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TYPICAL PROPERTIES OF LEAVES OF CARTAN FOLIATIONS WITH EHRESMANN CONNECTION

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We consider a Cartan foliation (M, F) of an arbitrary codimension q admitting an Ehresmann connection such that all leaves of (M, F) are embedded submanifolds of M. We prove that for any foliation (M, F) there exists an open, not necessarily connected, saturated, and everywhere dense subset M_0 of M and a manifold L_0 such that the induced foliation (M_0, F_{M_0}) is formed by the fibers of a locally trivial fibration with the standard fiber L_0 over (possibly, non-Hausdorff) smooth q-dimensional manifold. In the case of codimension 1, the induced foliation on each connected component of the manifold M_0 is formed by the fibers of a locally trivial fibration over a circle or over a line. Bibliography: 17 titles.

1 Introduction

We recall that a G_δ*-subset* of M is an at most countable intersection of open everywhere dense subsets of M. By a *typical property* we mean any generic property of leaves of a foliation (M, F) that belong to some everywhere dense G_δ-subset of M. The problem for finding typical properties of leaves of proper foliations was stated in [1]. We resolve this problem for Cartan foliations, i.e., for leaves admitting a transverse Cartan geometry (cf. definitions in Section 2).

The choice of Cartan foliations is motivated by the fact that they involve large classes of foliations such as Riemannian, pseudo-Riemannian, conformal, projective, transversally homogeneous foliations, and foliations with transverse linear connection. As was proved by the author [2], if a foliation admits a noneffective transverse Cartan geometry, then it also admits an associated effective transverse Cartan geometry. Therefore, to study the topology of Cartan foliations, we can without loss of generality assume that their transverse Cartan geometries are effective. In this paper, we assume that the Cartan foliations under consideration admit Ehresmann connections in the sense of [3].

We recall that a subset of a foliated manifold is *saturated* if it can be represented as the union of leaves. A leaf L of a foliation (M, F) is *proper* if L is an embedded submanifold of the manifold M. A foliation (M, F) is *proper* if each its leaf is proper. A leaf L of a foliation (M, F) is *closed* if L is a closed subset of M. As is known, any closed leaf of a foliation is proper.

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By the *holonomy group* we understand the germ holonomy group [4]. The $\mathfrak{M}$-holonomy group was introduced in [3] for a foliation with an Ehresmann connection $\mathfrak{M}$. Genreally speaking, this group differs from the germ holonomy group. However, as was proved by the author [5, Theorem 4], these groups are isomorphic in the case of Cartan foliations with Ehresmann connections.

The main result is formulated in the following theorem.

Theorem 1.1. *Let (M, F) be an arbitrary proper Cartan foliation of codimension q admitting an Ehresmann connection. Then there exists an open, not necessarily connected, saturated, and everywhere dense subset M_0 and a manifold L_0 such that the foliation (M_0, F_{M_0}) is formed by the fibers of a locally trivial fibration $p : M_0 \to B$ with the standard fiber L_0 over a smooth q-dimensional (not necessarily Hausdorff) manifold B. For $q = 1$ the induced foliation on each connected component of M_0 is formed by the fibers of a locally trivial fibration over a circle or by the fibers of a locally trivial fibration over a line.*

Example 6.1 below shows that the condition of the existence of an Ehresmann connection cannot be omitted in Theorem 1.1. In the proof of Theorem 1.1, we essentially use Theorem 4.1 below and results due to Glimm [6].

Remark 1.1. The method of the proof of Theorem 1.1 can be applied to foliations with transverse rigid geometry in the sense of [5]. Therefore, a counterpart of Theorem 1.1 is valid for any foliation with transverse rigid geometry, in particular, a G-foliation of finite type.

We introduce the category of foliations, objects of which are smooth foliations (M, F), whereas morphisms of foliations $f : (M, F) \to (\widetilde{M}, \widetilde{F})$ are smooth maps of manifolds $f : M \to \widetilde{M}$ that transform leaves of one foliation to the corresponding leaves of the other foliation.

Definition 1.1. A leaf $L = L(x)$ of a foliation (M, F) of codimension q with the trivial holonomy group is *locally stable in the sense of Reeb* if there exists a saturated neighborhood U and an isomorphism in the category of foliations $h : (U, F_U) \to (L \times R^q, F_{\mathrm{tr}})$ between the induced foliation (U, F_U) and the trivial foliation $F_{\mathrm{tr}} := \{L \times \{z\} | z \in R^q\}$ on the product of manifolds $L \times R^q$.

Thus, by Theorem 1.1, the triviality of the germ holonomy group, the local stability in the sense of Reeb, and the property to be diffeomorphic to a fixed manifold L_0 are typical properties of leaves of any proper Cartan foliation with an Ehresmann connection.

The local stability of leaves of foliations in the sense of Reeb and Ehresmann was studied by many researchers, in particular, by the author [7]–[9]. The local stability of an arbitrary compact leaf without holonomy follows from the classical Reeb theorem [4].

As is known, any conformal foliation can be regarded as a Cartan foliation in the sense of [10] or (which is equivalent in this case) in the sense of [2]. As was proved by the author [11, Theorem 2], for any conformal foliation (M, F) of codimension $q > 2$ the existence of an Ehresmann connection is equivalent to the completeness of the foliation (M, F). Furthermore, the structure of complete conformal foliations is described in [11, Theorem 3]. By these results, the following assertion holds.

Theorem 1.2. *Let (M, F) be a proper foliation that is either a Riemannian foliation of an arbitrary codimension $q \geqslant 1$ or a conformal foliation of codimension $q \geqslant 3$. We assume that the foliation (M, F) admits an Ehresmann connection. Then there exists an open, connected, saturated, and everywhere dense subset M_0 of M and a manifold L_0 such that the induced foliation (M_0, F_{M_0}) is formed by the fibers of a locally trivial fibration $p : M_0 \to B$ with the standard fiber L_0 over a Hausdorff smooth q-dimensional manifold B.*

We emphasize that Theorem 1.2 is independently proved and improves Theorem 1.1 for the above-mentioned classes of Cartan foliations.

