the case $\mathfrak{a} = \mathfrak{h} \oplus \widehat{\mathfrak{n}}$. Here $\widehat{\mathfrak{n}} = \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}]$, $\mathfrak{n}$ is a maximal nilpotent subalgebra of a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$, $\tilde{\mathfrak{h}} \simeq \mathfrak{h} \otimes 1 \oplus \mathbb{C}d$ is the Cartan subalgebra of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}d$ and $d = x\mathrm{d}/\mathrm{d}x$ is the grading operator. Let $e_1, e_2, \ldots, e_l$ be the generators of $\mathfrak{n}$ and $e_i[j] = e_i \otimes t^j$ be the corresponding generators of $\widehat{\mathfrak{n}}$. We define currents $e_i(x) = \sum_{j \leq 0} e_i[j]x^{-j}$. Fix a basis $\{d, h_1, \ldots, h_l\}$ in $\tilde{\mathfrak{h}}$. As a representation W we take the induced module generated by the vacuum vector v satisfying $e_i[j]v = 0$ for $j > 0$. We have the character

$$\chi(q, z_1, \ldots, z_l) = \mathrm{tr}_W\left(q^{-d} z_1^{h_1} \cdots z_l^{h_l}\right).$$

1.3.2. The A_1 case We consider $\mathfrak{g} = \mathfrak{sl}_2$. In this case $\mathfrak{n}$ is one-dimensional and is spanned by an element e. We fix W to be an induced $\mathfrak{a}$ module with a cyclic vector v satisfying $\mathfrak{h} \cdot v = 0$ and $e[j]v = 0$, $j > 0$. Let S be the subspace spanned by coefficients of the expansion of $e(x)^s$, $s \geq 0$ as series in x. Then we obtain

$$\chi(q, z) = \frac{1}{(z)_\infty},$$

$$\chi(q, z; y) = \sum_{m \geq 0} \frac{y^m z^m q^{m^2}}{(q)_m (z)_m},$$

where $(z)_m = \prod_{n=1}^m (1 - q^{n-1}z)$. We prove in Appendix A [see (A.5)] that $\chi(q, z; y)$ differs from the generating function of J_β^λ (or, equivalently, of I_m) by a simple factor. We now give fermionic expression for the quantity $\chi(q, z; y)$.

Introduce an algebra generated by Fourier coefficients $c_j[s]$ of the currents $c_j(x) = \sum_{s \leq 0} c_j[s]x^{-s}$, $j = 0, 1, 2, \ldots$ The defining relations are $[c_{j_1}[s_1], c_{j_2}[s_2]] = 0$ for all j_1, j_2, s_1, s_2, and $c_j(x)^2 = 0$. Note that $c_j(x)$ can be constructed as vertex operators with momentum p_j such that $\langle p_j, p_j \rangle = 2$ and $\langle p_i, p_j \rangle = 0$ for $i \neq j$. Let

$$e(\varepsilon, x) = \sum_j \varepsilon^j c_j(x),$$

where ε is a formal variable. Let $\mathcal{A}$ be an algebra over $\mathbb{C}[[\varepsilon]]$, the ring of formal power series in ε, which is generated by $e(\varepsilon, x)$. In $\mathcal{A}$ there exists a subspace S spanned by the elements $\{e(\varepsilon, x)^n; n \geq 0\}$, and a filtration $\mathcal{F}_j$ such that $\mathcal{F}_0 = \mathbb{C} \cdot 1$ and $\mathcal{F}_j = \mathcal{F}_{j-1} + S \cdot \mathcal{F}_{j-1}$. The associated graded space $\overline{\mathcal{A}}$ naturally has a structure of commutative algebra. It is generated by the space $\mathcal{F}_1/\mathcal{F}_0$. Actually, $\overline{\mathcal{A}}$ is a quadratic algebra. It is a free module over $\mathbb{C}[[\varepsilon]]$. Let us consider the specialization $\overline{\mathcal{A}}_0$ at $\varepsilon = 0$. The algebra $\overline{\mathcal{A}}_0$ is generated by the currents $d_1(x) = c_0(x)$, $d_2(x) = c_0(x)c_1(x)$, $d_3(x) = c_0(x)c_1(x)c_2(x)$, etc. The defining relations in $\overline{\mathcal{A}}_0$ are quadratic and takes the form involving derivatives of the currents:

$$d_i(x) \cdot d_j(x)^{(l)} = 0, \quad 0 \leq l \leq 2\min(i, j). \tag{1.7}$$

The representation W for the algebra $\mathcal{A} = \{e(\varepsilon, x)\}$ is also defined on the free module over the ring $\mathbb{C}[[\varepsilon]]$, and after substituting $\varepsilon = 0$ we get a representation over $\overline{\mathcal{A}}_0$. It is quadratic with simple relations, and this construction gives us a fermionic formula for the refined character χ:

$$\chi(q, z; y) = \sum_{\{m_j\}} \frac{q^{2\sum \min(i,j)m_i m_j} z^{\sum j m_j} y^{\sum m_j}}{\prod (q)_{m_j}}.$$

1.3.3. General $\mathfrak{g}$ and multi-filtration In order to treat the general case we need to replace the filtration F_i by a "multi-filtration". Let S_i, $i = 1, \ldots, l$, be the subspaces of $U(\widehat{\mathfrak{n}})$ spanned by coefficients of the expansion of $e_i(x)^s$, $s \geq 0$, as series in x,

and let $R_i = \oplus_{k \le i} S_k$, where $R_0 = S_0 = \mathbb{C} \cdot 1$. Note that $R_1 \subset R_2 \subset \cdots \subset R_l$. Then, we define

$$F_{j_1} = R_1 \cdot F_{j_1-1}, \quad F_0 = \mathbb{C} \cdot v,$$
$$F_{j_1,j_2} = R_2 \cdot F_{j_1,j_2-1}, \quad F_{j_1,0} = F_{j_1},$$
$$\cdots$$

We define

$$\overline{W} = \oplus_{j_1,\ldots,j_l} \overline{W}_{j_1,\ldots,j_l},$$
$$\overline{W}_{j_1,\ldots,j_l} = F_{j_1,\ldots,j_l} / \sum_i F_{j_1,\ldots,j_i-1,\ldots,j_l}$$

and denote by b_i the operator which gives the constant j_i on $\overline{W}_{j_1,\ldots,j_l}$.

In this way, we get the subspaces $F_{j_1,\ldots,j_l}$ and the associated graded spaces $\overline{W}_{j_1,\ldots,j_l}$; the latter have the actions of the grading operators $b_1, \ldots, b_l$. Since the operators $d, h_1, \ldots, h_l$ act on $\overline{W}$ in an evident way, we can write

$$\chi(q, z_1, \ldots, z_l; y_1, \ldots, y_l) = \mathrm{tr}_{\overline{W}} \left(q^{-d} z_1^{h_1} \cdots z_l^{h_l} y_1^{b_1} \cdots y_l^{b_l} \right).$$

We conjecture that this refined character gives an eigenfunction of the conjugated quantum Toda Hamiltonian (see [3,12], Appendix A). We now explain how to obtain a fermionic formula in the general case.

1.3.4. General fermionic formula Let L_1 be the vacuum representation of $\widehat{\mathfrak{g}}$ of level 1 with highest weight vector v, and let V be the principal subspace in L_1, i.e., $V = U(\widehat{\mathfrak{n}}) \cdot v$. The space V, as a representation of $U(\widehat{\mathfrak{n}})$ can be described by using the space W as

$$V = W / \sum_i e_i(x)^2 W.$$

Consider the tensor product of infinitely many copies of V labeled by $j = 0, 1, 2 \ldots$, and denote by $a_i^{(j)}$ the operator e_i acting on the jth copy:

$$a_i^{(j)} = 1 \otimes \cdots \otimes \underbrace{e_i}_{j\text{th}} \otimes \cdots$$

Now, set

$$e_i(\varepsilon, x) = \sum_{\alpha \ge 0} \varepsilon^\alpha a_i^{(\alpha)}(x).$$

After substituting $\varepsilon = 0$ we get an algebra generated by $c_i^{(\beta)}(x) = a_i^{(0)}(x) a_i^{(1)}(x) \cdots a_i^{(\beta-1)}(x)$. The currents $c_i^{(\beta)}(x)$ generate an algebra with the quadratic relations

$$c_i^{(\beta)}(x) c_j^{(\gamma)}(x)^{(l)} = 0 \quad \text{if} \quad l \le \min(\alpha, \beta) \cdot C_{i,j} \tag{1.8}$$

where $(C_{i,j})$ is the Cartan matrix of $\mathfrak{g}$. As a result we have a fermionic formula for the refined character of W:

$$\chi(q, z_1, \ldots, z_l; y_1, \ldots, y_l) = \sum_{\{n_i^{(t)}\}} \frac{q^B}{\prod_{t \geq 0} \prod_{i=1}^l (q_i; q_i)_{n_i^{(t)}}} \prod_{i=1}^l z_i^{\sum_{t \geq 0} t n_i^{(t)}} \prod_{i=1}^l y_i^{\sum_{t \geq 0} n_i^{(t)}},$$

(1.9)

$$B = \sum_{t \geq 0} t \left(\sum_{i,j=1}^l \frac{1}{2} b_{i,j} n_i^{(t)} n_j^{(t)} - \sum_{i=1}^l d_i n_i^{(t)} \right) + \sum_{t < t'} t \sum_{i,j=1}^l b_{i,j} n_i^{(t)} n_j^{(t')},$$

where $b_{i,j} = (\alpha_i, \alpha_j)$, $d_i = b_{i,i}/2$ and $q_i = q^{d_i}$. We note that if $\mathfrak{g}$ is simply laced the formula (1.9) corresponds to (1.4) and (1.5).

1.4. PLAN OF THE PAPER

Now let us outline the content of our paper.

In Section 2 we introduce the fermionic sums. We study quasi-classical limits of such formulas and various recursion relations.

In Section 3 we prove the fermionic formulas for the scalar products of the Whittaker vectors with dual ones. We also discuss the general procedure (based on the center of the quantum group), which produces equations satisfied by J_β^λ.

In Section 4 we study the quasi-classical decompositions using the representation theory of $U_v(\mathfrak{g})$. We prove the recursion relations and vanishing properties from Section 2.

In A_l case there is an alternative simple way to prove that the fermionic formula satisfies the Toda equation. We give this proof in Appendix A.

In Appendix B, we prove a proposition on the singular vectors in the tensor product of two Verma modules. We need this lemma to prove the vanishing property.

2. Fermionic Sums

2.1. FERMIONIC SUMS ON A FINITE INTERVAL

Let $l \in \mathbb{Z}_{\geq 1}$ be a positive integer, and let $[r, s] = \{t \in \mathbb{Z} \mid r \leq t \leq s\}$ be a finite interval in $\mathbb{Z}$. Here r, s are integers, but in later subsections we also consider the case $r = -\infty$ and/or $s = \infty$.

Let $C = (C_{i,i'})_{1 \leq i, i' \leq l}$ be a symmetric matrix, and let $\mu = (\mu_{i,t})_{1 \leq i \leq l, t \in [r,s]}$ be a vector. Let $m = (m_1, \ldots, m_l)$ be a set of non-negative integers. We will define a fermionic sum $I_{C,\mu,m}(q, z|r, s)$ on the interval $[r, s]$ corresponding to the data C, μ and m. Each fermionic sum is a rational function in q and $z = (z_1, \ldots, z_l)$. It is defined as a sum of rational functions parameterized by a configuration of particles $\mathbf{m}$, which we will explain below.

We call a tuple of non-negative integers

$$\mathbf{m} = \{m_{i,t} : \ 1 \leq i \leq l, r \leq t \leq s\}$$

a configuration of particles. An integer $i \in [1, l]$ is called color and $t \in [r, s]$ is called weight of the particle. The non-negative integer $m_{i,t}$ represents the number of particles with color i and weight t. For a configuration of particles $\mathbf{m}$, we associate a vector $\overline{\mathbf{m}} = (m_i)_{1 \leq i \leq l}$ by

$$m_i = \sum_{t \in [r,s]} m_{i,t}. \tag{2.1}$$

The number m_i is the number of particles with color i.

We set $(w)_n = \prod_{i=1}^{n}(1 - q^{i-1}w)$ and define

$$(q)_{\mathbf{m}} = \prod_{(i,t) \in [1,l] \times [r,s]} (q)_{m_{i,t}}$$

for any tuple $\mathbf{m} = (m_{i,t})$. We also use the standard scalar product $(\mathbf{m}, \mathbf{n}) = \sum_{(i,t) \in [1,l] \times [r,s]} m_{i,t} n_{i,t}$.

DEFINITION 2.1 Let $m = (m_i) \in Z_{\geq 0}^{l}$ and $\mu = (\mu_{i,t}) \in \mathbb{Z}^{l(s-r+1)}$ be two vectors. A fermionic sum $I_{C,\mu,m}(q, z|r, s)$ is a function in q and $z = (z_1, \ldots, z_l)$ defined by

$$I_{C,\mu,m}(q, z|r, s) = \sum_{\overline{\mathbf{m}} = m} \prod_{i=1}^{l} z_i^{\sum_{t=r}^{s} t m_{i,t}} \frac{q^{Q_C(\mathbf{m}) + (\mu, \mathbf{m})}}{(q)_{\mathbf{m}}}, \tag{2.2}$$

where

$$Q_C(\mathbf{m}) = \frac{1}{2} \left\{ ((C \otimes G)\mathbf{m}, \mathbf{m}) - (\operatorname{diag}(C \otimes G), \mathbf{m}) \right\}, \tag{2.3}$$

the matrix G is defined by

$$G = (G_{t,t'})_{t,t' \in [r,s]}, \qquad G_{t,t'} = \min(t, t'),$$

and for a matrix $X = (X_{i,j})$, $\operatorname{diag}(X)$ signifies the vector consisting of diagonal entries $X_{i,i}$.

