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A Parabolic Equation with Nonlocal Diffusion without a Smooth Inertial Manifold

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Abstract—A family of parabolic integro-differential equations with nonlocal diffusion on the circle which have no smooth inertial manifold is presented.

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INTRODUCTION

The theory of inertial manifolds is an extreme implementation of the concept, going back to Hopf [1], of the finite-dimensional large-time behavior of solutions of distributed evolution systems with dissipation. This means that the eventual dynamics of a dissipative system with infinitely many degrees of freedom can, in a sense, be controlled by finitely many parameters. The main object of study is the class of semilinear parabolic equations with Hilbert state space. Paradoxically, although the existence of an inertial manifold has been proved only for a narrow class of such problems, all known examples in which the absence of an inertial manifold is guaranteed are sophisticated and look rather artificial. Anyway, no such examples for real problems of mathematical physics have been found so far. The present paper is a step in this direction; namely, we present a family of integro-differential equations of parabolic type with nonlocal diffusion on the circle which have no smooth inertial manifold.

1. PRELIMINARIES

We consider evolution equations of the form

$$\partial_t u = -Au + F(u) \tag{1.1}$$

with linear part A and nonlinear part F in a real separable Hilbert space X with norm $\|\cdot\|$. The general theory of such equations is presented in the book [2]. A closed linear operator A on X with dense domain $\mathscr{D}(A)$ is said to be *sectorial* if the semigroup $\{e^{-At}\}_{t>0}$ generated by A is analytic; in this case, the spectrum $\sigma(A)$ lies in a half-plane $\operatorname{Re}\lambda > \delta$. The property of being sectorial is stable with respect to bounded perturbations. We assume without loss of generality that $\delta > 0$ and denote the one-sided scale of Hilbert spaces corresponding to A by $\{X^\alpha\}_{\alpha\geq 0}$, where $X^\alpha = \mathscr{D}(A^\alpha)$ and $\|u\|_\alpha = \|A^\alpha u\|$ for $u \in X^\alpha$. We have

$$X^0 = X, \qquad X^1 = \mathscr{D}(A), \qquad X^\beta \subset X^\alpha \quad \text{for} \quad \beta > \alpha.$$

We make the following basic assumptions about Eq. (1.1).

Condition (H1). The linear operator A is sectorial, its resolvent is compact, and the spectrum $\sigma(A)$ lies in a half-plane $\operatorname{Re}\lambda > \delta > 0$.

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Condition (H2). For some $\theta \in [0,1)$, the nonlinear function F maps X^θ to X and satisfies the condition

$$\|F(u)-F(v)\| \le L(r)\|u-v\|_\theta \qquad \text{for} \quad \|u\|_\theta \le r, \quad \|v\|_\theta \le r.$$

Condition (H3). Equation (1.1) generates a continuous dissipative semiflow $\{\Phi_t\}_{t\ge 0}$ in X^θ.

The dissipativity of the solution semiflow means that

$$\sup \lim_{t\to+\infty} \|\Phi_t u\|_\theta \le a$$

uniformly in u on bounded subsets of X^θ. We denote the closed ball $\|u\|_\theta \le r$ in the state space X^θ by B_r and say that the ball B_a is *absorbing*. We refer to the number θ as the *nonlinearity exponent* of Eq. (1.1). The embeddings $X^\beta \subset X^\alpha$ with $0 \le \alpha < \beta$ are dense and compact; in particular,

$$\|u\|_\alpha \le C(\alpha,\beta)\|u\|_\beta \qquad \text{for} \quad u \in X^\beta.$$

It is easy to prove by using the construction in [2, Theorem 3.3.6] that, under Conditions (H1)–(H3), the evolution operators Φ_t with $t>0$ are compact.

In real problems, the operator A often turns out to be self-adjoint, and the compactness of its resolvent is characteristic of parabolic partial differential equations in bounded domains $\Omega \subset \mathbb{R}^m$.