Convention. Except for the Glimm theorem, it is assumed that all neighborhoods are open and all manifolds are connected and Hausdorff with a countable base, unless otherwise stated.

Notation. Following [12], we denote by $P(B,H)$ the principal H-bundle $p: P \to B$. The module of vector fields on a manifold M is denoted by $\mathfrak{X}(M)$, and the set of tangent vector fields to a distribution $\mathfrak{M}$ on M is denoted by $\mathfrak{X}_{\mathfrak{M}}(M)$.

2 Cartan Foliations

2.1. Cartan geometries. We recall the definition of a Cartan geometry [13]. Assume that G is a Lie group, H is a closed subgroup of G, $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{h}$ is the Lie subalgebra of H. Let $p: P \to N$ be a principal H-bundle given by a free right action of the group H on a manifold P. The action of an element $a \in H$ on P is denoted by R_a. A nondegenerate $\mathfrak{g}$-valued 1-form ω on P is called a *Cartan connection* if the following two conditions hold:

1) $\omega(A^*) = A$ for $A \in \mathfrak{h}$, where A^* is the fundamental vector field on P corresponding to A,

2) the 1-form ω is H-equivariant, i.e., $(R_a)^*\omega = Ad_G(a^{-1})\omega$ for all $a \in H$, where Ad_G is the adjoint representation of the group G in the Lie algebra $\mathfrak{g}$.

A principal H-bundle $P(N,H)$ equipped with a Cartan connection ω is called a *Cartan geometry of type* (G,H) and is denoted by $\xi = (P(N,H),\omega)$. The pair (N,ξ) is called a *Cartan manifold*. A Cartan geometry of type (G,H) is *effective* if the group G effectively acts by left translations on G/H.

If $\xi = (P(N,H),\omega)$ is a Cartan geometry of type (G,H), then the Cartan geometry of the same type $\xi_U = (P_U(U,H),\omega|P_U)$ is induced on each open subset U of the manifold N, where $P_U := p^{-1}(U)$.

2.2. Isomorphisms of Cartan geometries. Let $\xi = (P(N,H),\omega)$ and $\widetilde{\xi} = (\widetilde{P}(\widetilde{N},H),\widetilde{\omega})$ be two effective Cartan geometries of the same type (G,H). A diffeomorphism $\Gamma: P \to \widetilde{P}$ such that $\Gamma^*\widetilde{\omega} = \omega$ and $\Gamma \circ R_a = R_a \circ \Gamma$, where R_a is a right action of an element $a \in H$ on the space of the corresponding H-bundle, is called an *isomorphism of Cartan geometries* ξ and $\widetilde{\xi}$.

Each isomorphism $\Gamma: P \to \widetilde{P}$ of Cartan geometries $\xi = (P(N,H),\omega)$ and $\widetilde{\xi} = (\widetilde{P}(\widetilde{N},H),\widetilde{\omega})$ defines a projection $\gamma: N \to \widetilde{N}$ such that $\Gamma \circ \widetilde{p} = p \circ \gamma$, called an *isomorphism* of the Cartan manifolds (N,ξ) and $(\widetilde{N},\widetilde{\xi})$. Since the Cartan geometries ξ and $\widetilde{\xi}$ are effective, each isomorphism $\gamma: N \to \widetilde{N}$ of Cartan manifolds is the projection of an exactly one isomorphism $\Gamma: P \to \widetilde{P}$ of the Cartan geometries ξ and $\widetilde{\xi}$.

2.3. Definition of foliation by an N-cocycle. Let N be a (not necessasrily connected) manifold of dimension q. We say that an N-cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ is given on an n-dimensional manifold M, $n > q$, if we are given an open covering $\{U_i | i \in J\}$ of the manifold M and submersions $f_i: U_i \to N$ in N with connected fibers such that

(i) if $U_i \cap U_j \neq \varnothing$, then there exists a diffeomorphism $\gamma_{ij}: f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ satisfying the identity $f_i = \gamma_{ij} \circ f_j$ on $U_i \cap U_j$;

(ii) $\gamma_{ik} = \gamma_{ij} \circ \gamma_{jk}$ for all $x \in f_k(U_i \cap U_j \cap U_k)$, where $i,j,k \in J$.

We assume that the family η is maximal, i.e., contains all U_i, f_i, γ_{ij} possessing the above properties and $N = \{\bigcup U_i | i \in J\}$. Then the set of fibers of the submersions $\{f_i^{-1}(x) | x \in N, i \in J\}$ forms a base of the new topology Υ in M, called *foliated.* The connected components of the topological space (M, Υ) form a partition F of the manifold M, called a *foliation of codimension q defined by a cocycle* η and is denoted by (M, F).

2.4. Cartan foliations. Assume that (M, F) is a foliation defined by an N–cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in J}$. Let $\xi = (P(N, H), \omega)$ be an effective Cartan geometry of type (G, H) on the manifold N. If each local diffeomorphism γ_{ij} is the projection of some local isomorphism Γ_{ij} of the induced Cartan geometries $\xi_{f_j(U_i \cap U_j)}$ and $\xi_{f_i(U_i \cap U_j)}$, then the foliation (M, F) is called a *Cartan foliation of type* (G, H) and ξ is called its *transverse Cartan geometry.* In this case, we say that the foliation (M, F) is modelled on the Cartan geometry ξ.