The quantity $I_{C,\mu,m}(q, z|r, s)$ is a Laurent polynomial in $z = (z_1, \ldots, z_l)$ with coefficients which are Laurent polynomials in $q^{C_{i,i'}}, q^{\mu_{i,t}}$ and rational functions in q.

We define the following formal power series in $y = (y_1, \ldots, y_l)$.

$$F_{C,\mu}(q, z, y|r, s) = \sum_{m \in \mathbb{Z}_{\geq 0}^{l}} y^m I_{C,\mu,m}(q, z|r, s). \tag{2.4}$$

Here we use the notation $y^m = \prod_{i=1}^{l} y_i^{m_i}$ for l-component vectors y and m. We use the convention $F_{C,\mu}(q, z, y|r, s) = 1$ if $r = s + 1$. We denote the functions $F_{C,\mu}(q, z, y|r, s)$, $I_{C,\mu,m}(q, z|r, s)$ in the case of $\mu = 0$ by $F_C(q, z, y|r, s)$, $I_{C,m}(q, z|r, s)$, dropping the parameter μ.

In what follows we also need another parametrization of configurations of particles. Namely, with each $\mathbf{m} = (m_{i,t})$, $1 \le i \le l$, $t \in [r, s]$, we associate the vector

$$\mathbf{p} = (p_{i,j}), \quad 1 \le i \le l, 1 \le j \le m_i$$

defined by two conditions:

- $p_{i,1} \le p_{i,2} \le \cdots \le p_{i,m_i}$, $1 \le i \le l$,
- $m_{i,t} = \#\{j : p_{i,j} = t\}$.

It is easy to see that the correspondence $\mathbf{m} \leftrightarrow \mathbf{p}$ is one-to-one. In the following lemma we rewrite powers of z and q in (2.2) in terms of $\mathbf{p}$. To avoid confusion we denote the function $Q_C(\mathbf{m})$ written in $\mathbf{p}$ coordinates by $\overline{Q}_C(\mathbf{p})$.

LEMMA 2.2. *We have*

$$z_i^{\sum_{t=r}^{s} t m_{i,t}} = z_i^{\sum_{j=1}^{m_i} p_{i,j}}, \tag{2.5}$$

$$Q_C(\mathbf{m}) = \overline{Q}_C(\mathbf{p}) = \frac{1}{2} \left(\sum_{(i,j),(i',j')} C_{i,i'} \min(p_{i,j}, p_{i',j'}) - \sum_{(i,j)} C_{i,i} p_{i,j} \right). \tag{2.6}$$

If we shift the parameters $\mu_{i,t}$ to $\mu_{i,t} + t\nu_i + \kappa_i$, the sum $I_{C,\mu,m}(q, z|r, s)$ responds by a simple prefactor and q-shifts of z_i:

$$I_{C,\mu+t\nu+\kappa,m}(q, z|r, s) = q^{\kappa \cdot m} I_{C,\mu,m}(q, q^\nu z|r, s). \tag{2.7}$$

Here, $\nu = (\nu_i)_{i \in [1,l]}$ and $(\mu + t\nu + \kappa)_{i,t} = \mu_{i,t} + t\nu_i + \kappa_i$. We use the abbreviated notation $\kappa \cdot m = \sum_{i=1}^{l} \kappa_i m_i$ and $q^\nu z = (q^{\nu_i} z_i)_{i \in [1,l]}$.

If $\mu = 0$ and we shift the interval, we get a simple factor:

$$I_{C,m}(q, z|r+k, s+k) = \left(z^m q^{W_{C,m}} \right)^k I_{C,m}(q, z|r, s),$$

where

$$W_{C,m} = \frac{1}{2}(Cm \cdot m - \operatorname{diag} C \cdot m).$$

2.2. FERMIONIC SUMS ON A SEMI-INFINITE INTERVAL

We replace the interval $[r, s]$ in the construction in Section 2.1 by an infinite interval. Let us consider the case $[r, s] = [0, \infty)$. We use the abbreviation

$$I_{C,\mu,m}(q, z) = I_{C,\mu,m}(q, z|0, \infty), \quad I_{C,m}(q, z) = I_{C,0,m}(q, z|0, \infty). \tag{2.8}$$

In this case, the sum in the right hand side of (2.2) becomes an infinite sum. We are interested in the case where the parameters $\mu = (\mu_{i,t})_{(i,t) \in [1,l] \times [0,\infty)}$ are specialized so that the infinite sum is well-defined as a rational function in $q, z_i, q^{C_{i,i'}}, q^{\mu_{i,t}}$.

LEMMA 2.3. *Suppose that there exists $t_0 \geq 0$ and ν_i, κ_i $(1 \leq i \leq l)$ such that*

$$\mu_{i,t} = t\nu_i + \kappa_i \quad if \quad t \geq t_0. \tag{2.9}$$

Then, the sum (2.2) is a polynomial in $q^{\mu_{i,t}}$ $((i,t) \in [1,l] \times [0, t_0 - 1])$ and q^{κ_i} $(1 \leq i \leq l)$ with coefficients which are rational functions in $q, z_i, q^{C_{i,i'}}, q^{\nu_i}$ $(1 \leq i, i' \leq l)$.

The proof is easy, and we omit it.

There are simple relations between the fermionic sums over semi-infinite intervals with $\mu = 0$. Namely, we can reduce all cases to $[0, \infty)$. We have

$$I_{C,m}(q, z | k, \infty) = (z^m q^{W_{C,m}})^k I_{C,m}(q, z), \tag{2.10}$$

$$I_{C,m}(q, z | -\infty, k) = (z^m q^{W_{C,m}})^k I_{C,m}(q, z^{-1} q^{-Cm + \mathrm{diag}\, C}). \tag{2.11}$$

Here we used the abbreviation $z^{-1} q^m = (z_1^{-1} q^{m_1}, \ldots, z_l^{-1} q^{m_l})$.

The following proposition determines the fermionic sums recursively.

PROPOSITION 2.4. *The rational functions $I_{C,m}(q, z)$ satisfy the recursion*

$$I_{C,m}(q, z) = \sum_{0 \leq a \leq m} \frac{z^a q^{W_{C,a}}}{(q)_{m-a}} I_{C,a}(q, z). \tag{2.12}$$

The solution of this recursion is unique if we fix $I_{C,0}(q, z) = 1$.

Proof. We subdivide the fermionic sum in the left hand side into parts labeled by $a = (a_1, \ldots, a_l) \in \mathbb{Z}_{\geq 0}^l$, with a_i being the number of color i particles whose weights are larger than or equal to 1. Then this corresponds to the sum in the right hand side because of (2.10). The uniqueness is clear because the equation can be written as

$$(1 - z^m q^{W_{C,m}}) I_{C,m}(q, z) = \sum_{0 \leq a < m} \frac{z^a q^{W_{C,a}}}{(q)_{m-a}} I_{C,a}(q, z).$$

$\square$

We call (2.12) the fermionic recursion.

In what follows we study functions $I_{C,\mu,m}(q, z)$ for some special values of μ of the type (2.9). We describe μ in terms of corners and angles.

We say μ has a corner at $t \in [1, \infty)$ if the vector $\mu[t]$ given by $\mu_i[t] = \mu_{i,t+1} + \mu_{i,t-1} - 2\mu_{i,t}$ is not zero. We call $\mu[t]$ the angle of μ at t. We define $\mu_i[0] = \mu_{i,1} - \mu_{i,0}$, and call $\mu[0]$ the angle of μ at 0. We say μ has a corner at 0 if $\mu[0] \neq 0$. Let us discuss simple cases.

First, consider the case where $\mu_{i,t} = t\nu_i + \kappa_i$ for all $t \geq 0$. It reduces to the basic case $I_{C,m}(q, z)$ by (2.7).

Second, we define the case with one corner. Namely, we consider

$$\mu_{i,t} = \begin{cases} 0 & \text{if } t \le k; \\ (t-k)\nu_i & \text{if } t \ge k. \end{cases} \tag{2.13}$$

We denote the fermionic sum corresponding to this μ by $J_{C,m}^{(k,\nu)}(q,z)$.

LEMMA 2.5. *There exist mutual relations between* $I_{C,m}(q,z), J_{C,m}^{(k,\nu)}(q,z), I_{C,m}$
$(q,z|0,k)$. *We have*

$$J_{C,m}^{(0,\nu)}(q,z) = I_{C,m}(q,q^\nu z), \tag{2.14}$$

$$J_{C,m}^{(k,\nu)}(q,z) = \sum_{0 \le a \le m} \left(z^a q^{W_{C,a}} \right)^k I_{C,m-a}(q, q^{Ca} z|0, k-1) I_{C,a}(q, q^\nu z), \tag{2.15}$$

$$= \sum_{0 \le a \le m} q^{a \cdot \nu} \left(z^a q^{W_{C,a}} \right)^{k+1} I_{C,m-a}(q, q^{Ca} z|0, k) I_{C,a}(q, q^\nu z). \tag{2.16}$$

Proof. The relation (2.14) is trivial. In order to prove the other two relations we cut the interval of weights $[0, \infty)$ into two parts. For (2.15) these parts are $[0, k-1]$ and $[k, \infty)$ and for (2.16) they are $[0, k]$ and $[k+1, \infty)$. The rest is straightforward.   □

If we set $k = 0$ in (2.15) we obtain (2.14).

Let us denote the limit $q^\nu \to 0$ symbolically by $\nu \to \infty$ (see Lemma 2.3). From (2.16) follows that

$$\lim_{\nu \to \infty} J_{C,m}^{(k,\nu)}(q,z) = I_{C,m}(q,z|0,k).$$

2.3. QUASI-CLASSICAL DECOMPOSITION OF FERMIONIC SUMS

In this subsection, we discuss the decomposition of fermionic sums with respect to the dependence on some large parameters. We call it the quasi-classical decomposition. We first explain the idea in simple examples, and then discuss more general cases.

EXAMPLE 1. Consider the simplest case $l = 1, [r,s] = [0,k], \mu = 0, m = 1$. Then one gets

$$I_{C,1}(q,z|0,k) = \frac{1 + z + \cdots + z^k}{1-q} = \frac{1}{(1-q)(1-z)} - \frac{z^{k+1}}{(1-q)(1-z)}.$$

The result consists of two terms. It is a linear combination of 1 and z^k with rational function coefficients independent of k. This decomposition can be explained by examining the large k behavior. There is one particle in the interval $[0,k]$. The

weight p of the particle is restricted to this interval, $0 \leq p \leq k$. If k is large, the sum over p for $0 \leq p \ll k$ and that for $0 \leq k - p \ll k$ does not overlap. Considering the sum over p for $0 \leq p$ and that for $p \leq k$ we obtain $I_{C,1}(q, z|0, \infty)$ and $I_{C,1}(q, z|-\infty, k)$. In fact, the above decomposition is the same as

$$I_{C,1}(q, z|0, k) = I_{C,1}(q, z|0, \infty) + I_{C,1}(q, z|-\infty, k).$$

The sums in the right hand side contains more terms than the sum over $0 \leq p \leq k$. However, since $\sum_{p \in \mathbb{Z}} z^p = 0$ as rational function, we have the equality.

EXAMPLE 2. Let $l = 1, [r, s] = [0, k], \mu = 0$ as before but $m = 2$. Since $l = 1$ the matrix C is simply a scalar. We denote it by c. One can check the following equality.

$$I_{C,2}(q, z|0, k) = I_{C,2}(q, z|0, \infty) + I_{C,1}(q, qz|0, \infty) I_{C,1}(q, z|-\infty, k) +$$
$$+ I_{C,2}(q, z|-\infty, k).$$

Each of the terms in the right hand side has the distinction that the dependence on k enters only through $1, z^k, (z^2 q^c)^k$, respectively. These sums are over p_1, p_2 in the regions

$$\{0 \leq p_1 \leq p_2\}, \quad \{0 \leq p_1, 0 \leq k - p_2\}, \quad \{0 \leq k - p_2 \leq k - p_1\}.$$

They are obtained by extending the following regions in the original sum:

$$\{0 \leq p_1 \leq p_2 \ll k\}, \quad \{0 \leq p_1 \ll k, 0 \leq k - p_2 \ll k\}, \quad \{0 \leq k - p_2 \leq k - p_1 \ll k\}.$$

In general, we call such a decomposition quasi-classical. We conjecture that the quasi-classical decompositions are exact for the fermionic sums. In later sections, we prove the conjecture in some cases.

Here is the quasi-classical decomposition for the general case when the interval is finite.

CONJECTURE 2.6. We have

$$I_{C,m}(q, z|r, s) =$$
$$= \left(z^m q^{W_{C,m}} \right)^r \sum_{0 \leq a \leq m} \left(z^a q^{W_{C,a}} \right)^{s-r} I_{C,m-a}(q, q^{Ca}z) I_{C,a}(q, z^{-1} q^{-Ca+\text{diag}C}).$$

$$(2.17)$$

Let us explain how one can obtain the right hand side. Recall (2.6) and (2.5). Consider the fermionic sum on the interval $[r, s]$ where $r \to -\infty$ and $s \to \infty$. Let $0 \leq a_i \leq m_i$ for $1 \leq i \leq l$, and consider the sum over p in the region $r \leq p_{i,1} \leq \cdots \leq p_{i,m_i-a_i} \ll s$ and $r \ll p_{i,m_i-a_i+1} \leq \cdots \leq p_{i,m_i} \leq s$. We denote the vector of $p_{i,j}$ from

the first region by $\mathbf{p}_{m-a}$ and from the second region by $\mathbf{p}_a$. Two groups of variables are separated: if $p_{i,j}$ is in the first group and $p_{i',j'}$ is in the second group, then we have

$$\min(p_{i,j}, p_{i',j'}) = p_{i,j}.$$

Therefore, we have

$$\overline{Q}_C(\mathbf{p}) = \overline{Q}_C(\mathbf{p}_{m-a}) + \overline{Q}_C(\mathbf{p}_a) + \sum_{i'=1}^{l} C_{i,i'} a_{i'} \sum_{j=1}^{m_i - a_i} p_{i,j}.$$

Extending the regions for summation by removing the bounds of the form $\cdots \ll s$ or $r \ll \cdots$, we obtain

$$I_{C,m}(q, z|r, s) = \sum_{0 \le a \le m} I_{C,m-a}(q, q^{Ca} z|r, \infty) I_{C,a}(q, z| - \infty, s). \tag{2.18}$$

We rewrite this to (2.17) by using (2.10) and (2.11).