A set $U \subset X^\theta$ is said to be *invariant* if

$$\Phi_t U = U \qquad \text{for} \quad t > 0.$$

The *global attractor* $\mathscr{A}$ of a semiflow $\{\Phi_t\}_{t\ge 0}$ was defined in [3], [4] as the union of all entire (existing for $t \in (-\infty,+\infty)$) bounded trajectories of the infinite-dimensional dynamical system (1.1) in the state space X^θ. The global attractor (which we call simply *attractor* in what follows) is a connected compact (by virtue of the compactness of the evolution operators Φ_t) invariant set in X^θ and uniformly attracts all balls in X^θ as $t\to+\infty$. In particular, $\mathscr{A}$ contains all possible limit modes (such as equilibrium points, cycles, invariant tori, etc.) of the solution semiflow. By virtue of the *smoothing property* of the parabolic equation, we have $\Phi_t X^\theta \subset X^1$ for $t>0$; therefore, any invariant set (in particular, any attractor) lies in X^1.

Let us modify the function $F(u)$ (without the loss of Lip or C^k regularity, $1\le k\le\infty$) outside the absorbing ball B_a so that the new function $\widetilde{F}(u)$ identically vanishes outside the ball B_{a+1}. Such a "truncation" procedure, described in detail in [4], allows us to pass to the equation $u_t = -Au + \widetilde{F}(u)$ with globally Lipschitz function $\widetilde{F}(u)$, dissipative phase semiflow in X^θ, and the same eventual dynamics as Eq. (1.1). Assuming that this passage has already been made, we return to the initial notation $F(u)$ and impose the condition

$$\|F(u)-F(v)\| \le L\|u-v\|_\theta \tag{1.2}$$

on the nonlinear component of Eq. (1.1). Note that [2] the phase semiflow $\{\Phi_t\}$ inherits the smoothness of the function $F(u)$.

Conditions (H1) and (H2) ensure the local (in $t>0$) solvability of Eq. (1.1) in X^θ. The dissipativity of the corresponding dynamical system is a technical but important point.

Lemma 1.1. *If Conditions* (H1) *and* (H2) *hold and* $\|F(u)\| \le M$ *for* $u \in X^\theta$*, then Eq.* (1.1) *is dissipative in* X^θ*.*

Proof. Let us write (1.1) in the form of a Duhamel integral equation as

$$u(t) = e^{-At}u(0) + \int_0^t e^{-A(t-\tau)} F(u(\tau))\,d\tau.$$

Since $\operatorname{Re}\sigma(A) > \delta > 0$, it follows from the well-known estimates

$$\|e^{-At}u\|_\theta \le Ce^{-\delta t}\|u\|_\theta, \quad \|e^{-At}u\|_\theta \le Ct^{-\theta}e^{-\delta t}\|u\| \qquad \text{for} \quad u \in X^\theta$$

that

$$\|u(t)\|_\theta \le Ce^{-\delta t}\|u(0)\|_\theta + CM\int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\theta}\,d\tau.$$

We see that the norm $\|u(t)\|_\theta$ remains bounded on the existence domain of the solution $u(t)$; hence, according to [2, Theorem 3.3.4], the solution can be extended to $[0,\infty)$. Thus, the initial equation has absorbing ball $B_a \subset X^\theta$ of radius

$$a = CM\int_0^\infty e^{-\delta s}s^{-\theta}\,ds,$$

which proves the lemma. □

2. INERTIAL MANIFOLDS

We shall consider semilinear parabolic equations of the form (1.1) with self-adjoint linear operator A, nonlinear function $F \in C^1(X^\theta, X)$, $0 \le \theta < 1$, and solution semiflow $\{\Phi_t\}_{t\ge 0}$ in the state space X^θ. An *inertial manifold* is a smooth or Lipschitz finite-dimensional invariant surface $\mathscr{M} \subset X^\theta$ containing the attractor $\mathscr{A}$ and exponentially attracting all solutions $u(t)$ at large times.

Most of the known methods for constructing an n-dimensional inertial manifold (beginning with those proposed in the fundamental papers [5], [6]) require the spectral jump condition

$$\lambda_{n+1} - \lambda_n > kL(\lambda_{n+1}^\theta + \lambda_n^\theta), \tag{2.1}$$

where L is the constant in inequality (1.2), the λ_n are the eigenvalues of A in nondecreasing order (with multiplicities taken into account), and k is an absolute constant.

As is known [7], [8], in the case $\mathscr{M} \in \mathrm{Lip}$, we can take $k = 1$, and this value cannot be decreased [8]. The construction of a C^1-smooth inertial manifold usually employs slightly larger values of k, but there are reasons to believe (see [9, p. 17]) that $k = 1$ is the optimal constant in this case, too.