Remark 2.1. The definition of a Cartan foliation with a noneffective transverse Cartan geometry is given in [2]. In [2, Proposition 1], it is shown that such a foliation also admits an induced effective Cartan geometry. Without loss of generality we will further assume that the Cartan geometries under consideration are effective.

3 Foliated Bundle of a Proper Cartan Foliation

3.1. Ehresmann connections for foliations. We recall the definition of an Ehresmann connection for a foliation introduced in [3]. As in [2], we use the term *vertical-horizontal homotopy* proposed in [14].

Let (M, F) be a foliation of an arbitrary codimension q. A distribution $\mathfrak{M}$ on a manifold M is *transverse* to the foliation F if $T_xM = T_xF \oplus \mathfrak{M}_x$ for any $x \in M$, where $\oplus$ is the symbol of the direct sum of vector subspaces. The vectors in $\mathfrak{M}_x$, $x \in M$, are said to be *horizontal.* A piecewise smooth curve σ is *horizontal* if all its tangent vectors are horizontal. In other words, a piecewise smooth curve is horizontal if each its smooth piece is an integral curve of the distribution $\mathfrak{M}$. The distribution TF tangent to leaves of the foliation F is called *vertical.* We say that a curve h is *vertical* if h lies in one leaf of the foliation F.

A *vertical-horizontal homotopy* is a piecewise smooth map $H : I_1 \times I_2 \to M$, $I_1 = I_2 = [0, 1]$, such that for any $(s, t) \in I_1 \times I_2$ the curve $H|_{I_1 \times \{t\}}$ is horizontal and the curve $H|_{\{s\} \times I_2}$ is vertical. A pair of curves $(H|_{I_1 \times \{0\}}, H|_{\{0\} \times I_2})$ is called the *base* of the vertical-horizontal homotopy H. Two paths (σ, h) with common start point $\sigma(0) = h(0)$, where σ is a horizontal path and h is a vertical path, is called an *admissible pair of paths.* As is known, for any admissible pair of paths (σ, h) there exists at most one vertical-horizontal homotopy with base (σ, h).

A distribution $\mathfrak{M}$ transverse to a foliation F is called an *Ehresmann connection* for F if for any admissible pair of paths (σ, h) there exists a vertical-horizontal homotopy with base (σ, h).

3.2. Foliated bundle. First of all, we prove the following assertion.

Theorem 3.1. *Let a proper Cartan foliation (M, F) of type (G, H) with an arbitrary codimension q admit an Ehresmann connection. Then the following assertions hold.*

1. *A principal right H-bundle $\pi : \mathscr{R} \to M$ with H-invariant transversally parallelizable foliation $(\mathscr{R}, \mathscr{F})$ is defined.*

2. *The foliation $(\mathscr{R}, \mathscr{F})$ is formed by the fibers of a locally trivial fibration $\pi_b : \mathscr{R} \to W$ with the standard fiber L_0.*

3. *The restriction of the projection* $\pi|_{\mathscr{L}} : \mathscr{L} \to L$ *to an arbitrary leaf* $\mathscr{L}$ *of the foliation* $(\mathscr{R}, \mathscr{F})$ *is a regular covering map on the corresponding leaf* $L = L(x)$ *of the foliation* (M, F)*; moreover, the group of desk transformations is isomorphic to the germ holonomy group* $\Gamma(L, x)$ *and the* $\mathfrak{M}$*-holonomy group* $H_{\mathfrak{M}}(L, x)$ *of the leaf* L *as well.*

4. *A locally free right action* R^W *of the Lie group* H *is induced on the manifold* W *by the rule* $R^W(w, a) = \pi_b(R_a u)$*, where* $w = \pi_b(u)$*,* $u \in \mathscr{R}$*, and* R_a *is the action of* $a \in H$ *on* $\mathscr{R}$*.*

5. *A stationary subgroup* H_w *of the group* H *at any point* $w \in \pi_b(\pi^{-1}(L))$ *is isomorphic to the holonomy group* $\Gamma(L)$ *of the leaf* L *and is a discrete subgroup of the Lie group* H*.*

Proof. Let (M, F) be a Cartan foliation modelled on an effective transverse Cartan geometry of type (G, H). For the foliation (M, F) it is possible to define (cf. [2, Proposition 2]) an H-bundle with the projection $\pi : \mathscr{R} \to M$ and H-invariant e-foliation $(\mathscr{R}, \mathscr{F})$; moreover, π is a morphism of the foliation $(\mathscr{R}, \mathscr{F})$ to (M, F) in the category of foliations.

By assumption, the foliation (M, F) admits an Ehresmann connection $\mathfrak{M}$. In this case, it is easy to verify that the distribution $\widetilde{\mathfrak{M}} := \pi^*\mathfrak{M}$ is a connection for the e-foliation $(\mathscr{R}, \mathscr{F})$.

We note that foliations admitting effective Cartan geometries form a subclass of foliations with transverse rigid geometry in the sense of [5]. It can be seen that Theorem 4 in [5] remains valid if the completeness condition is replaced with the weaker condition of the existence of an Ehresmann connection for this foliation. Therefore, Theorem 4 in [5] is applicable to the foliation (M, F). Hence the restriction of the projection $\pi|_{\mathscr{L}}$ on an arbitrary leaf $\mathscr{L}$ of the foliation $(\mathscr{R}, \mathscr{F})$ is a regular covering map for the leaf $L := \pi(\mathscr{L})$ of the foliation (M, F) with the group of desk transformations isomorphic to the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L, x)$ and the germ holonomy group $\Gamma(L, x)$ of the leaf $L = L(x)$ as well. We emphasize that the proof of Theorem 4 in [5] is essentially based on the fact that the pseudogroup of holonomy of a Cartan foliation is quasianalytic.