In the special cases $r = s$ and $r = s + 1$ Conjecture 2.6 reads as follows.

$$\sum_{0 \le a \le m} I_{C,m-a}(q, q^{Ca} z) I_{C,a}(q, z^{-1} q^{-Ca + \mathrm{diag} C}) = \frac{1}{(q)_m}, \tag{2.19}$$

$$\sum_{0 \le a \le m} \left(z^a q^{W_{C,a}} \right)^{-1} I_{C,m-a}(q, q^{Ca} z) I_{C,a}(q, z^{-1} q^{-Ca + \mathrm{diag} C}) = 0. \tag{2.20}$$

The same quasi-classical decomposition procedure can be applied to $J_{C,m}^{(k,\nu)}(q, z)$, where $k \to \infty$. For this purpose we need fermionic sums on $\mathbb{Z} = (-\infty, \infty)$. We define

$$X_{C,m}^{(k,\nu)}(q, z) = I_{C,\mu,m}(q, z| - \infty, \infty), \tag{2.21}$$

where μ, which depends on (k, ν), is given by (2.13).

LEMMA 2.7. $X_{C,m}^{(k,\nu)}(q, z)$ is a rational function in q, z. It satisfies the relations:

$$X_{C,m}^{(k,\nu)}(q, z) = \left(z^m q^{W_{C,m}} \right)^k X_{C,m}^{(0,\nu)}(q, z)$$

and

$$X_{C,m}^{(0,\nu)}(q, z) = X_{C,m}^{(0,\nu)}(q, z^{-1} q^{-\nu - Cm + \mathrm{diag} C}). \tag{2.22}$$

Proof. The proof is straightforward. We only note that in order to prove (2.22) one needs to write the fermionic sum in $\mathbf{p}$ variables and change the summation variable $p_{i,j}$ to $-p_{i,m_i+1-j}$ for all i, j. $\square$

The following proposition is an analogue of Lemma 2.5.

PROPOSITION 2.8.

$$X_{C,m}^{(0,v)}(q,z) = \sum_{0 \le a \le m} \left(z^a q^{W_{C,a} + C a \cdot (m-a)} \right)^{-1} I_{C,a}(q, z^{-1} q^{-Cm + \mathrm{diag}C}) I_{C,m-a}(q, q^v z),$$

(2.23)

$$= \sum_{0 \le a \le m} z^a q^{W_{C,a} + v \cdot a} I_{C,m-a}(q, z^{-1} q^{-Cm + \mathrm{diag}C}) I_{C,a}(q, q^v z). \quad (2.24)$$

Proof. We use two cuttings of the infinite interval of weights $(-\infty, \infty)$ into semi-infinite intervals: $(-\infty, \infty) = (-\infty, -1] \sqcup [0, \infty)$ or $(-\infty, \infty) = (-\infty, 0] \sqcup [1, \infty)$. This leads to (2.23) and (2.24). $\qquad\qquad\square$

Applying the procedure of the quasi-classical decomposition, we obtain

CONJECTURE 2.9.

$$J_{C,m}^{(k,v)}(q,z) = \sum_{0 \le a \le m} I_{C,m-a}(q, q^{Ca} z) X_{C,a}^{(k,v)}(q,z).$$

In the right hand side the summation variable a_i where $a = (a_1, \ldots, a_l)$, represents the number of color i particles whose weights are "close to k". The weights of the remaining particles are "small" compared to k. We conjecture that this is exact for finite k. In particular, setting $k = 0$ and $k = -1$ and using (2.14), we obtain

$$I_{C,m}(q, q^v z) = \sum_{0 \le a \le m} I_{C,m-a}(q, q^{Ca} z) X_{C,a}^{(0,v)}(q,z), \qquad (2.25)$$

$$q^{v \cdot m} I_{C,m}(q, q^v z) = \sum_{0 \le a \le m} I_{C,m-a}(q, q^{Ca} z) \left(z^a q^{W_{C,a}} \right)^{-1} X_{C,a}^{(0,v)}(q,z). \qquad (2.26)$$

In Section 4 we prove these equalities in the case where C is a simply-laced Cartan matrix. We also give a generalization of these equalities in the case where C is the symmetrization of a non simply-laced Cartan matrix.

Finally, we give the quasi-classical decomposition for the fermionic sum on $(-\infty, \infty)$ with two corners at 0 and k, with angle v_1 and v_2, respectively. We denote this quantity by $X_{C,m}^{(0,v_1;k,v_2)}(q,z)$. We conjecture that

$$X_{C,m}^{(0,v_1;k,v_2)}(q,z) = \sum_{0 \le a \le m} \left(z^a q^{W_{C,a}} \right)^k X_{C,m-a}^{(0,v_1)}(q, q^{Ca} z) X_{C,a}^{(0,v_2)}(q,z).$$

Restricting to $k = 0$ we have

$$X_{C,m}^{(0,v_1+v_2)}(q,z) = \sum_{0 \le a \le m} X_{C,m-a}^{(0,v_1)}(q, q^{Ca} z) X_{C,a}^{(0,v_2)}(q,z) \qquad (2.27)$$

2.4. THE CASE OF $\mathfrak{sl}_2$

In this subsection, we restrict to the $\mathfrak{sl}_2$ case, i.e., $l=1$ and $C=2$, and write some of the fermionic sums and their relations explicitly. Moreover, we discuss vanishing theorems which arise in connection with integrality of angle variables. In the following we drop C in the notation because it is fixed to $C=2$. We also use the q binomial coefficients defined by

$$\begin{bmatrix} \gamma \\ n \end{bmatrix} = \frac{(q^{\gamma-n+1})_n}{(q)_n}.$$

Here n is a non-negative integer, but γ is arbitrary, possibly a formal variable.

First, we recall a known result and its proof [9].

PROPOSITION 2.10. *We have*

$$I_m(q,z) = \frac{1}{(q)_m (z)_m}. \tag{2.28}$$

Proof. The recursion (2.12) in this case reads as

$$I_m(q,z) = \sum_{0 \le a \le m} \frac{z^a q^{a(a-1)}}{(q)_{m-a}} I_a(q,z).$$

The fermionic sum is uniquely determined by this recursion with the initial condition $I_0(q,z)=1$. Therefore, it is enough to prove this recursion for (2.28). After substitution, we want to prove

$$\frac{1}{(z)_m} = \sum_{0 \le a \le m} \begin{bmatrix} m \\ a \end{bmatrix} \frac{z^a q^{a(a-1)}}{(z)_a}.$$

Using $\begin{bmatrix} m \\ a \end{bmatrix} = q^a \begin{bmatrix} m-1 \\ a \end{bmatrix} + \begin{bmatrix} m-1 \\ a-1 \end{bmatrix}$, we obtain

$$(\mathrm{RHS}) = \sum_{a=0}^{m-1} \begin{bmatrix} m-1 \\ a \end{bmatrix} \left(\frac{z^a q^{a^2}}{(z)_a} + \frac{z^{a+1} q^{a(a+1)}}{(z)_{a+1}} \right) =$$

$$= \frac{1}{1-z} \sum_{a=0}^{m-1} \frac{\begin{bmatrix} m-1 \\ a \end{bmatrix} (qz)^a q^{a(a-1)}}{(qz)_a} = (\mathrm{LHS}).$$

$\square$

The following proposition holds for an arbitrary value of ν.

PROPOSITION 2.11.

$$X_m^{(0,\nu)}(q,z) = \frac{\begin{bmatrix} \nu \\ m \end{bmatrix}}{(z^{-1} q^{2(1-m)})_m (q^\nu z)_m}.$$

Proof. We use the decomposition (2.24). Substituting the expressions for $I_{C,a}(q, z | -\infty, k)$ and $I_{C,m-a}(q, q^v z)$ given by (2.11) and (2.28), we obtain

$$X_m^{(0,v)}(q, z) = \frac{1}{(q)_m} \sum_{0 \leq a \leq m} \frac{\left[{m \atop a} \right] (q^v z)^a q^{a(a-1)}}{(z^{-1}q^{2-2m})_{m-a}(q^v z)_a}.$$

The rest of the proof goes similarly as in Proposition 2.10. $\qquad \square$

If n is a non-negative integer, the range for m where $X_m^{(0,n)}(q, z)$ is non-zero is restricted.

COROLLARY 2.12. *If $n \in \mathbb{Z}_{\geq 0}$ and $m > n$, then we have $X_m^{(0,n)}(q, z) = 0$.*

We identify n with the highest weight of the irreducible representation V_n of $\mathfrak{sl}_2$. The above statement says the fermionic sum is non-zero only if $n - 2m$ is a weight of V_n. In Proposition 4.8 we establish vanishing theorems of the form $X_{C,m}^{(0,v)} = 0$ for the case where C is a simply-laced Cartan matrix, the parameter v corresponds to a dominant integral weight, and $v - \sum_i m_i \alpha_i$ is not a weight of V_v. Our tool is the representation theory of $U_v(\mathfrak{g})$ with $v^2 = q$, where $\mathfrak{g}$ is a simple Lie algebra associated with C. We expect vanishing theorems of this kind are valid in a much wider class of C, though it is beyond the scope of this paper.

Two recursions (2.25) and (2.26) reads as follows.

PROPOSITION 2.13. *For an arbitrary value of v, we have*

$$I_m(q, q^v z) = \sum_{0 \leq a \leq m} \frac{\left[{v \atop a} \right]}{(z^{-1}q^{2(1-a)})_a (q^v z)_a} I_{m-a}(q, q^{2a} z), \tag{2.29}$$

$$q^{vm} I_m(q, q^v z) = \sum_{0 \leq a \leq m} \frac{z^{-a} q^{-a(a-1)} \left[{v \atop a} \right]}{(z^{-1}q^{2(1-a)})_a (q^v z)_a} I_{m-a}(q, q^{2a} z). \tag{2.30}$$

The quasi-classical decomposition generates many more identities than we discussed above. Here we give an example. Let $0 \leq k_1 \leq \cdots \leq k_n$ be non-negative integers. Set

$$\mu_t = \sum_{i=1}^{n} (t - k_i)_+ \quad \text{where } (t)_+ = \begin{cases} t & \text{if } t \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Namely, the linear coefficient μ has corners at k_i with angle $\sum_{r=1}^{n} \delta_{k_i, k_r}$. Suppose that the quasi-classical decomposition is exact. Then for $n \geq m$ we have

$$I_{\mu,m}(q, z) = \sum_{0 \leq a \leq m} I_a(q, q^{2(m-a)} z) X_{\mu, m-a}(q, z), \tag{2.31}$$

$$X_{\mu,m}(q,z) = \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n=0,1 \\ \varepsilon_1+\cdots+\varepsilon_n=m}} X_{\varepsilon_1,\ldots,\varepsilon_n}^{k_1,\ldots,k_n}, \tag{2.32}$$

$$X_{\varepsilon_1,\ldots,\varepsilon_n}^{k_1,\ldots,k_n} = \prod_{i:\ \varepsilon_i=1} \frac{q^{-\sum_{r=1}^{i-1}k_r}(q^{\varepsilon(i)}z)^{k_i}}{g(q^{\varepsilon(i)}z)}. \tag{2.33}$$

Here $g(z) = (1-z^{-1})(1-qz)$, $\varepsilon(i) = 2\sum_{r=i+1}^{n}\varepsilon_r + i - 1$. Note that $X_{1,0}^{(0,1)} = 1/g(z)$.
For example, setting $n=3$, $k_1=k_2=k_3=0$, $m=2$, we obtain

$$\frac{\begin{bmatrix}3\\2\end{bmatrix}}{(q^3z)_2(q^{-2}z^{-1})_2} = \frac{1}{g(zq^2)}\left(\frac{1}{g(zq)}+\frac{1}{g(zq^2)}+\frac{1}{g(zq^3)}\right).$$

3. Fermionic and Toda Recursions

In this section, we develop the representation theoretic approach to fermionic formulas. In certain cases we show that they coincide with scalar products of Whittaker vectors, which are eigenfunctions of the quantum Toda Hamiltonian.

Quantum deformation of Whittaker vectors has been introduced and studied by Sevostyanov [19]. An independent construction was given by Etingof [7] from a slightly different point of view. A geometric interpretation of the eigenvectors of the quantum Toda Hamiltonian as Shapovalov scalar product of Whittaker vectors has been given by Braverman and Finkelberg [3], reproducing the main results of Givental and Lee [12]. In this section we give a review of this subject, following closely the algebraic framework of [19] with minor modifications. We shall show that fermionic formulas naturally arise from Drinfeld's Casimir element.

3.1. QUANTUM GROUPS

We fix the notation as follows. Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{h}$ the Cartan subalgebra, $\alpha_1,\ldots,\alpha_l$ the simple roots and $\omega_1,\ldots,\omega_l$ the fundamental weights. Set $P = \oplus_{i=1}^{l}\mathbb{Z}\omega_i$, $Q = \oplus_{i=1}^{l}\mathbb{Z}\alpha_i$, $P_+ = \oplus_{i=1}^{l}\mathbb{Z}_{\geq 0}\omega_i$, $Q_+ = \oplus_{i=1}^{l}\mathbb{Z}_{\geq 0}\alpha_i$. Let further Δ_+ denote the set of positive roots. We fix a non-degenerate invariant bilinear form $(\ ,\ ):\mathfrak{h}\times\mathfrak{h}\to\mathbb{C}$ such that $(P,Q)\subset\mathbb{Z}$, and identify $\mathfrak{h}^*$ with $\mathfrak{h}$ via $(\ ,\ )$. We set

$$d_i = \frac{1}{2}(\alpha_i,\alpha_i), \quad \alpha_i^\vee = d_i^{-1}\alpha_i, \quad \rho = \sum_{r=1}^{l}\omega_r.$$

We choose $(\ ,\ )$ so that $d_1,\ldots,d_l$ are relatively prime positive integers.