As shown in [8], estimate (2.1) with $k = 1$ makes it possible to construct an n-dimensional Lipschitz inertial manifold of Eq. (1.1) in the form

$$\mathscr{M} = \{u \in X^\theta : u = y + h(y),\ y \in P_nX^\theta\};$$

here P_n is the spectral projection of A corresponding to the part $\{\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n\}$ of the spectrum, $h\colon P_nX^\theta \to (I-P_n)X^\theta$, where $I = \mathrm{id}$, and

$$\|h(y) - h(y')\|_\theta \le d\|y-y'\|_\theta \qquad \text{for} \quad y, y' \in P_nX^\theta.$$

In this situation, to each $u \in X^\theta$ there corresponds $\overline{u} \in \mathscr{M}$ such that

$$\|\Phi_t u - \Phi_t\overline{u}\|_\theta \le C\|u-\overline{u}\|_\theta e^{-\gamma t} \qquad \text{for} \quad t > 0, \quad \text{where} \quad \gamma = \lambda_{n+1} - \lambda_{n+1}^\theta L > 0.$$

The constant C does not depend on u and $\overline{u}$. Since the manifold $\mathscr{M}$ is invariant, it follows that $\mathscr{M} \subset X^1$.

The limit dynamics of the dynamical system $\{\Phi_t\}_{t\ge 0}$ with state space X^θ is completely described by the *inertial form*

$$y_t = -Ay + P_nF(y + h(y)), \qquad y \in P_nX^\theta,$$

which is an ordinary differential equation in $P_nX^\theta \simeq \mathbb{R}^n$. In this case, we say that the initial equation (1.1) is *asymptotically n-dimensional*.

The spectral jump condition (2.1) makes it possible to prove the existence of an inertial manifold for the dissipative equation (1.1) with fixed linear part A and any nonlinear function F satisfying the Lipschitz condition (1.2) under the "spectrum scatteredness" assumption

$$\sup_{n\ge 1}\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}^\theta + \lambda_n^\theta} = \infty. \tag{2.2}$$

In the case where the linear part $-A$ of the parabolic equation (1.1) is the Laplace operator Δ with standard boundary conditions in $L^2(\Omega)$, $\Omega \subseteq \mathbb{R}^m$, this assumption becomes restrictive because of the well-known asymptotics $\lambda_n \sim cn^{2/m}$ of the eigenvalues $\lambda_n \in \sigma(-\Delta)$. The attempts to avoid assumption (2.2) have been successful only in isolated special cases (see, e.g., [10], [11]). So far, asymptotic finite-dimensionality has not been proved even for simple problems, such as a parabolic equation of the form

$$u_t = u_{xx} + f(x, u, u_x)$$

on the circle or the reaction-diffusion equation

$$u_t = u_{xx} + f(x, u)$$

with standard boundary conditions in the disk.

On the other hand, very little is known about examples of evolution equations of the form (1.1) with no inertial manifold. In [12], a system of two coupled one-dimensional parabolic pseudodifferential equations without a smooth inertial manifold was constructed from the following considerations. Let $F'(u)$ denote the Fréchet derivative of the smooth map $F\colon X^\theta \to X$ at a point $u \in X^\theta$. The linear operators $F'(u)$ are continuous operators from X^θ to X, i.e., $F'(u) \in \operatorname{End}(X^\theta, X)$, and the Lipschitz condition (1.2) is equivalent to the inequality

$$\|F'(u)\|_{\mathrm{op}} \le L \qquad \text{for} \quad u \in X^\theta.$$

Let $\sigma(T(u))$ be the spectrum of the unbounded linear operator $T(u) = F'(u) - A$ on X with domain X^1. Since

$$F'(u) = F'(u)A^{-\theta}A^\theta \qquad \text{with} \quad \theta < 1,$$

where $F'(u)A^{-\theta} \in \operatorname{End} X$, it follows that the operator $-T(u)$ inherits sectoriality and the property of having compact resolvent from A (see [2, Sec. 1.4]). Thus, $\sigma(T(u))$ consists of eigenvalues of finite multiplicity. The number $l(u)$ of positive eigenvalues in $\sigma(T(u))$ (counting algebraic multiplicities) is finite. Finally, let E be the set of stationary points $u \in X^\theta$ of Eq. (1.1) for which the spectrum $\sigma(T(u))$ contains no real eigenvalues $\lambda \le 0$.