It is known [15] that any foliation has a leaf with the trivial holonomy group. Since the foliation (M, F) is proper, it has a proper leaf L with the trivial holonomy group. Taking into account the above interpretation of the holonomy group, we find that the leaf $\mathscr{L}$ of the foliation $(\mathscr{R}, \mathscr{F})$ lying over L is also proper. We recall that all germ holonomy groups of any e-foliation are trivial. Thus, $(\mathscr{R}, \mathscr{F})$ is a proper Riemannian foliation with an Ehresmann connection such that all its holonomy groups are trivial. Consequently [7, Theorem 2], the leaf space $\mathscr{R}/\mathscr{F}$ is a smooth manifold, denoted by W, and the projection $\pi_b : \mathscr{R} \to W$ on the leaf space form a locally trivial fibration. Thus, Assertions 1–3 are proved.

A verification shows that the rule in Assertion 4 determines a smooth action of the Lie group H on the manifold W.

We take any points $w \in W$, $x \in \pi(\pi_b^{-1}(w))$ and $u \in \pi^{-1}(x)$. Let H_w be a stationary subgroup at the point w of the action R^W of the group H on W. For a leaf $\mathscr{L} = \mathscr{L}(u)$ of a foliation $(\mathscr{R}, \mathscr{F})$ we introduce the subgroup $H(u) := \{a \in H | R_a(\mathscr{L}) = \mathscr{L}\}$ of the group H which preserves $\mathscr{L}$. By [5, Theorem 4], the group $H(u)$ is isomorphic to the holonomy group $\Gamma(L, x)$ of the leaf $L = L(x)$. The restriction $\pi_b|\pi^{-1}(x) : \pi^{-1}(x) = uH \to wH$ is a submersion of the orbit uH to the orbit wH. Since $\pi_b|\pi^{-1}(x) = uH \cap \mathscr{L} = uH(u)$, we have an isomorphism of groups $\chi(u) : H(u) \to H_w : a \mapsto R^W_a$, $a \in H$. Since the leaf $\mathscr{L}$ is proper, the intersection $uH \cap \mathscr{L}$ is a discrete subset of the leaf $\pi^{-1}(x) = uH$. Consequently, $H(u)$ is a discrete subgroup of the Lie group H. Hence the stationary subgroup H_w is discrete. Assertions 4 and 5 are proved. The theorem is proved. □

We recall that an H-bundle $\mathscr{R}(M,H)$ with a projection $\pi:\mathscr{R}\to M$ is called a *foliated bundle* and $(\mathscr{R},\mathscr{F})$ is referred to as a *lifted foliation* relative to the initial Cartan foliation (M,F).

Corollary 3.1. *Any leaf L of a Cartan foliation (M,F) having the trivial germ holonomy group $\Gamma(L,x)$ is diffeomorphic to the manifold L_0.*

Corollary 3.2. *The connected components of orbits of a locally free action R^W of a Lie group H on a manifold W form a smooth foliation (W,F^H).*

4 Proof of Theorem 1.1

4.1. The Glimm theorem. Under rather general assumptions, Glimm [6] established equivalence of seven conditions, but we indicate only two of these conditions which will be used below. As in [6], we say that a space M is *locally compact* if each neighborhood of a point in M contains a compact neighborhood of this point in M. We recall that a topological space N satisfies the separability axiom T_0 and is called a *T_0-space* if for any two distinct points in N there exists a neighborhood of at least one of these points that does not contain the other point.

Theorem (Glimm)**.** *Let G be a locally compact Hausdorff topological transformation group acting on a locally compact topological space M. We assume that the topologies in G and M have countable bases and every nonempty locally compact G-invariant subspace V of M contains a nonempty relatively open Hausdorff subset. Then the following conditions are equivalent.*

(1) *The orbit space M/G is a T_0-space.*

(2) *For each neighborhood N of the unit element e of the group G, any nonempty G-invariant locally compact subspace V of M, and any nonempty relatively compact subset V_0 of V there exists a nonempty relatively open subset U of V_0 such that, at each point $m\in U$,*

$$Nm\cap U = Gm\cap U. \tag{4.1}$$

4.2. Locally free actions of Lie groups. An action of a Lie group H on a manifold is *locally free* if all stationary subgroups of this action are discrete subgroups of H. For smooth actions of Lie groups on manifolds the Glimm theorem is satisfied. Applying this theorem, we prove the following assertion.

Theorem 4.1. *Let an action of a Lie group H on a smooth manifold W is smooth and locally free, and let the orbit space W/H is a T_0-space. Then there exists an open and everywhere dense submanifold W_0 of W that is invariant under the action of the group H and possesses the following properties*:

(1) *the action of the group H on W_0 is free,*

(2) *the orbit space W_0/H is a smooth, q-dimensional* (*not necessarily connected and Hausdorff*) *manifold, where $q=\dim(W)-\dim(H)$,*

Proof. By Theorem 3.1, the smooth action of the group H on the manifold W is locally free. Therefore, the connected components of orbits form a smooth foliation (W, F^H). Consequently, at any point $v \in W$, there exists a foliated chart $(\mathscr{U}, \psi)$ with respect to this foliation such that $\psi(\mathscr{U}) = R^m \times R^q$, where $m = \dim(H)$ and $q = \dim(W) - m$.

We note that foliated neighborhoods at each point $v \in W$ form a neighborhood base for the point v in the topological space W. Therefore, by the continuity of the action of the group H on the manifold W, for a neighborhood $\mathscr{U}$ of the point v there exists a connected neighborhood N of the unity e of the group H and a foliated neighborhood $V_0 \subset \mathscr{U}$ of v such that $NV_0 \subset \mathscr{U}$.