Let $\mathcal{N}$ be a positive integer satisfying $(P,P)\subset(1/\mathcal{N})\mathbb{Z}$. The quantum group $U_v(\mathfrak{g})$ is a unital associative algebra over $\mathbb{K}=\mathbb{C}(v^{1/\mathcal{N}})$, with generators E_i, F_i ($1\leq i\leq l$),

K_μ ($\mu \in P$) and the standard defining relations

$$K_\mu K_{\mu'} = K_{\mu+\mu'}, \quad K_0 = 1,$$

$$K_\mu E_i K_\mu^{-1} = v^{(\mu,\alpha_i)} E_i, \quad K_\mu F_i K_\mu^{-1} = v^{-(\mu,\alpha_i)} F_i,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{v_i - v_i^{-1}},$$

$$\sum_{s=0}^{r} (-1)^s E_i^{(r-s)} E_j E_i^{(s)} = 0 \qquad (r = 1 - (\alpha_i^\vee, \alpha_j), \ i \neq j),$$

$$\sum_{s=0}^{r} (-1)^s F_i^{(r-s)} F_j F_i^{(s)} = 0 \qquad (r = 1 - (\alpha_i^\vee, \alpha_j), \ i \neq j).$$

Here $K_i = K_{\alpha_i}$, $v_i = v^{d_i}$, $X_i^{(s)} = X_i^s / [s]_{v_i}!$ ($X = E, F$) and $[s]_v! = \prod_{p=1}^{s} (v^p - v^{-p})/(v - v^{-1})$. We choose the coproduct

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i,$$

$$\Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

$$\Delta K_\mu = K_\mu \otimes K_\mu,$$

antipode

$$S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_\mu) = K_\mu^{-1}$$

and counit

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_\mu) = 1.$$

We shall also consider the quantum group $U_{v^{-1}}(\mathfrak{g})$ with parameter v^{-1}. Denote the generators by $\bar{E}_i$, $\bar{F}_i$, $\bar{K}_\mu$. We choose the opposite coproduct,

$$\Delta \bar{E}_i = \bar{E}_i \otimes \bar{K}_i + 1 \otimes \bar{E}_i,$$

$$\Delta \bar{F}_i = \bar{F}_i \otimes 1 + \bar{K}_i^{-1} \otimes \bar{F}_i,$$

$$\Delta \bar{K}_\mu = \bar{K}_\mu \otimes \bar{K}_\mu,$$

antipode

$$S(\bar{E}_i) = -\bar{E}_i \bar{K}_i^{-1}, \quad S(\bar{F}_i) = -\bar{K}_i \bar{F}_i, \quad S(\bar{K}_\mu) = \bar{K}_\mu^{-1},$$

and counit

$$\varepsilon(\bar{E}_i) = \varepsilon(\bar{F}_i) = 0, \quad \varepsilon(\bar{K}_\mu) = 1.$$

There is a $\mathbb{K}$-linear anti-isomorphism of algebras given by

$$\sigma : U_v(\mathfrak{g}) \to U_{v^{-1}}(\mathfrak{g}), \quad E_i \mapsto \bar{F}_i, \ F_i \mapsto \bar{E}_i, \ K_\mu \mapsto \bar{K}_\mu^{-1}. \tag{3.1}$$

We have

$$\Delta \circ \sigma = \sigma \otimes \sigma \circ \Delta. \tag{3.2}$$

3.2. VERMA MODULES

For $\lambda \in P$, let $\mathcal{V}^\lambda$ be the Verma module over $U_v(\mathfrak{g})$ generated by the highest weight vector $\mathbf{1}^\lambda$ with defining relations

$$E_i \mathbf{1}^\lambda = 0 \quad (1 \le i \le l), \qquad K_\mu \mathbf{1}^\lambda = v^{(\mu,\lambda)} \mathbf{1}^\lambda \quad (\mu \in P).$$

Similarly let $\overline{\mathcal{V}}^\lambda$ be the Verma module over $U_{v^{-1}}(\mathfrak{g})$ generated by the highest weight vector $\bar{\mathbf{1}}^\lambda$ with defining relations

$$\bar{E}_i \bar{\mathbf{1}}^\lambda = 0 \quad (1 \le i \le l), \qquad \bar{K}_\mu \bar{\mathbf{1}}^\lambda = v^{-(\mu,\lambda)} \bar{\mathbf{1}}^\lambda \quad (\mu \in P).$$

We have obvious gradings $\mathcal{V}^\lambda = \oplus_{\beta \in Q^+} (\mathcal{V}^\lambda)_\beta$, $\overline{\mathcal{V}}^\lambda = \oplus_{\beta \in Q^+} (\overline{\mathcal{V}}^\lambda)_\beta$, where

$$(\mathcal{V}^\lambda)_\beta = \{w \in \mathcal{V}^\lambda \mid K_\mu w = v^{(\mu,\lambda-\beta)} w \quad (\mu \in P)\}, \tag{3.3}$$

$$(\overline{\mathcal{V}}^\lambda)_\beta = \{w \in \overline{\mathcal{V}}^\lambda \mid \bar{K}_\mu w = v^{-(\mu,\lambda-\beta)} w \quad (\mu \in P)\}. \tag{3.4}$$

There exists a unique non-degenerate $\mathbb{K}$-bilinear pairing $(\ ,\ ) : \mathcal{V}^\lambda \times \overline{\mathcal{V}}^\lambda \to \mathbb{K}$, such that $(\mathbf{1}^\lambda, \bar{\mathbf{1}}^\lambda) = 1$ and

$$(xw, w') = (w, \sigma(x)w') \tag{3.5}$$

for all $x \in U_v(\mathfrak{g})$ and $w \in \mathcal{V}^\lambda$, $w' \in \overline{\mathcal{V}}^\lambda$. The weight components (3.3), (3.4) are mutually orthogonal with respect to $(\ ,\ )$. We extend the scalar product on tensor products of Verma modules as

$$(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)(u_2, v_2) \quad (u_i \in \mathcal{V}^{\lambda_i}, \ v_i \in \bar{\mathcal{V}}^{\lambda_i}).$$

3.3. WHITTAKER VECTORS

Whittaker vectors are defined by giving the following data: an orientation of the Dynkin graph, a set of elements $v_i \in P$, and non-zero scalars $c_i \in \mathbb{K}$. An orientation is represented by a skew-symmetric matrix $\epsilon = (\epsilon_{i,j})$, where $\epsilon_{i,j} = 1$ if there is an arrow pointing from node i to node j, and $\epsilon_{i,j} = 0$ if i and j are disconnected. Given ϵ, choose $v_i \in P$ such that

$$(v_i, \alpha_j) - (v_j, \alpha_i) = \epsilon_{i,j}(\alpha_i, \alpha_j). \tag{3.6}$$

For instance one can take $v_i = \sum_{k=1}^{i-1} \omega_k \epsilon_{i,k}(\alpha_i, \alpha_k^\vee)$.

A Whittaker vector associated with the data ϵ, $v = (v_i)$ and $c = (c_i)$ is an element

$$\theta^\lambda = \theta^\lambda(\epsilon, v, c) = \sum_{\beta \in Q_+} \theta_\beta^\lambda, \qquad \theta_\beta^\lambda \in (\mathcal{V}^\lambda)_\beta,$$

which belongs to a completion $\widehat{\mathcal{V}}^\lambda = \prod_{\beta \in Q_+} (\mathcal{V}^\lambda)_\beta$ of the Verma module. It is defined by the conditions $\theta_0^\lambda = \mathbf{1}^\lambda$ and

$$E_i K_{v_i} \ \theta^\lambda = \frac{c_i}{1 - v_i^2} \ \theta^\lambda. \tag{3.7}$$

It is known that the Whittaker vector exists and is unique [19]. Fixing ϵ and changing v, c results in a simple scalar multiple of the weight components of $\theta^\lambda(\epsilon, v, c)$. Explicitly we have the transformation law

$$\theta_\beta^\lambda(\epsilon, v + \gamma, c') = v^{(1/2)\sum_{i=1}^l (\beta, \omega_i^\vee)(\beta - 2\lambda, \gamma_i)} \theta_\beta^\lambda(\epsilon, v, c), \tag{3.8}$$

$$\theta_\beta^\lambda(\epsilon, v, c'') = v^{-(\kappa, \beta)} \theta_\beta^\lambda(\epsilon, v, c), \tag{3.9}$$

where $\gamma_i \in P$ is such that $(\gamma_i, \alpha_j) = (\gamma_j, \alpha_i)$ holds, and $c_i' = v^{-(\gamma_i, \alpha_i)/2} c_i$, $c_i'' = v^{-(\kappa, \alpha_i)} c_i$.

Similarly one defines the dual Whittaker vector

$$\bar\theta^\lambda = \bar\theta^\lambda(\epsilon, v, c) \in \widehat{\bar{\mathcal{V}}}^\lambda, \qquad \widehat{\bar{\mathcal{V}}}^\lambda = \prod_{\beta \in Q_+} (\bar{\mathcal{V}}^\lambda)_\beta,$$

imposing $\bar\theta_0^\lambda = \bar{\mathbf{1}}^\lambda$ and

$$\bar{E}_i \bar{K}_{v_i} \bar\theta^\lambda = \frac{c_i^{-1}}{1 - v_i^{-2}} \bar\theta^\lambda \tag{3.10}$$

in place of (3.7).

The main object of our interest is the scalar product

$$J_\beta^\lambda = v^{-(\beta,\beta)/2 + (\lambda,\beta)} \ (\theta_\beta^\lambda(\epsilon, v, c), \bar\theta_\beta^\lambda(\epsilon, v, c)). \tag{3.11}$$

We set $J_\beta^\lambda = 0$ unless $\beta \in Q_+$.

From (3.8) and (3.9), we see that (3.11) is independent of the choice of v, c. As it will turn out, it is actually independent of the orientation ϵ as well (see Theorem 3.1 below). Anticipating this fact, we suppress the dependence on ϵ, v, c from the notation.

3.4. FERMIONIC RECURSION

Now we state the main result of this section. Set $q = v^2$, $q_i = q^{d_i}$, and

$$(q)_\beta = \prod_{i=1}^l (q_i; q_i)_{m_i} \quad \text{for } \beta = \sum_{i=1}^l m_i \alpha_i.$$

The following is a counterpart of the fermionic recursion given in Proposition 2.4.

THEOREM 3.1. *The quantities J_β^λ are uniquely characterized by $J_0^\lambda = 1$ and the recursion relation*

$$J_\beta^\lambda = \sum_\gamma \frac{1}{(q)_{\beta-\gamma}} q^{(\gamma,\gamma)/2 - (\lambda+\rho,\gamma)} J_\gamma^\lambda. \tag{3.12}$$

In particular, J_β^λ is independent of the choice of orientation which enters the definition.

The J_β^λ are determined as rational functions in the variables q and $z_i = q^{-(\lambda,\alpha_i)}$. In accordance with the previous section, let us introduce the following sum for a (possibly infinite) interval $[r, s]$.

$$J_\beta^\lambda[r, s] = \sum_{\sum_{t=r}^s \gamma^{(t)} = \beta} \frac{q^{B(\gamma)}}{\prod_{t=r}^s (q)_{\gamma^{(t)}}}, \tag{3.13}$$

where

$$B(\gamma) = \frac{1}{2} \sum_{r \le t, t' \le s} \min(t, t')(\gamma^{(t)}, \gamma^{(t')}) - \left(\lambda + \rho, \sum_{t=r}^s t\gamma^{(t)}\right).$$

We have

$$J_\beta^\lambda[0, 0] = \frac{1}{(q)_\beta}, \tag{3.14}$$

$$J_\beta^\lambda[r + 1, s + 1] = q^{(\beta,\beta)/2 - (\lambda+\rho,\beta)} J_\beta^\lambda[r, s], \tag{3.15}$$

$$J_\beta^\lambda[r, s] = \sum_{\alpha + \gamma = \beta} J_\alpha^{\lambda-\gamma}[r, u] J_\gamma^\lambda[u + 1, s] \qquad (r \le u < s). \tag{3.16}$$

THEOREM 3.2. *In the notation above, we have*

$$J_\beta^\lambda = J_\beta^\lambda[0, \infty).$$

Proof. This is a restatement of Theorem 3.1.   □

Suppose C is a simply laced Cartan matrix, and let $\beta = \sum_{i=1}^l m_i \alpha_i$, $z_i = q^{-(\lambda,\alpha_i)}$. Since $(Cm, m) = (\beta, \beta)$ and $(\text{diag} C, m) = (2\rho, \beta)$, we have

$$J_\beta^\lambda[0, \infty) = I_{C,m}(q, z)$$

where the right hand side is defined in (2.8). We shall discuss the interpretation of $J_\beta^\lambda[r, s]$ for finite interval $[r, s]$ in the next section (see Theorem 4.13). When C is non-simply laced, (3.13) gives a generalization of the fermionic sum considered in the previous section due to the denominator $(q_i; q_i)_n$.

COROLLARY 3.3.   (i) *The rational function J_β^λ is regular at $z_1 = \cdots = z_l = 0$ and*

$$J_\beta^\lambda \Big|_{z_1 = \cdots = z_l = 0} = \prod_{r=1}^l \frac{1}{(q_i; q_i)_{m_i}}. \tag{3.17}$$

(ii) *We have the symmetry property*

$$J_\beta^\lambda \Big|_{q \to q^{-1}} = q^{(\beta,\beta)/2 - (\lambda,\beta)} J_\beta^\lambda. \tag{3.18}$$

(iii) *The set $\{J_\beta^\lambda\}_{\beta \in Q_+}$ is linearly independent over $\mathbb{C}(q)$.*

Proof. Assertion (i) is a direct consequence of (3.13) since $J_\beta^\lambda = J_\beta^\lambda[0, \infty)$. In the definition of J_β^λ, θ^λ and $\bar{\theta}^\lambda$ enter in a symmetric way. Therefore assertion (ii) follows from the definition (3.11). (Note that the change of variable $q \to q^{-1}$ in the left hand side of (ii) implies $z_i \to z_i^{-1}$).