We apply the Glimm theorem to $V = W$ and fixed neighborhoods N and V_0. Then there exists a neighborhood $U \subset V_0$ such that $Nw \cap U = Hw \cap U$ for all points $w \in U$. Without loss of generality we can assume that U is the foliated neighborhood of $w_0 \in U$ corresponding to the chart $(U, \psi|_U)$. Let $\psi(U) = A_1 \times A_2$, where A_1 and A_2 are open subsets of R^m and R^q respectively, $\psi(w_0) = (a_1, a_2) \in A_1 \times A_2$. Then $D := \psi^{-1}(\{a_1\} \times R^q)$ is a q-dimensional transversal disk at the point w_0.

Since N is connected and $NV_0 \subset \mathscr{U}$, for any $w \in D \subset V_0$ the set Nw belongs to one local leaf of the foliation (W, F^H) in the foliated neighborhood $\mathscr{U}$. Consequently, $Nw \cap D = \{w\}$ and $Nw \cap U = Hw \cap U$ imply $Nw \cap D = Hw \cap D = \{w\}$ for any point w in D.

Thus, in any foliated neighborhood $\mathscr{U}$ there exists a subordinated foliated neighborhood $U \subset \mathscr{U}$, such that each orbit of the group H intersects U at most one local leaf. Let $f^H : W \to W/H$ be the quotient map on the orbit space. Then $f^H|_D : D \to W/H$ is a homeomorphism to an open subset $B := f^H(D) = f^H(U)$ in the orbit space W/H. The map $\chi := pr_2 \circ \psi \circ (\psi|_D)^{-1}$, where $pr_2 : R^m \times R^q \to R^q$ is the canonical projection on the second factor, is a homeomorphism from B to an open subset A_2 in R^q.

Let $N(U)$ be the saturation of U, i.e., the union of all orbits intersecting U. Since $N(U) = (f^H)^{-1}(f^H(U))$, we conclude that $N(U)$ is open in W and invariant under the action of the group H. We denote by W_0 the union of saturations of all open subsets U obtained as above. In each foliated neighborhood there exist points in W_0. Therefore, W_0 is open and everywhere dense in W. Hence $f^H(W_0)$ is open and everywhere dense in the orbit space W/H.

The above-defined pair (B, χ) is a chart in $f^H(W_0)$. A verification shows that the family $\mathscr{A} := \{(B, \chi) | B = f^H(U), U \subset W_0\}$ of all such charts forms a smooth atlas on the set $f^H(W_0)$. Thus, $f^H(W_0)$ becomes a smooth q-dimensional manifold. □

Remark 4.1. Theorem 4.1 remains valid if the left action of the group H on the manifold W is replaced with the right action.

4.3. Lemmas. Let (M, F) be an arbitrary foliation, and let V be any subset of M. The union of all leaves of the foliation (M, F) intersecting V is called the *saturation* of V and is denoted by $N(V)$. As is known, the saturation $N(V)$ of any open subset is open in M.

We denote by $f : M \to M/F$ the quotient map on the leaf space of a foliation (M, F). A leaf L of the foliation (M, F), regarded as a point of the leaf space M/F, is denoted by $[L]$; moreover, $f(L) = [L]$.

Lemma 4.1. *If a foliation (M, F) is proper, then the leaf space M/F satisfies the separability axiom T_0.*

Proof. We assume that (M, F) is a proper folliation. Let $[L_1]$ and $[L_2]$ be any two distinct points in the leaf space M/F. By [4, Theorem 4.11], a leaf L_1 is proper if and only if there exists a foliated neighborhood U intersecting L_1 by one local fiber.

Case 1. The leaf L_2 does not intersect the neighborhood U. Then L_2 does not intersect the saturation $N(U)$. Consequently, $[L_2]$ does not belong to the neighborhood $f(N(U))$ of the point $[L_1]$ in M/F.

Case 2. The intersection $U \cap L_2$ is not empty. Then $\widetilde{U} := U \setminus L_1$ is an open subset of M that intersects L_2 and does not intersect L_1. Therefore, $f(\widetilde{U})$ is a neighborhood of $[L_2]$ in the leaf space M/F that does not contain the point $[L_1]$. □

In the following lemma, we use the notation introduced in Theorem 3.1.

Lemma 4.2. *Let (M, F) be a proper Cartan foliation of type (G, H) admitting an Ehresmann connection. Let $f^H : W \to W/H$ be the projection onto the orbit space of the action R^W of the group H on the basic manifold W. Then there exists a homeomorphism $\theta : M/F \to W/H$ of topological spaces satisfying the commutative diagram*

$$\begin{array}{ccccc} & & \mathscr{R} & & \\ & \swarrow^{\pi} & & \searrow^{\pi_b} & \\ M & & & & W \\ \downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle f^H} \\ M/F & & \xrightarrow{\quad\theta\quad} & & W/H \end{array} \qquad (4.2)$$

Proof. We denote by $[wH]$ the orbit of a point $w \in W$ regarded as a point in W/H. Then $f^H(wH) = [wH]$. For any leaf L of the foliation (M, F) we set $\theta([L]) := [wH]$, where $w \in \pi_b(\pi^{-1}(L))$, $\pi_b : \mathscr{R} \to W$. It is easy to see that the above equality defines a map $\theta : M/F \to W/H$. By the definition of the action R^W of the Lie group H on the basic manifold W, the map $\theta : M/F \to W/H$ is bijective. By the definition of θ, the diagram (4.2) is commutative. Since the maps f, f^H, π, and π_b are simultaneously continuous and open, the commutativity of the diagram (4.2) implies that θ is an open continuous map. Thus, $\theta : M/F \to W/H$ is a homeomorphism of topological spaces. □

In the proof of the following lemma, we use the known facts: if a foliation (M, F) is formed by the fibers of a submersion $p : M \to B$, then the Ehresmann connection for this foliation is the Ehresmann connection for the submersion $p : M \to B$ (cf. [3, Proposition 1.3]), and the foliation (M, F) is formed by the fibers of a locally trivial fibration.

Lemma 4.3. *Let (M, F) be a smooth foliation of codimension* 1 *admitting an Ehresmann connection. We assume that each leaf has the trivial holonomy group and is locally stable in the sense of Reeb. Then either (M, F) is isomorphic to the trivial foliation $F_{\mathrm{tr}} = \{L_0 \times \{t\} | t \in \mathbb{R}^1\}$ of the product $L_0 \times \mathbb{R}^1$ or (M, F) is formed by the fibers of a locally trivial fibration over a circle.*

Proof. Let $\mathfrak{M}$ be an Ehresmann connection for a foliation (M, F). Since the codimension of (M, F) is equal to 1, the distribution $\mathfrak{M}$ is integrable and determines a foliation, denoted by (M, F^T). We consider the universal covering map $k : \widetilde{M} \to M$ and the induced foliations $\widetilde{F}$, $\widetilde{F^T}$ respectively on $\widetilde{M}$. By the Kashiwabara theorem [16], the manifold $\widetilde{M}$ is diffeomorphic to the product $N \times \mathbb{R}^1$ and $\widetilde{F} = \{N \times \{t\} | t \in \mathbb{R}^1\}$, $\widetilde{F}^T = \{\{v\} \times \mathbb{R}^1 | v \in N\}$. The fundamental group $\pi_1(M, x)$, $x \in M$, acts on $\widetilde{M} = N \times \mathbb{R}^1$ as the group of desk transformations preserving the structure of product. Therefore, with the help of the canonical projection on the second factor $pr : N \times \mathbb{R}^1 \to \mathbb{R}^1$, the group of diffeomorphisms Ψ is induced on $\mathbb{R}^1$. Assume that $\tau : \mathbb{R}^1 \to \mathbb{R}^1/\Psi$ is the projection on the orbit space of the group Ψ and $f : M \to M/F$ is the quotient map on the leaf space of the foliation (M, F). We note that there exists a homeomorphism $h : M/F \to \mathbb{R}^1/\Psi$ such that $\tau \circ pr = h \circ f \circ k$. We identify (by using h) the topological spaces M/F and $\mathbb{R}^1/\Psi$.

Case 1. The group Ψ is trivial, i.e., $\Psi = \{id_{\mathbb{R}^1}\}$. In this case, $M/F \cong \mathbb{R}^1/\Psi \cong \mathbb{R}^1$. Consequently, the foliation (M, F) is formed by the fibers of a submersion $f : M \to \mathbb{R}^1$ with an Ehresmann connection. Therefore, $f : M \to \mathbb{R}^1$ is a locally trivial fibration with the standard fiber L_0 over a contractible base isomorphic to the trivial fibration $L_0 \times \mathbb{R}^1 \to \mathbb{R}^1$.

Case 2. The group Ψ is not trivial. By assumption, each leaf of a foliation (M, F) has the trivial holonomy group and is locally stable in the sense of Reeb. Consequently, the action of the group Ψ on the line $\mathbb{R}^1$ is completely discontinuous, i.e., for each point $t \in \mathbb{R}^1$ there exists a neighborhood $U = U(t)$ such that $U \cap \psi(U) = \varnothing$ for any nontrivial element $\psi \in \Psi$. It is possible only if the group Ψ is isomorphic to the group $\mathbb{Z}$ of integers and the orbit space $\mathbb{R}^1/\Psi$ is diffeomorphic to the circle $\mathbb{S}^1$. As was shown above, there exists a homeomorphism $h : M/F \to W/H$. Therefore, the leaf space M/F is also diffeomorphic to a circle. Thus, the foliation (M, F) is formed by the fibers of a submersion $f : M \to \mathbb{S}^1$ with an Ehresmann connection and, consequently, by the fibers of a locally trivial fibration. □

4.4. Proof of Theorem 1.1. We use the notation introduced in Theorem 3.1. Let (M, F) be an arbitrary proper Cartan foliation of type (G, H) with an Ehresmann connection $\mathfrak{M}$. By Lemma 4.1, the leaf space M/F satisfies the separability axiom T_0. By Lemma 4.2, there exists a homeomorphism $\theta : M/F \to W/H$ from the leaf space M/F to the orbit space W/H of the action R^W of the group H on the basic manifold W. Hence W/H is a T_0-space. By Theorem 3.1, the associated action of the Lie group H on W is locally free and, consequently, we can apply Theorem 4.1 to the action R^W of the group H on W. By Theorem 4.1, there exists an open subset $f^H(W_0)$ which is a smooth q-dimensional, not necessarily Hausdorff, and everywhere dense manifold in W/H. Hence $S := \theta^{-1}(f^H(W_0))$ is an open and everywhere dense subset of M/F which is a smooth q-dimensional (possibly, non-Hausdorff) manifold. Consequently, the preimage $M_0 := f^{-1}(S) = \pi(\pi_b^{-1}(W_0))$ of the set S under the quotient map $f : M \to M/F$ is an open, saturated, and everywhere dense subset of M and the induced foliation (M_0, F_{M_0}) is formed by the fibers of the submersion $f|_{M_0} : M_0 \to S$. Consequently, all germ holonomy groups of the foliation (M_0, F_{M_0}) are trivial. Therefore, by assertion (3) of Theorem 3.1, each leaf in M_0 is diffeomorphic to the manifold L_0.