To see (iii), it suffices to show that $J_\beta^\lambda \big|_{q \to q^{-1}}$ constitute a linearly independent set. Property (ii) implies that each of them has distinct leading power $z_1^{m_1} \cdots z_l^{m_l}$ in $z_1, \ldots, z_l$. Hence the independence is evident. $\qquad\square$

3.5. DERIVATION OF THE FERMIONIC RECURSION

In this subsection we give a derivation of Theorem 3.1.

First we recall the Cartan–Weyl basis and the product formula for the universal R matrix due to Khoroshkin and Tolstoy [15]. By definition, a total order $<$ on Δ_+ is said to be normal if $\alpha, \beta, \alpha + \beta \in \Delta_+$ and $\alpha < \beta$ imply $\alpha < \alpha + \beta < \beta$. Normal orders are in one-to-one correspondence with reduced decompositions of the longest element of the Weyl group. Moreover an arbitrary total order on the set of simple roots can be extended to a normal order on Δ_+ [22,23]. To a normal order $<$, one associates root vectors

$$E_\beta^<, \quad F_\beta^< \qquad (\beta \in \Delta_+) \tag{3.19}$$

by induction on $h(\beta)$, where $h(\sum_{i=1}^l n_i \alpha_i) = \sum_{i=1}^l n_i$. When $h(\beta) = 1$, define $E_{\alpha_i}^< = E_i$, $F_{\alpha_i}^< = F_i$. Let γ be an element with $h(\gamma) = n$, and suppose that (3.19) are already defined for $h(\beta) < n$. Choose a decomposition $\gamma = \alpha + \beta$ in such a way that there are no other roots $\alpha', \beta' \in \Delta_+$ satisfying $\gamma = \alpha' + \beta'$, $\alpha \leq \alpha' < \beta' \leq \beta$. Then define

$$E_\gamma^< = E_\alpha^< E_\beta^< - v^{(\alpha,\beta)} E_\beta^< E_\alpha^<, \tag{3.20}$$
$$F_\gamma^< = c_\gamma (F_\beta^< F_\alpha^< - v^{-(\alpha,\beta)} F_\alpha^< F_\beta^<). \tag{3.21}$$

The scalar $c_\gamma \in \mathbb{K}$ can be so chosen that $[E_\gamma^<, F_\gamma^<] = (K_\gamma - K_\gamma^{-1})/(v_\gamma - v_\gamma^{-1})$. Here and after we set $v_\gamma = v^{(\gamma,\gamma)/2}$.

The product formula is written in terms of the q-exponential function

$$\exp_q(x) = \sum_{n=0}^\infty \frac{((1-q)x)^n}{(q;q)_n},$$

which satisfies $\exp_q(x) \exp_{q^{-1}}(-x) = 1$.

PROPOSITION 3.4. [15] *Fix a normal order $<$, and consider the element*

$$\Theta = \overset{\frown}{\prod_{\beta \in \Delta_+}} \exp_{v_\beta^2} \left(-(v_\beta - v_\beta^{-1}) F_\beta^< \otimes E_\beta^< \right), \tag{3.22}$$

where the product is so ordered that β appears to the right of β' if $\beta < \beta'$. Then Θ does not depend on the choice of the normal order $<$.

The universal R matrix is given by $\mathcal{R} = \Theta_{21} v^{-T}$, where $\Theta_{21} = \tau(\Theta)$, $\tau(a \otimes b) = b \otimes a$, and $T \in \mathfrak{h} \otimes \mathfrak{h}$ stands for the canonical element.

The following construction is well known.

PROPOSITION 3.5. [6] *Set $u = m(S \otimes \mathrm{id})\Theta$, where $m(a \otimes b) = ab$ stands for the multiplication. Then u is a well defined operator on $\mathcal{V}^\lambda$. It acts on each $(\mathcal{V}^\lambda)_\beta$ as a scalar,*

$$u \Big|_{(\mathcal{V}^\lambda)_\beta} = v^{-(\beta,\beta)+2(\lambda+\rho,\beta)} \times \mathrm{id}_{(\mathcal{V}^\lambda)_\beta}.$$

The formal element $v^{\sum_{i=1}^l \alpha_i^\vee \omega_i + 2\rho} u$ is sometimes referred to as the Drinfeld (quantum) Casimir element. It acts on $\mathcal{V}^\lambda$ as a scalar: $v^{\sum_{i=1}^l \alpha_i^\vee \omega_i + 2\rho} u \big|_{\mathcal{V}^\lambda} = v^{(\lambda+2\rho,\lambda)} \times \mathrm{id}_{\mathcal{V}^\lambda}$.

A normal order $<$ is said to be compatible with an orientation ϵ of the Dynkin graph if $\epsilon_{j,i} = 1$ implies $\alpha_i < \alpha_j$.

PROPOSITION 3.6. [19] *Let (3.19) be the root vectors with respect to a normal order $<$, and let $\theta^\lambda(\epsilon, v, c)$ be the Whittaker vector. If the order $<$ is compatible with the orientation ϵ, then we have*

$$E_\beta^< \; \theta^\lambda(\epsilon, v, c) = 0 \qquad \textit{for all non-simple roots } \beta \in \Delta_+. \tag{3.23}$$

Proof. Since the root vectors are defined by multiple v-commutators (3.20), (3.21), it is sufficient to show (3.23) when $\gamma = \alpha_i + \alpha_j$. This can be verified by a direct calculation using (3.6) and (3.7).   □

Proof of Theorem 3.1. By Proposition 3.5, we have

$$(u \; \theta_\beta^\lambda, \bar{\theta}_\beta^\lambda) = v^{-(\beta,\beta)+2(\lambda+\rho,\beta)}(\theta_\beta^\lambda, \bar{\theta}_\beta^\lambda).$$

Suppose that $\gamma_1 < \cdots < \gamma_l$ are the simple roots appearing in the chosen normal order. Expanding the formula (3.22), we obtain

$$u \; \theta_\beta^\lambda = \sum_{n_1,\ldots,n_l} \prod_{i=1}^l \left(\frac{v_i^{-n_i}(1-v_i^2)^{2n_i}}{(v_i^2; v_i^2)_{n_i}} \right) \times S(F_{\gamma_1})^{n_1} \cdots S(F_{\gamma_l})^{n_l} E_{\gamma_l}^{n_l} \cdots E_{\gamma_1}^{n_1} \; \theta_\beta^\lambda.$$

In view of Proposition 3.6, we have retained only those terms consisting of simple roots. Setting $\gamma = \sum_{i=1}^{l} n_i \gamma_i$ and using (3.5), we obtain

$$(u \ \theta_\beta^\lambda, \bar{\theta}_\beta^\lambda) = \sum_{n_1,\ldots,n_l} \prod_{i=1}^{l} \left(\frac{(-v_i)^{-n_i}(1-v_i^2)^{2n_i}}{(v_i^2; v_i^2)_{n_i}} \right) v^{(\gamma,\gamma)/2+(\gamma,\lambda-\beta-\rho)} \times$$
$$\times (E_{\gamma_l}^{n_l} \cdots E_{\gamma_1}^{n_1} \theta_\beta^\lambda, \bar{E}_{\gamma_l}^{n_l} \cdots \bar{E}_{\gamma_1}^{n_1} \bar{\theta}_\beta^\lambda).$$

Now apply the relations following from (3.7), (3.10),

$$v^{(v_i,\lambda-\beta)} E_i \theta_\beta^\lambda = \frac{c_i}{1-v_i^2} \ \theta_{\beta-\alpha_i}^\lambda, \qquad v^{-(v_i,\lambda-\beta)} \bar{E}_i \bar{\theta}_\beta^\lambda = \frac{c_i^{-1}}{1-v_i^{-2}} \ \bar{\theta}_{\beta-\alpha_i}^\lambda.$$

Since the generators E_{γ_i}, $\bar{E}_{\gamma_i}$ are arranged in the same order, the powers of v cancel out. Substituting the definition (3.11) of J_β^λ and simplifying the formula, we obtain the desired result. □

3.6. TODA RECURSION

As shown in [7,19], the generating function

$$F(q,z,y) = \sum_\beta J_\beta^\lambda \prod_{i=1}^{l} y_i^{(\beta,\omega_i)/d_i}$$

is an eigenfunction of the quantum difference Toda Hamiltonian: $H_{\text{Toda}} F = \varepsilon F$. For example, when $\mathfrak{g} = \mathfrak{sl}_{l+1}$, the Hamiltonian and the eigenvalue are given by

$$H_{\text{Toda}} = \sum_{i=0}^{l} D_{i+1} D_i^{-1} (1-y_i) \prod_{j=i+1}^{l} \left(q^{-1} z_j \right), \quad \varepsilon = \sum_{i=0}^{l} \prod_{j=i+1}^{l} \left(q^{-1} z_j \right), \qquad (3.24)$$

where D_i denotes the shift operator $(D_i f)(y_1, \ldots, y_l) = f(y_1, \ldots, qy_i, \ldots, y_l)$, with $D_0 = D_{l+1} = 1$, $y_0 = y_{l+1} = 0$. In this subsection we give an account on this point for completeness. The argument is similar to the one for Theorem 3.1. In place of the Drinfeld Casimir element, we use other central elements including the quadratic Casimir element.

The following construction is standard.

PROPOSITION 3.7. [8] *Let $\pi_V : U_v(\mathfrak{g}) \to \text{End}(V)$ be a finite dimensional representation, and set $\Theta_V = (\text{id} \otimes \pi_V)(\Theta)$, $\Theta_V' = (\text{id} \otimes \pi_V)(\Theta_{21})$ and $v^{\rho_V} = \text{id} \otimes \pi_V(v^\rho)$. Denote further by φ_V the operator on $\mathcal{V}^\lambda \otimes V$ which acts as $\text{id} \otimes \pi_V(v^{\lambda-\beta})$ on each $(\mathcal{V}^\lambda)_\beta \otimes V$. Then, for any $k \in \mathbb{Z}$, the operator on $\mathcal{V}^\lambda$ given by*

$$\mathcal{C}_V^{(k)} = \text{tr}_V \left(v^{2\rho_V} (\Theta_V' \circ \varphi_V^{-1} \circ \Theta_V \circ \varphi_V^{-1})^k \right) \qquad (3.25)$$

acts as a scalar.

In the following we take $k = -1$ and π_V to be the vector representation for the series A_l, B_l, C_l, D_l. In terms of orthonormal vectors ϵ_i, the simple roots are given by

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_l = \epsilon_l - \epsilon_{l+1} \quad \text{for } A_l,$$

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{l-1} = \epsilon_{l-1} - \epsilon_l, \alpha_l = \epsilon_l \quad \text{for } B_l,$$

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{l-1} = \epsilon_{l-1} - \epsilon_l, \alpha_l = 2\epsilon_l \quad \text{for } C_l,$$

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{l-1} = \epsilon_{l-1} - \epsilon_l, \quad \alpha_l = \epsilon_{l-1} + \epsilon_l \quad \text{for } D_l.$$

Unlike the fermionic recursion, the Toda recursion depends on the choice of the orientation. Here we give the formulas for the standard orientation compatible with the order $\alpha_1 < \cdots < \alpha_l$.

PROPOSITION 3.8. *For algebras of type A_l, B_l, C_l with the above orientation, we have*

$$\left(\mathrm{tr}_V q^{\lambda+\rho-\beta} - \mathrm{tr}_V q^{\lambda+\rho}\right) J_\beta^\lambda = \sum_{i=1}^{l} v^{-d_i} \mathrm{tr}_V \left(q^{\lambda+\rho-\beta} v^{\alpha_i} E_i F_i\right) \ J_{\beta-\alpha_i}^\lambda.$$

For algebra of type D_l, the same recursion holds where the right hand side has an additional term

$$-v^{-2} \mathrm{tr}_V \left(q^{\lambda+\rho-\beta} v^{\alpha_{l-1}+\alpha_l} E_{l-1} E_l F_{l-1} F_l\right) \ J_{\beta-\alpha_{l-1}-\alpha_l}^\lambda.$$

Proof. The scalar $\mathcal{C}_V^{(-1)}$ can be evaluated on the highest weight vector $\mathbf{1}^\lambda$, giving

$$(\mathcal{C}_V^{(-1)}\theta_\beta^\lambda, \bar\theta_\beta^\lambda) = \mathrm{tr}_V(v^{2(\lambda+\rho)}) \ (\theta_\beta^\lambda, \bar\theta_\beta^\lambda).$$

On the other hand, inserting (3.22) into $\mathcal{C}_V^{(-1)}$, we obtain a sum of terms comprising scalar products

$$((F_{\gamma_1}^<)^{m_1} \cdots (F_{\gamma_t}^<)^{m_t} (E_{\gamma_1}^<)^{n_1} \cdots (E_{\gamma_t}^<)^{n_t} \theta_\beta^\lambda, \ \bar\theta_\beta^\lambda),$$

where $\gamma_1 < \cdots < \gamma_t$ are the positive roots. From the rules (3.20), (3.21) we observe that $\sigma(F_\gamma^<)$ is proportional to $\bar E_\gamma^<$, and hence kills $\bar\theta^\lambda$ if γ is non-simple. Therefore, for both θ_β^λ and $\bar\theta_\beta^\lambda$ we need to retain only root vectors corresponding to the simple roots. Then we must have $m_i = n_i$, so we obtain

$$(\mathcal{C}_V^{(-1)}\theta_\beta^\lambda, \bar\theta_\beta^\lambda) = \sum_{n_1,\ldots,n_i \geq 0} \prod_{i=1}^{l} \frac{v_i^{-2n_i^2+4n_i}(1-v_i^2)^{4n_i}}{(v_i^2; v_i^2)_{n_i}^2} \ v^{-(\gamma,\lambda-\beta+\gamma)} \times$$

$$\times \mathrm{tr}_V \left(v^{2(\lambda+\rho-\beta)+\gamma} E_1^{n_1} \cdots E_l^{n_l} F_1^{n_1} \cdots F_l^{n_l}\right) \times$$

$$\times (E_1^{n_1} \cdots E_l^{n_l} \theta_\beta^\lambda, \bar E_l^{n_l} \cdots \bar E_1^{n_1} \bar\theta_\beta^\lambda).$$

Here we have set $\gamma = \sum_{i=1}^{l} n_i \alpha_i$.

When π_V is the vector representation, $E_i^n = F_i^n = 0$ for $n > 1$. A simple check shows that in the standard ordering the trace is non-zero only when $\sum_{i=1}^{l} n_i \leq 1$, except for D_l where $n_{l-1} = n_l = 1$ also contributes. $\qquad\square$

Here are the more explicit expressions.

$$\sum_{i=1}^{l+1} q^{(\lambda,\epsilon_i)-i}(q^{-(\beta,\epsilon_i)} - 1)\, J_\beta^\lambda = \sum_{i=1}^{l} q^{(\lambda-\beta,\epsilon_i)-i} J_{\beta-\alpha_i}^\lambda \qquad \text{for } A_l,$$

$$\sum_{i=1}^{l} \left(q^{(\lambda,\epsilon_i)+l-i+1/2}(q^{-(\beta,\epsilon_i)} - 1) + q^{-(\lambda,\epsilon_i)-l+i-1/2}(q^{(\beta,\epsilon_i)} - 1) \right)\, J_\beta^\lambda =$$

$$= \sum_{i=1}^{l-1} \left(q^{(\lambda-\beta,\epsilon_i)+l-i+1/2} + q^{-(\lambda-\beta,\epsilon_{i+1})-l+i+1/2} \right) J_{\beta-\alpha_i}^\lambda +$$

$$+ (q^{1/2} + 1)(q^{(\lambda-\beta,\epsilon_l)+1/2} + q^{-1/2}) J_{\beta-\alpha_l}^\lambda \qquad \text{for } B_l,$$

$$\sum_{i=1}^{l} \left(q^{(\lambda,\epsilon_i)+l-i+1}(q^{-(\beta,\epsilon_i)} - 1) + q^{-(\lambda,\epsilon_i)-l+i-1}(q^{(\beta,\epsilon_i)} - 1) \right)\, J_\beta^\lambda =$$

$$= \sum_{i=1}^{l-1} \left(q^{(\lambda-\beta,\epsilon_i)+l-i+1} + q^{-(\lambda-\beta,\epsilon_{i+1})-l+i} \right) J_{\beta-\alpha_i}^\lambda + q^{(\lambda-\beta,\epsilon_l)+1} J_{\beta-\alpha_l}^\lambda \qquad \text{for } C_l,$$

$$\sum_{i=1}^{l} \left(q^{(\lambda,\epsilon_i)+l-i}(q^{-(\beta,\epsilon_i)} - 1) + q^{-(\lambda,\epsilon_i)-l+i}(q^{(\beta,\epsilon_i)} - 1) \right)\, J_\beta^\lambda =$$

$$= \sum_{i=1}^{l-1} \left(q^{(\lambda-\beta,\epsilon_i)+l-i} + q^{-(\lambda-\beta,\epsilon_{i+1})-l+i+1} \right)\, J_{\beta-\alpha_i}^\lambda +$$

$$+ \left(q^{(\lambda-\beta,\epsilon_{l-1})+1} + q^{(\lambda-\beta,\epsilon_l)} \right)\, J_{\beta-\alpha_l}^\lambda - q^{(\lambda-\beta,\epsilon_{l-1})+1}\, J_{\beta-\alpha_{l-1}-\alpha_l}^\lambda \qquad \text{for } D_l.$$

4. Functions $\mathcal{X}_\alpha^{\mu,\lambda}$, $\bar{\mathcal{X}}_\alpha^{\lambda,\mu}$, $X_\beta^{\lambda_1,\lambda_2}$ and Recurrence Relations

In this section we give an interpretation of the fermionic sum (2.21) in the context of Whittaker vectors, and derive various relations among them and J_β^λ.

4.1. FUNCTIONS $\mathcal{X}_\alpha^{\mu,\lambda}$, $\bar{\mathcal{X}}_\alpha^{\lambda,\mu}$

Let $\lambda, \mu \in P$. We are going to define rational functions in the variables $z_i = q^{-(\lambda,\alpha_i)}$. The definition is given under the assumption $\lambda + \mu + 2\rho \in -P_+$. This restriction is dropped afterwards.

If $\lambda + \mu + 2\rho \in -P_+$, we have the decomposition into isotypic components (see Corollary B.2)

$$\mathcal{V}^\mu \otimes \mathcal{V}^\lambda = \oplus_{\alpha \in Q_+} \mathcal{W}^{\mu+\lambda-\alpha}, \tag{4.1}$$

$$\bar{\mathcal{V}}^\mu \otimes \bar{\mathcal{V}}^\lambda = \oplus_{\alpha \in Q_+} \bar{\mathcal{W}}^{\mu+\lambda-\alpha}, \tag{4.2}$$

with each summand $\mathcal{W}^{\mu+\lambda-\alpha}$ (resp. $\bar{\mathcal{W}}^{\mu+\lambda-\alpha}$) being isomorphic to a direct sum of Verma modules $\mathcal{V}^{\mu+\lambda-\alpha}$ (resp. $\bar{\mathcal{V}}^{\mu+\lambda-\alpha}$).

Choosing $v = (v_i)$ as in (3.6), let us consider the decomposition of the vector

$$\mathbf{1}^\mu \otimes \theta^\lambda(\epsilon, v, c) = \sum_{\alpha \in Q_+} \eta^{\mu+\lambda-\alpha}(\epsilon, v, c) \quad \in \mathcal{V}^\mu \otimes \widehat{\mathcal{V}}^\lambda \tag{4.3}$$

corresponding to (4.1). Consider similarly

$$\bar{\mathbf{1}}^\mu \otimes \bar{\theta}^\lambda(\epsilon, v, c) = \sum_{\alpha \in Q_+} \bar{\eta}^{\mu+\lambda-\alpha}(\epsilon, v, c) \quad \in \bar{\mathcal{V}}^\mu \otimes \widehat{\bar{\mathcal{V}}}^\lambda.$$

We define

$$\mathcal{X}_\alpha^{\mu,\lambda} = v^{-(\alpha,\alpha)/2+(\alpha,\lambda)} \left((\eta^{\mu+\lambda-\alpha}(\epsilon, v, c))_0, (\bar{\eta}^{\mu+\lambda-\alpha}(\epsilon, v, c))_0 \right). \tag{4.4}$$

Here $(\cdot)_0$ stands for the highest component.

In a similar manner, let us introduce

$$\theta^\lambda(\epsilon, v, c) \otimes \mathbf{1}^\mu = \sum_{\alpha \in Q_+} \xi^{\lambda+\mu-\alpha}(\epsilon, v, c) \quad \in \widehat{\mathcal{V}}^\lambda \otimes \mathcal{V}^\mu,$$

$$\bar{\theta}^\lambda(\epsilon, v, c) \otimes \bar{\mathbf{1}}^\mu = \sum_{\alpha \in Q_+} \bar{\xi}^{\lambda+\mu-\alpha}(\epsilon, v, c) \quad \in \widehat{\bar{\mathcal{V}}}^\lambda \otimes \bar{\mathcal{V}}^\mu.$$

We define

$$\bar{\mathcal{X}}_\alpha^{\lambda,\mu} = v^{(\alpha,\alpha)/2-(\alpha,\lambda+2\rho)} \left((\xi^{\lambda+\mu-\alpha}(\epsilon, v, c))_0, (\bar{\xi}^{\lambda+\mu-\alpha}(\epsilon, v, c))_0 \right). \tag{4.5}$$

Later it will be shown that (4.4), (4.5) are independent of the data ϵ, v, c (see Proposition 4.3 and Proposition 4.10).

PROPOSITION 4.1. *For any $\mu \in P$ we have*

$$J_\beta^\lambda = \sum_{\alpha \in Q_+} \mathcal{X}_\alpha^{\mu,\lambda} J_{\beta-\alpha}^{\mu+\lambda-\alpha} \tag{4.6}$$

$$= \sum_{\alpha \in Q_+} \bar{\mathcal{X}}_\alpha^{\lambda,\mu} J_{\beta-\alpha}^{\lambda+\mu-\alpha} q^{-(\alpha,\alpha)/2+(\alpha,\lambda+\mu+\rho)-(\mu,\beta)}. \tag{4.7}$$

Proof. The vector $\mathbf{1}^\mu \otimes \theta^\lambda(\epsilon, v, c)$ is a joint eigenvector of $E_i K_{v_i}$ with eigenvalue $c_i' = v^{(\mu,v_i+\alpha_i)} c_i$. Therefore each isotypic component $\eta^{\mu+\lambda-\alpha}(\epsilon, v, c)$ is proportional to $\theta^{\mu+\lambda-\alpha}(\epsilon, v, c')$, where the proportionality being determined by the

highest component $(\eta^{\mu+\lambda-\alpha}(\epsilon, v, c))_0$. Similarly, $\mathbf{1}^\mu \otimes \bar{\eta}^\lambda(\epsilon, v, c)$ is a joint eigenvector of $\bar{E}_i \bar{K}_{v_i}$ with eigenvalue $c_i'' = v^{-(\mu, v_i)} c_i$, so that $\bar{\eta}^{\mu+\lambda-\alpha}(\epsilon, v, c)$ is proportional to $\bar{\theta}^{\mu+\lambda-\alpha}(\epsilon, v, c'')$. Formula (4.6) follows from these facts. The case of (4.7) is similar. $\qquad\square$

4.2. FUNCTION $X_\beta^{\lambda_1, \lambda_2}$

Fix $\lambda_1, \lambda_2 \in P$ and $\beta \in Q_+$. Consider the Whittaker and dual Whittaker vectors

$$\theta^{(1)} = \theta^{\lambda_1}(\epsilon, v^-, c^{(1)}) \quad \in \widehat{\mathcal{V}}^{\lambda_1},$$
$$\theta^{(2)} = \theta^{\lambda_2}(-\epsilon, -v, c^{(2)}) \quad \in \widehat{\mathcal{V}}^{\lambda_2},$$
$$\bar{\theta}^{(1)} = \bar{\theta}^{\lambda_1}(\epsilon, v, \bar{c}^{(1)}) \quad \in \widehat{\bar{\mathcal{V}}}^{\lambda_1},$$
$$\bar{\theta}^{(2)} = \bar{\theta}^{\lambda_2}(-\epsilon, -v^+, \bar{c}^{(2)}) \quad \in \widehat{\bar{\mathcal{V}}}^{\lambda_2},$$

where $v_i^\pm = v_i \pm \alpha_i$, and $c^{(i)}, \bar{c}^{(i)}$ are chosen to satisfy

$$\bar{c}_i^{(1)} = v_i^{-1} c_i^{(1)}, \quad \bar{c}_i^{(2)} = v_i^3 c_i^{(2)}, \tag{4.8}$$
$$c_i^{(1)} / c_i^{(2)} = -v^{(v_i, \lambda_1 + \lambda_2 - \beta - \alpha_i) + (\alpha_i, \alpha_i)}. \tag{4.9}$$

Note that for all $\beta \in Q_+$ we have

LEMMA 4.2. *The following conditions are satisfied for all i.*

$$E_i \left(\theta^{(1)} \otimes \theta^{(2)} \right)_\beta = 0, \tag{4.10}$$
$$\bar{E}_i \left(\bar{\theta}^{(1)} \otimes \bar{\theta}^{(2)} \right)_\beta = 0. \tag{4.11}$$

Proof. Noting that

$$\Delta(E_i) = v^{(\alpha_i, v_i - \alpha_i)} K_{-v_i + \alpha_i} E_i K_{v_i - \alpha_i} \otimes 1 + K_i \otimes v^{-(\alpha_i, v_i)} K_{v_i} E_i K_{-v_i},$$

and using the defining relation (3.7) for Whittaker vectors, we find

$$E_i \left(\theta^{(1)} \otimes \theta^{(2)} \right) = \frac{1}{1 - v_i^2} \left(v^{(\alpha_i, v_i - \alpha_i)} c_i^{(1)} + v^{-(\alpha_i, v_i)} c_i^{(2)} \ K_{v_i} \otimes K_{v_i} \right) \times$$
$$\times \left(K_{-v_i + \alpha_i} \theta^{(1)} \otimes \theta^{(2)} \right).$$

Therefore with the choice of (4.9) the condition (4.10) is satisfied. Similarly (4.11) holds if

$$\bar{c}_i^{(1)} / \bar{c}_i^{(2)} = -v^{(v_i, \lambda_1 + \lambda_2 - \beta - \alpha_i) - (\alpha_i, \alpha_i)}.$$

Because of (4.8) and (4.9), it is also satisfied. $\qquad\square$

We define

$$X_\beta^{\lambda_1,\lambda_2} = \left((\theta^{(1)} \otimes \theta^{(2)})_\beta, (\bar\theta^{(1)} \otimes \bar\theta^{(2)})_\beta \right). \tag{4.12}$$

PROPOSITION 4.3. *We have*

$$X_\beta^{\lambda_1,\lambda_2} = \sum_{\alpha \in Q_+} J_{\beta-\alpha}^{\lambda_1} J_\alpha^{\lambda_2} q^{(\alpha,\alpha)/2-(\alpha,\lambda_2+\rho)}. \tag{4.13}$$

In particular, $X_\beta^{\lambda_1,\lambda_2}$ is independent of the data ϵ, ν and $c^{(i)}, \bar c^{(i)}$.