For any leaf $L \subset M_0$ there exists a neighborhood $\mathscr{V}$ of the point $[L]$ in the manifold S that is homeomorphic to R^q. Moreover, $\mathfrak{U} := f^{-1}(\mathscr{V})$ is an open saturated neighborhood of the fiber L in M. Let $\mathfrak{M}$ be an Ehresmann connection for the foliation (M, F). In this case, $\mathfrak{M}_{\mathfrak{U}} := \mathfrak{M}|_{\mathfrak{U}}$ is an Ehresmann connection for the foliation $(\mathfrak{U}, F_{\mathfrak{U}})$. As is known (cf. [3, Proposition 1.3]), an Ehresmann connection for a foliation formed by the fibers of a submersion is an Ehresmann connection for this submersion. Since any submersion with an Ehresmann connection over a Hausdorff manifold forms a locally trivial fibration, the foliation $(\mathfrak{U}, F_{\mathfrak{U}})$ is formed by fibers of a locally trivial fibration over a contractible base. This means that the neighborhood $\mathfrak{U}$ of the leaf L is diffeomorphic to the product of manifolds $L_0 \times R^q$ and the foliation $(\mathfrak{U}, F_{\mathfrak{U}})$ is isomorphic in the category of foliations to the trivial foliation $F_{\mathrm{tr}} := \{L_0 \times \{z\} | z \in R^q\}$ of this product, i.e., the leaf L is locally stable in the sense of Reeb.

Assume that a foliation (M, F) has codimension 1. In this case, applying Lemma 4.3 to each connected component of the submanifold M_0, we arrive at the required assertion.

Remark 4.2. As is established in [1, Theorem 3], for any smooth proper foliation (M, F) there exists an open and everywhere dense subset of M formed by the union of leaves without holonomy. However, the proof of Lemma 3 in [1], used in the proof of the above assertion, is not correct. Therefore, the results of [1] will not be used here.

5 Proper Riemannian and Conformal Foliations

5.1. Conformal foliations. Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds. We recall that a diffeomorphism $f : N_1 \to N_2$ is *conformal* if $f^* g_2 = e^{\lambda} g_1$, where λ is a smooth function on N_1. If $\lambda = \text{const}$, then f is called *similarity* of the Riemannian manifolds (N_1, g_1) and (N_2, g_2).

If an N-cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in J}$ determines a foliation (M, F) and there exists a Riemannian metric on N with respect to which any element γ_{ij} is an isometry of Riemannian manifolds induced on the corresponding open subsets, then (M, F) is called a *Riemannian foliation*.

A folliation given by a cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in J}$ is called *conformal* if each element γ_{ij} of the cocycle η is a local conformal diffeomorphism of conformal structures induced on the corresponding open subsets.

5.2. Proof of Theorem 1.2. We assume that (M, F) is a Riemannian foliation of an arbitrary codimension q with an Ehresmann connection $\mathfrak{M}$. Then, on the manifold M, there exists a transversally projectable Riemannian metric g (cf. [17]) relative to the foliation (M, F) such that $\mathfrak{M}$ is a q-dimensional distribution that is the orthogonal complement of TF. Furthermore, the lengths of horizontal curves remain unchanged under translations along any vertical curves relative to the Ehresmann connection. This property of vertical-horizontal homotopies with respect to a complete Riemannian foliation was used in [7]. Therefore, all the results of [7] remain valid provided that the condition of the transverse completeness of a Riemannian foliation is replaced with the weaker condition of the existence of an Ehresmann connection for this foliation.

By assumption, the foliation (M, F) is proper, i.e., all its leaves are proper. Consequently, by [7, Theorems 1 and 3], all leaves of the foliation (M, F) are closed, have the finite holonomy groups, and are locally stable in the sense of Reeb and Ehresmann, whereas the leaf space M/F is a smooth q-dimensional orbifold. Let $f : M \to M/F$ be a quotient map on the leaf space. We denote by $[L]$ a leaf L of the foliation (M, F), regarded as a point in the space M/F. Then $f(L) = [L]$. Any leaf L with the trivial holonomy group $\Gamma(L, x)$ is mapped to a regular point of the orbifold M/F. The converse assertion is also valid: any regular point of the orbifold M/F is the image of some leaf with the trivial holonomy group. As is known, the set of all regular points of any smooth orbifold is a q-dimensional stratum which is a connected, Hausdorff, and everywhere dense subset of the orbifold which is a smooth q-dimensional manifold. We denote by B the q-dimensional stratum of the orbifold M/F. Then $M_0 := f^{-1}(B)$ is an open, connected, and everywhere dense submanifold of M and the induced foliation (M_0, F_{M_0}) is formed by the fibers of a locally trivial fibration $F_{M_0} : M_0 \to B$ over the connected Hausdorff manifold B.

Let (M, F) be a non-Riemannian conformal foliation of codimension $q \geqslant 3$ admitting an Ehresmann connection $\mathfrak{M}$. By assumption, (M, F) is a proper folliation. By [11, Theorems 1 and 3], the foliation (M, F) has a global attractor $\mathscr{M}$ that is either a single closed leaf or the union of two closed leaves of this foliation. Hence $M \setminus \mathscr{M}$ is a connected open subset of M. Furthermore, by [11, Theorems 1 and 3], the induced foliation $(M \setminus \mathscr{M}, F_{M\setminus \mathscr{M}})$ is a Riemannian foliation with the Ehresmann connection $\mathfrak{M}_{M\setminus\mathscr{M}}$. As was shown above, there exists an open, connected, saturated, and everywhere dense subset M_0 of $M \setminus \mathscr{M}$ such that the induced foliation (M_0, F_{M_0}) is formed by the fibers of a locally trivial fibration $M_0 \to B$ over a Hausdorff manifold B. To complete the proof, we note that M_0 is open and everywhere dense in M.