Proof. This follows from the definition along with the transformation laws (3.8), (3.9).   □

The relation (4.13) is a counterpart of (2.23). In order to describe the identification we introduce two vectors λ and $\bar\nu$ defined by

$$z_i = q^{-(\lambda,\alpha_i)}, \quad \bar\nu = \sum_{i=1}^l \nu_i \omega_i.$$

PROPOSITION 4.4. *We have*

$$X_\beta^{\lambda_1,\lambda_2} = X_{C,m}^{(0,\nu)}(q,z),$$

where C is a Cartan matrix of ADE type and

$$\beta = \sum_{i=1}^l m_i \alpha_i, \quad \lambda_1 = \lambda - \bar\nu, \quad \lambda_2 = \beta - \lambda - 2\rho. \tag{4.14}$$

Proof. Follows from (4.13) and (2.23).   □

COROLLARY 4.5. *Let $\mathfrak{g}$ be of ADE type. Then we have*

$$X_\beta^{\lambda_1,\lambda_2} = X_\beta^{\lambda_2,\lambda_1}. \tag{4.15}$$

Proof. The equality (4.15) is obtained from (2.22) substituting (4.14).   □

CONJECTURE 4.6. Relation (4.15) holds for arbitrary $\mathfrak{g}$.

PROPOSITION 4.7. *We have*

$$X_\beta^{\lambda_1,\lambda_2} \Big|_{q \to q^{-1}} = X_\beta^{\lambda_2,\lambda_1} \cdot q^{(\beta,\rho)}. \tag{4.16}$$

Proof. The relation (4.16) follows from (4.13) and (3.18). $\qquad\square$

The following vanishing property of $X_\beta^{\lambda_1,\lambda_2}$ will play a key role in the sequel. For $\mu \in P$, we denote by L^μ the irreducible quotient of $\mathcal{V}^\mu$.

PROPOSITION 4.8. *Let $\beta \in Q_+ \backslash \{0\}$. Assume that $-\lambda_1 - \lambda_2 - 2\rho + \beta \in P_+$, and that either $\lambda_1 - \beta \in P_+$ or $\lambda_2 - \beta \in P_+$. Then we have*

$$X_\beta^{\lambda_1,\lambda_2} = 0 \quad \text{if } -\lambda_1 - \lambda_2 - 2\rho \text{ is not a weight of } L^{-\lambda_1-\lambda_2-2\rho+\beta}. \tag{4.17}$$

Proof. We apply Proposition B.3 in Appendix 4.2, choosing $\lambda_1 = \lambda - \mu$, $\lambda_2 = \beta - \lambda - 2\rho$ and $\mu = -\lambda_1 - \lambda_2 - 2\rho + \beta$. Under our assumption $v = (\theta^{(1)} \otimes \theta^{(2)})_\beta$ can be written as $\sum_{i=1}^l F_i v_i$. Since $\bar{v} = (\bar{\theta}^{(1)} \otimes \bar{\theta}^{(2)})_\beta$ is a singular vector, we have

$$X_\beta^{\lambda_1,\lambda_2} = \sum_{i=1}^l (F_i v_i, \bar{v}) = \sum_{i=1}^l (v_i, \bar{E}_i \bar{v}) = 0.$$

$\qquad\square$

LEMMA 4.9. *For any $\mu \in P$, the following recurrence relations hold.*

$$X_\beta^{\lambda_1,\lambda_2} = \sum_{\alpha \in Q_+} \mathcal{X}_\alpha^{\mu,\lambda_1} X_{\beta-\alpha}^{\lambda_1+\mu-\alpha,\lambda_2} \tag{4.18}$$

$$= \sum_{\alpha \in Q_+} \bar{\mathcal{X}}_\alpha^{\lambda_2,\mu} X_{\beta-\alpha}^{\lambda_1,\lambda_2+\mu-\alpha}. \tag{4.19}$$

Proof. This follows from substituting (4.6),(4.7) into (4.13). $\qquad\square$

We can now state the relationship between $\mathcal{X}_\beta^{\mu,\lambda}$, $\bar{\mathcal{X}}_\beta^{\lambda,\mu}$ and $X_\beta^{\lambda_1,\lambda_2}$.

PROPOSITION 4.10. *We have*

$$\begin{aligned} X_\beta^{\lambda_1,\lambda_2} &= \mathcal{X}_\beta^{\beta-\lambda_1-\lambda_2-2\rho,\lambda_1} \quad if \quad \lambda_2 - \beta \in P_+ = \\ &= \bar{\mathcal{X}}_\beta^{\lambda_2,\beta-\lambda_1-\lambda_2-2\rho} \quad if \quad \lambda_1 - \beta \in P_+. \end{aligned}$$

Proof. We first assume $\lambda_2 - \beta \in P_+$. In the relation (4.18), choose $\mu = \beta - \lambda_1 - \lambda_2 - 2\rho$ and apply Proposition 4.8. Then the summand is non-zero only if $\alpha - \beta$ is a weight of L^0, i.e., only if $\alpha = \beta$. The first equality of Proposition follows from this. Likewise the second follows from (4.19). $\qquad\square$

In summary, we obtain the following relations.

THEOREM 4.11. *For any $\mu \in P$ we have*

$$J_\beta^\lambda = \sum_{\alpha \in Q_+} X_\alpha^{\lambda, \alpha - \mu - 2\rho} J_{\beta - \alpha}^{\mu - \alpha} \tag{4.20}$$

$$= \sum_{\alpha \in Q_+} X_\alpha^{\alpha - \mu - 2\rho, \lambda} J_{\beta - \alpha}^{\mu - \alpha} q^{-(\alpha, \alpha)/2 + (\alpha, \mu + \rho) - (\mu - \lambda, \beta)}, \tag{4.21}$$

$$X_\beta^{\lambda_1, \lambda_2} = \sum_{\alpha \in Q_+} X_{\beta - \alpha}^{\mu - \alpha, \lambda_2} X_\alpha^{\lambda_1, \alpha - \mu - 2\rho} \tag{4.22}$$

$$= \sum_{\alpha \in Q_+} X_{\beta - \alpha}^{\lambda_1, \mu - \alpha} X_\alpha^{\alpha - \mu - 2\rho, \lambda_2}. \tag{4.23}$$

Proof. By Proposition 4.10, Theorem is a restatement of the relations (4.6), (4.7), (4.18), (4.19) applied with shifted μ.   □

Identity (4.20) corresponds to (2.25), while identity (4.22) corresponds to (2.27). We have thus shown that these quasi-classical decompositions are exact in the case where C is a Cartan matrix of ADE type.

As an application we prove the following

THEOREM 4.12. *For any $\beta \in Q_+$, we have*

$$\sum_{\alpha \in Q_+} J_\alpha^{\alpha - \lambda - 2\rho} J_{\beta - \alpha}^{\lambda - \alpha} q^{-(\alpha, \alpha)/2 + (\lambda + \rho, \alpha)} = \delta_{\beta, 0}. \tag{4.24}$$

If $\mathfrak{g}$ is of ADE type, we also have

$$\sum_{\alpha \in Q_+} J_\alpha^{\alpha - \lambda - 2\rho} J_{\beta - \alpha}^{\lambda - \alpha} = \frac{1}{(q)_\beta}. \tag{4.25}$$

Proof. We first note that we expect (4.25) to hold for general $\mathfrak{g}$. The only reason we restrict ourselves to the ADE case is that the proof of (4.25) uses Corollary 4.15.

Substituting (4.13) to (4.20), we obtain

$$J_\beta^{\lambda - \mu} = \sum_{\alpha, \gamma} J_{\alpha - \gamma}^{\lambda - \mu} J_\gamma^{\beta - 2\rho - \lambda} J_{\beta - \alpha}^{\lambda - \alpha} \, q^{(\gamma, \gamma)/2 - (\gamma, \beta - \rho - \lambda)}.$$

This is a linear relation among $\{J_\delta^{\lambda - \mu}\}_{\delta \in Q_+}$ viewed as functions of μ. Since this is a linearly independent set, we can compare the coefficients of $J_0^{\lambda - \mu}$. In the right hand side only the term with $\gamma = \alpha$ contributes, so that

$$\delta_{\beta, 0} = \sum_\alpha J_\alpha^{\beta - 2\rho - \lambda} J_{\beta - \alpha}^{\lambda - \alpha} \, q^{(\alpha, \alpha)/2 - (\alpha, \beta - \rho - \lambda)}.$$

Renaming α to $\beta - \alpha$, then changing λ to $\beta - 2\rho - \lambda$, we arrive at (4.24).

Similarly, after using the symmetry (4.15) and substituting (4.13) to (4.20), we find

$$J_\beta^\lambda = \sum_{\alpha,\gamma} J_{\alpha-\gamma}^{\alpha-\mu-2\rho} J_\gamma^\lambda q^{(\gamma,\gamma)/2-(\gamma,\lambda+\rho)} J_{\beta-\alpha}^{\mu-\alpha}.$$

Specializing $q^{-\lambda}$ to 0, the left hand side simplifies due to (3.17), while only the term with $\gamma=0$ remains in the right hand side. This proves (4.25). □

Finally we give the counterpart of the identity (2.15) in Section 2.

THEOREM 4.13. *For a non-negative integer k we have*

$$\sum_{\alpha\in Q_+} J_\alpha^{\alpha-\lambda-2\rho} J_{\beta-\alpha}^{\lambda-\alpha} q^{k((\alpha,\alpha)/2-(\lambda+\rho,\alpha))} = J_\beta^\lambda[0,k], \tag{4.26}$$

where the right hand side is defined in (3.13).

Proof. Let us denote the left hand side of (4.26) by $\bar{J}_\beta^\lambda[0,k]$, and set

$$\bar{J}_\beta^\lambda[r,s] = q^{r\left(\frac{(\beta,\beta)}{2}-(\lambda+\rho,\beta)\right)} \bar{J}_\beta^\lambda[0,s-r].$$

The previous formula (4.25) states that $\bar{J}_\beta^\lambda[0,0] = J_\beta^\lambda[0,0]$. Using (3.16) for $J_\beta^\lambda = J_\beta^\lambda[0,\infty)$, it is easy to verify that

$$\bar{J}_\beta^\lambda[0,k] = \sum_{\beta_1+\beta_2=\beta} \bar{J}_{\beta_1}^{\lambda-\beta_2}[0,0] \bar{J}_{\beta_2}^\lambda[1,k].$$

The same relation holds for $J_\beta^\lambda[0,k]$ by (3.16). Hence by induction we obtain $\bar{J}_\beta^\lambda[0,k] = J_\beta^\lambda[0,k]$. □

We remark that (4.26) in the limit $k\to\infty$ reproduces the fermionic formula for J_β^λ.

Appendix A: Direct Proof of Toda Recursion for $\mathfrak{sl}_{l+1}$

We give here a direct proof that the fermionic sum $I_{C,m}(q,z)$ for the Cartan matrix C of type A_l satisfies the Toda recursion. Since we fix C, we drop it and denote $I_{C,m}(q,z), W_{C,m}$ by $I_m(q,z), W_m$.

PROPOSITION A.1. *The rational functions $I_m(q,z)$ $(m\in\mathbb{Z}_{\geq0}^l)$ are characterized by the Toda recursion*

$$\left\{\sum_{i=0}^l \left(q^{m_{i+1}-m_i}-1\right)\prod_{j=i+1}^l \left(q^{-1}z_j\right)\right\} I_m(q,z) =$$

$$= \sum_{i=1}^l \left\{q^{m_{i+1}-m_i}\prod_{j=i+1}^l \left(q^{-1}z_j\right)\right\} I_{m_1,\dots,m_i-1,\dots,m_l}(q,z). \tag{A.1}$$

Moreover, they satisfy the symmetry relation

$$I_{m_l,\ldots,m_1}(q^{-1},z_l^{-1},\ldots,z_1^{-1}) = I_{m_1,\ldots,m_l}(q,z_1,\ldots,z_l)(qz)^m q^{W_m}. \tag{A.2}$$

Proof. Let $\tilde{I}_m(q,z)$, $m \in \mathbb{Z}_{\geq 0}^l$ be a set of rational functions in $q, z = (z_1,\ldots,z_l)$ such that $\tilde{I}_0(q,z) = 1$. It is straightforward to show by induction on m that for $\tilde{I}_m(q,z)$ the Toda recursion (A.1) implies the symmetry relation (A.2). Now, we want to show that the former also implies the fermionic recursion (2.12). Since the solution is unique for both (A.1) and (2.12), the statement of Proposition follows.

Let $\mathbb{C}(q,z)$ be the field of rational functions in $q, z = (z_1,\ldots,z_l)$. Consider the vector space over $\mathbb{C}(q,z)$ consisting of formal power series in $y = (y_1,\ldots,y_l)$ with coefficients in $\mathbb{C}(q,z)$. We denote it by $\mathcal{F}$.

We consider the $\mathbb{C}(q,z)$-linear actions y_i, D_i $(i = 1,\ldots,l)$ on $\mathcal{F}$:

$$y_i \cdot f(y_1,\ldots,y_l) = y_i f(y_1,\ldots,y_l),$$
$$D_i \cdot f(y_1,\ldots,y_l) = f(y_1,\ldots,qy_i,\ldots,y_l),$$

and set formally $D_0 = D_{l+1} = 1$, $y_0 = y_{l+1} = 0$.

Let $\tilde{I}_m(q,z) \in \mathbb{C}(q,z)$. We assume that $\tilde{I}_0(q,z) = 1$. Set

$$F(q,z,y) = \sum_m y^m \tilde{I}_m(q,z)$$

and

$$G(q,z,y) = \sum_m y^m \tilde{I}_m(q,z) z^m q^{W_m}.$$

They belong to $\mathcal{F}$. Set

$$H = \sum_{i=0}^l \left(D_{i+1} D_i^{-1}(1 - y_i) - 1 \right) \prod_{j=i+1}^l \left(q^{-1} z_j \right).$$

The Toda recursion reads as $HF = 0$, and the symmetry relation reads as

$$F\left(q^{-1}, z_l^{-1},\ldots,z_1^{-1}, q^{-1}y_l,\ldots,q^{-1}y_1 \right) = G(q,z,y). \tag{A.3}$$

Set

$$\Lambda = \prod_{i=1}^l \frac{1}{(y_i)_\infty} = \sum_m \frac{y^m}{(q)_m}. \tag{A.4}$$

The fermionic recursion reads as

$$F(q,z,y) = \Lambda G(q,z,y). \tag{A.5}$$

Our goal is to show that if $HF = 0$, and therefore (A.3) is valid, then (A.5) follows.