6 Examples

Example 6.1. We introduce the notation: $\mathbb{N}$ is the set of natural numbers, $\mathbb{Q} := \{q_m | m \in \mathbb{N}\}$ is the enumerated set of rational numbers, $\mathbb{R}^n$, $n \geqslant 3$, is the n-dimensional arithmetical space, $l_m := \{(m, 0, \ldots, 0), t \in \mathbb{R}^n | t \geqslant q_m\}$ is a ray, and $l := \bigcup_{m\in\mathbb{N}} l_m$. Then $M := \mathbb{R}^n \setminus l$ is open, simply connected, and everywhere dense in $\mathbb{R}^n$. The foliation (M, F), where $F := \{L_t := (\mathbb{R}^{n-1} \times \{t\}) \cap M | t \in \mathbb{R}^1\}$, is formed by the fibers of a submersion $p : M \to \mathbb{R}^1 : (x_1, ..., x_{n-1}, t) \mapsto t$. Therefore, (M, F) is a proper foliation without holonomy and every leaf of (M, F) is diffeomorphic to the manifold $\mathbb{R}^n \setminus \mathbb{N}$. The folliation (M, F) does not have locally stable leaves in the sense of Reeb. Indeed, if L is a locally stable leaf, then for any small number $\varepsilon < 1/3$ there exists a neighborhood $(a-\varepsilon, a+\varepsilon)$ such that the open submanifold $p^{-1}(a-\varepsilon, a+\varepsilon)$ is diffeomorphic to the product of manifolds $(\mathbb{R}^n \setminus \mathbb{N}) \times (a - \varepsilon, a + \varepsilon)$, which is impossible. We note that the foliation (M, F) does not have Ehresmann connection. Otherwise, in view of [16], the manifold M should be diffeomorphic to the product $(\mathbb{R}^n \setminus \mathbb{N}) \times \mathbb{R}^1$, which contradicts the definition of M.

A construction similar to Example 6.1 was used in [1].

Example 6.2. We consider the unit circle as a set of points in the plane $\mathbb{S}^1 = \{e^{\frac{\pi t i}{2}} | t \in [-2, 2]\}$. The map $f : \mathbb{S}^1 \to \mathbb{S}^1 : e^{\frac{\pi t i}{2}} \mapsto e^{\frac{\pi h(t) i}{2}}$, where

$$h(t) := \begin{cases} t, & t \in [-2, 0], \\ t + e^{\frac{1}{(t-1)^2 - 1}}, & t \in (0, 2], \end{cases}$$

is a C^∞-diffeomorphism. We define the action of the group $\mathbb{Z}$ of integers on the cylinder $\mathbb{S}^1 \times \mathbb{R}^1$ by the formula

$$n \cdot (x, y) := (f^n(x), y + n) \quad \forall (x, y) \in \mathbb{S}^1 \times \mathbb{R}^1, \forall n \in \mathbb{Z}.$$

Since this action is free and properly discontinuous, we can define the quotient manifold $M = (\mathbb{S}^1 \times \mathbb{R}^1)/\mathbb{Z}$ diffeomorphic to the torus $\mathbb{T}^2$. Let $p : \mathbb{S}^1 \times \mathbb{R}^1 \to M$ be the natural projection. Then $F := \{p(\mathbb{S}^1 \times \{y\}) | y \in \mathbb{R}^1\}$ is a proper foliation with an integrable Ehresmann connection on the manifold M. The union of leaves without holonomy forms an open and everywhere dense subset M_0 of M with two connected components. All leaves are diffeomorphic to the circle $\mathbb{S}^1$ in one of these components and to the line $\mathbb{R}^1$ in the other. Thus, unlike Theorem 1.1 and Corollary 3.1, the standard fiber L_0 does not exist for the foliation (M_0, F_{M_0}). This situation is caused by the fact that the foliation (M, F) admits no transverse Cartan geometry. Otherwise, the pseudogroup generated by the diffeomorphism f of the circle $\mathbb{S}^1$ which is the holonomy pseudogroup of this foliation should be quasianalytic, i.e., the equality $f^k|_U = id_U$ for some $k \in \mathbb{Z}$ on some open subset U in $\mathbb{S}^1$ implies $f^k = id_{\mathbb{S}^1}$,

Example 6.3. We define the action of the group $\mathbb{Z}$ of integers on $\mathbb{R}^1 \times \mathbb{R}^q$, $q \geqslant 2$, by the formula $n \cdot (t, z) := (t + n, 2^n z)$ for any point $(t, z) \in \mathbb{R}^1 \times \mathbb{R}^q$ and $n \in \mathbb{Z}$. Then we can introduce the quotient manifold $M := (\mathbb{R}^1 \times \mathbb{R}^q)/\mathbb{Z}$, the quotient map $p : \mathbb{R}^1 \times \mathbb{R}^q \to M$, and the proper foliation (M, F) with $F := \{p(\mathbb{R}^1 \times \{z\}) | z \in \mathbb{R}^q\}$. We note that (M, F) is a transversally homotetic foliation, regarded as a Cartan foliation (cf., for example, [2]). Let $L^* := p(\mathbb{R}^1 \times \{0\})$, where $\{0\}$ is the zero in $\mathbb{R}^q$. The set of leaves without holonomy of the foliation (M, F) is defined by $M_0 := M \setminus L^*$. This set is open, saturated, and everywhere dense in M.

For $q = 1$ the manifold M_0 has two connected components diffeomorphic to a cylinder, whereas the induced folliation on each component is formed by the fibers of a locally trivial fibration over a circle with the standard fiber L_0 diffeomorphic to the line.

For $q \geqslant 2$ the manifold M_0 is connected and the foliation (M_0, F_{M_0}) is formed by the fibers of a locally trivial fibration over the product $\mathbb{S}^{q-1} \times \mathbb{S}^1$ of the $(q-1)$-dimensional sphere $\mathbb{S}^{q-1}$ and the circle $\mathbb{S}^1$ with the standard fiber L_0 diffeomorphic to the line.

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