Suppose that $HF = 0$. By changing $q \to q^{-1}$, $z_i \to z_i^{-1}$, $y_i \to q^{-1} y_i$, $D_i \to D_i^{-1}$, we obtain

$$\sum_{i=0}^{l} \left(D_{i+1}^{-1} D_i (1 - q^{-1} y_i) - 1 \right) \prod_{j=i+1}^{l} \left(q z_j^{-1} \right) F \left(q^{-1}, z_1^{-1}, \ldots, z_l^{-1}, q^{-1} y_1, \ldots, q^{-1} y_l \right) = 0.$$

Using

$$\Lambda^{-1} D_i \Lambda = (1 - y_i) D_i,$$

we can rewrite this as

$$\left(\Lambda^{-1} H \Lambda \right) F \left(q^{-1}, z_l^{-1}, \ldots, z_1^{-1}, q^{-1} y_l, \ldots, q^{-1} y_1 \right) = 0. \tag{A.6}$$

Because of the uniqueness of the solution $HF(q, z, y) = 0$ with $F(q, z, 0) = 1$, we obtain

$$F(q, z, y) = \Lambda F \left(q^{-1}, z_l^{-1}, \ldots, z_1^{-1}, q^{-1} y_l, \ldots, q^{-1} y_1 \right).$$

From (A.3) and this equality follows the fermionic recursion (A.5). □

Appendix B: Proposition on Singular Vectors

The main goal of this Appendix is to prove a statement about singular vectors which is used in the main text. In what follows, for a $U_v(\mathfrak{g})$ module M, $[M]_v$ will denote its subspace of weight v.

We start with the following Lemma.

LEMMA B.1. *Let M be a $U_v(\mathfrak{g})$ module from the category $\mathcal{O}$. Let $p \in [M]_{-\mu - 2\rho}$, $\mu \in P_+$ be a singular vector such that*

$$p \notin \sum_{i=1}^{l} \mathrm{Im} F_i.$$

Then the Verma module $U_v(\mathfrak{g}) \cdot p$ generated by p is a direct summand in M.

Proof. We first note that since the Verma module $\mathcal{V}^{-\mu - 2\rho}$ is irreducible, the submodule $V = U_v(\mathfrak{g}) \cdot p$ generated by p is isomorphic to it. We now show that there exists a submodule $W \subset M$ such that $M = V \oplus W$.

Denoting by C_v the quantum Drinfeld Casimir element $v^{\sum_{i=1}^{l} \alpha_i^\vee \omega_i + 2\rho} u$ (see [6] and Proposition 3.5) we have the decomposition $M = \oplus_{z \in \mathbb{K}} M^z$ into the generalized eigenspaces

$$M^z = \{ m \in M \mid (C_v - z)^k m = 0 \text{ for some } k \}.$$

Setting $z_0 = v^{(\mu, \mu + 2\rho)}$ we have $V \subset M^{z_0}$ and $M = M^{z_0} \oplus \bigoplus_{z \neq z_0} M^z$. So it suffices to find a submodule $M_0 \hookrightarrow M^{z_0}$ such that $M^{z_0} = V \oplus M_0$.

Since $M \in \mathcal{O}$, there exists a sequence of submodules

$$L_0 \hookrightarrow L_1 \hookrightarrow L_2 \hookrightarrow \ldots, \quad L_j \hookrightarrow M^{z_0}/V, \quad \lim_{j \to \infty} L_j = M^{z_0}/V$$

such that each quotient L_j/L_{j-1} is a highest weight module with highest weight τ_j. For all $j \geq 0$, let $\bar{w}_j \in L_j$ be a vector of weight τ_j such that the image of $\bar{w}_j$ in L_j/L_{j-1} is highest weight vector. Note that $\tau_j \not< -\mu - 2\rho$. In fact, for all j we have $z_0 = v^{(\tau_j, \tau_j + 2\rho)}$. If we set $\alpha = -\mu - 2\rho - \tau_j$, we obtain

$$(\alpha, \alpha) + 2(\rho, \alpha) + 2(\mu, \alpha) = 0,$$

which implies that $\tau_j \notin -\mu - 2\rho - (Q_+ \backslash \{0\})$. Hence, if $\tau_j \neq -\mu - 2\rho$, there exists the unique vector $w_j \in M^{z_0}$ which is a lifting of $\bar{w}_j$. For j, such that $\tau_j = -\mu - 2\rho$, we fix arbitrary liftings $w_j \in M^{z_0}$. Setting

$$M_0 = \sum_{j \geq 0} U_v(\mathfrak{g}) \cdot w_j,$$

we obtain $V + M_0 = M^{z_0}$. Since $V = U_v(\mathfrak{g}) \cdot p$ is irreducible and $p \notin \sum_{i=1}^{l} \operatorname{Im} F_i$, the intersection $V \cap M_0$ is trivial. This proves the Lemma. $\qquad\square$

COROLLARY B.2. *Let M be a $U_v(\mathfrak{g})$ module from the category $\mathcal{O}$ such that $M = \bigoplus_{\mu \in P_+} [M]_{-\mu - 2\rho}$. Then M is isomorphic to a direct sum of Verma modules.*

Proof. Let $M_0 \subset M$ be the maximal submodule such that $M = M_0 + W$ is a decomposition into the direct sum of $U_v(\mathfrak{g})$ modules and M_0 is a direct sum of Verma modules. Let $w \in W$ be a singular vector such that $[W]_\lambda = 0$ for λ bigger than the weight of w. Then Lemma B.1 implies $W = (U_v(\mathfrak{g}) \cdot w) \oplus W'$ and thus M_0 is not maximal. $\qquad\square$

PROPOSITION B.3. *Let L^μ be an irreducible $U_v(\mathfrak{g})$ module with highest weight $\mu \in P_+$, and let $\lambda \in P$, $\beta \in Q_+$ be such that either $\lambda + 2\rho \in -P_+$ or $\mu + \beta - \lambda \in -P_+$. Assume further that $[L^\mu]_{\mu - \beta} = 0$. Then we have*

$$\left[\mathcal{V}^{\lambda - \mu} \otimes \mathcal{V}^{\beta - 2\rho - \lambda}\right]^{sing}_{-\mu - 2\rho} \subset \sum_{i=1}^{l} \operatorname{Im} F_i$$

where $(\,)^{sing}$ means the space of singular vectors.

Proof. Set $M = \mathcal{V}^{\lambda - \mu} \otimes \mathcal{V}^{\beta - 2\rho - \lambda}$, and suppose that the statement of the Proposition is not true. Then there exists a vector $p \in M^{sing}_{-\mu - 2\rho}$ such that $p \notin \sum_{i=1}^{l} \operatorname{Im} F_i$. Set $V = U_v(\mathfrak{g}) \cdot p$. Because of Lemma (B.1), there exists a submodule $W \subset M$ such that

$$M = V \oplus W. \tag{B.1}$$

Tensoring both sides of (B.1) by L^μ we obtain

$$\mathcal{V}^{\lambda-\mu} \otimes \mathcal{V}^{\beta-2\rho-\lambda} \otimes L^\mu = (\mathcal{V}^{-\mu-2\rho} \otimes L^\mu) \oplus (W \otimes L^\mu). \tag{B.2}$$

We show that the decomposition (B.2) is impossible by a homological argument.

In the following we set $U = U_v(\mathfrak{g})$. Let N (resp. B, H) be the subalgebra of U generated by $\{E_i\}_{1 \le i \le l}$ (resp. $\{E_i, K_i^{\pm 1}\}_{1 \le i \le l}$, $\{K_i^{\pm 1}\}_{1 \le i \le l}$). All these subalgebras are vector spaces over the field $\mathbb{K} = \mathbb{C}(v^{1/\mathcal{N}})$. We shall make use of the following facts.

(i) Let X be a B module and $\mathrm{Ind}_B^U X = U \otimes_B X$ be the induced U module. Then we have an isomorphism

$$\mathrm{Tor}_\bullet^{U,H}(\mathbb{K}, \mathrm{Ind}_B^U X) \simeq \mathrm{Tor}_\bullet^{B,H}(\mathbb{K}, X). \tag{B.3}$$

The proof is essentially given in [16], Lemma 3.1.14, which treats the classical case of (B.3). In order to prove (B.3) we only need to replace the classical (B, H) projective resolution of X from [16], Corollary 3.1.8 by an arbitrary B-free resolution.

(ii) Denoting by n the number of positive roots of $\mathfrak{g}$ we have

$$\dim[\mathrm{Tor}_n^N(\mathbb{K}, \mathcal{V}^\lambda)]_\nu = \delta_{\nu, \lambda+2\rho}, \tag{B.4}$$

$$\dim[\mathrm{Tor}_n^N(\mathbb{K}, L^\mu)]_\nu = \delta_{\nu, \mu+2\rho}. \tag{B.5}$$

The proof of these equalities is based on the quantum BGG resolution [14] (see also [17,18]), which generalizes the classical BGG resolution [1]. Let W be the Weyl group of $\mathfrak{g}$. For $w \in W$ and $\lambda \in P$ we denote by $l(w)$ the length of w and by $w * \lambda = w(\lambda + \rho) - \rho$ the shifted action of w. In order to prove (B.4) we use the quantum BGG resolution of the trivial $U_{v^{-1}}(\mathfrak{g})$ module

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to \mathbb{K} \to 0, \tag{B.6}$$

where $F_p = \bigoplus_{w \in W: l(w)=p} \overline{\mathcal{V}}^{w*0}$ is a direct sum of Verma modules $\overline{\mathcal{V}}^{w*0}$ over $U_{v^{-1}}(\mathfrak{g})$. Using the anti-isomorphism $\sigma : U_v(\mathfrak{g}) \to U_{v^{-1}}(\mathfrak{g})$ (3.1) we endow each F_p with the structure of right $U_v(\mathfrak{g})$ module. Thus (B.6) is a right B-free resolution of the trivial module $\mathbb{K}$ and $\mathrm{Tor}_n^N(\mathbb{K}, \mathcal{V}^\lambda)$ is equal to nth homology of the complex

$$0 \to F_n \otimes_N \mathcal{V}^\lambda \to F_{n-1} \otimes_N \mathcal{V}^\lambda \to \cdots \to F_0 \otimes_N \mathcal{V}^\lambda \to 0. \tag{B.7}$$

We note that F_n is the free B module with one generator of H-weight $-w_0 * 0$, where w_0 is the longest element in W. Since $\mathcal{V}^\lambda$ is irreducible we obtain that the space of nth homology of (B.7) is one-dimensional and is generated by the tensor product of highest weight vectors of $\overline{\mathcal{V}}^{w_0*0}$ and of $\mathcal{V}^\lambda$. Now the equality $w_0\rho = -\rho$ implies (B.4). The proof of (B.5) is very similar and uses the (left) quantum BGG resolution of the module L^μ.

For any λ we have $\mathcal{V}^\lambda = \mathrm{Ind}_B^U \mathbb{K}_\lambda$, where $\mathbb{K}_\lambda$ denotes the one-dimensional B module with trivial action of N and an action of K_i by $v^{(\lambda, \alpha_i)}$. This gives (see [16], Proposition 3.1.10)

$$\mathcal{V}^{\lambda-\mu} \otimes \mathcal{V}^{\beta-2\rho-\lambda} \otimes L^\mu = \mathrm{Ind}_B^U(\mathbb{K}_{\lambda-\mu} \otimes \mathcal{V}^{\beta-2\rho-\lambda} \otimes L^\mu) =$$
$$= \mathrm{Ind}_B^U(\mathbb{K}_{\beta-2\rho-\lambda} \otimes \mathcal{V}^{\lambda-\mu} \otimes L^\mu).$$

We conclude that

$$\mathrm{Tor}_n^{\mathrm{U,H}}(\mathbb{K}, \mathcal{V}^{\lambda-\mu} \otimes \mathcal{V}^{\beta-2\rho-\lambda} \otimes \mathrm{L}^\mu) = \left[\mathrm{Tor}_n^{\mathrm{N}}(\mathbb{K}, \mathcal{V}^{\beta-2\rho-\lambda} \otimes \mathrm{L}^\mu)\right]_{\mu-\lambda} \tag{B.8}$$

$$= \left[\mathrm{Tor}_n^{\mathrm{N}}(\mathbb{K}, \mathcal{V}^{\lambda-\mu} \otimes \mathrm{L}^\mu)\right]_{\lambda+2\rho-\beta}. \tag{B.9}$$

Suppose $\mu + \beta - \lambda \in -P_+$. Then we have the decomposition (see Corollary B.2)

$$\mathcal{V}^{\beta-2\rho-\lambda} \otimes L^\mu = \oplus_\nu \mathcal{V}^{\beta-\lambda-2\rho+\nu} \otimes [L^\mu]_\nu.$$

From (B.4) and the vanishing assumption, the right hand side of (B.8) is equal to $[L^\mu]_{\mu-\beta} = 0$. Now suppose $\lambda + 2\rho \in -P_+$. Then we have the decomposition (see Corollary B.2)

$$\mathcal{V}^{\lambda-\mu} \otimes L^\mu = \oplus_\nu \mathcal{V}^{\lambda-\mu+\nu} \otimes [L^\mu]_\nu.$$

Again, from (B.4) and the vanishing assumption, the right hand side of (B.9) is equal to $[L^\mu]_{\mu-\beta} = 0$.

Similarly, (B.5) implies

$$\mathrm{Tor}_n^{\mathrm{U,H}}(\mathbb{K}, \mathcal{V}^{-\mu-2\rho} \otimes \mathrm{L}^\mu) \simeq \left[\mathrm{Tor}_n^{\mathrm{N}}(\mathbb{K}, \mathrm{L}^\mu)\right]_{2\rho+\mu} = \mathbb{K}.$$

This shows that the decomposition (B.1) is impossible, and thus proves our Proposition. □

Acknowledgements

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