Lemma 2.1 (see [12]). *If the attractor $\mathscr{A}$ of Eq. (1.1) with nonlinear function $F \in C^1(X^\theta, X)$ is contained in a smooth invariant finite-dimensional manifold $\mathscr{M} \subset X^\theta$, then the number $l(u_1) - l(u_2)$ is even for any $u_1, u_2 \in E$.*

Note that the condition that the stationary points $u \in E$ are hyperbolic, which was imposed in [12], turns out to be redundant. A modification of Lemma 2.1 was used in the recent papers [9] and [13] to obtain a general construction of an abstract equation (1.1) with nonlinear function $F \in C^\infty$ and nonlinearity exponent $\theta = 0$ for which there is no smooth inertial manifold. In the same papers, by using a different (more delicate) argument, an equation of the form (1.1) with $F \in C^\infty$ which does not have even a Lipschitz inertial manifold was constructed . Apparently, these results can be extended to the general case of a nonlinearity exponent $\theta \in [0, 1)$. The counterexamples constructed in [9], [12], and [13] are rather unnatural; it is desirable to find a physically meaningful semilinear parabolic equation without the property of asymptotic finite-dimensionality. This problem is solved in this paper. The example presented below was announced by the author (in a less perfect form) in [14].

3. MAIN RESULT

Let H^ν, $\nu \ge 0$, denote the generalized L^2 Sobolev spaces [15] of real functions on the unit circle Γ; in particular, $H^0 = H = L^2(\Gamma)$. The differentiation $\partial_x u = u_x$ is a continuous operator from $H^{\nu+1}$ to H^ν, and $\partial_x\colon H^1 \to H_0$, where H_0 is the subspace functions in $L^2(\Gamma)$ with zero mean over Γ. For $\nu > 1/2$, we have continuous embeddings

$$H^\nu \subset C(\Gamma), \qquad H^{\nu+1} \subset C^1(\Gamma).$$

Consider the integro-differential parabolic equation

$$u_t = ((I+B)u_x)_x + f(x,u,u_x), \tag{3.1}$$

where $x \in \Gamma$, $I = \mathrm{id}$ and $B = B^*$ are bounded linear operators on the Hilbert space H with norm $\|\cdot\|$, and $f(x,s,p)$ is a function on $\Gamma \times \mathbb{R}^2$, which is assumed to be infinitely differentiable but not analytic. The operator $I+B$ plays the role of a nonlocal diffusion coefficient. To be more precise, let

$$(Bh)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \ln\left|\sin\frac{x+y}{2}\right| h(y)\,dy,$$

and let

$$(Jh)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\frac{x+y}{2} h(y)\,dy$$

for $h \in H$. The operator J is related to the Hilbert singular integral operator

$$(\mathscr{G}h)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\frac{y-x}{2}\, h(y)\,dy$$

by $(Jh)(x) = (\mathscr{G}h)(-x)$, $h \in H$. As is known [16, Chap. 6], $\mathscr{G}1 = 0$ and

$$\mathscr{G}\colon\ \cos nx \to -\sin nx, \qquad \mathscr{G}\colon\ \sin nx \to \cos nx$$

for integer $n \ge 1$; hence $J1 = 0$ and

$$J\colon\ \cos nx \to \sin nx, \qquad J\colon\ \sin nx \to \cos nx \tag{3.2}$$

for such n.

The integral operators J and B have the following properties:

(a) $J \in \operatorname{End} H$ and $J^2 = I$ on H_0;

(b) $B \in \operatorname{End}(H, H^1)$ and $\partial_x B = J$ on H.

Clearly, $J^* = J$, $B^* = B$, the operator B is compact on H, and

$$B\colon\ \cos nx \to -\frac{1}{n}\cos nx, \qquad B\colon\ \sin nx \to \frac{1}{n}\sin nx$$

for $n \ge 1$. We see that the subspace H_0 is invariant under B, and the least eigenvalue of the restriction of B to H_0 equals -1. Thus, the self-adjoint operator $I+B$ is nonnegative on H_0 and can be interpreted as a degenerate nonlocal "diffusion coefficient" in the evolution equation (3.1).

To write Eq. (3.1) in the standard form (1.1), we set

$$Au = u - u_{xx}, \qquad \mathscr{D}(A) = H^2,$$

and

$$F(u) = u + (Bu_x)_x + f(x,u,u_x). \tag{3.3}$$

Let $X = H$, and let $X^\alpha = \mathscr{D}(A^\alpha)$ for $\alpha > 0$. The self-adjoint positive linear operator A on X has compact resolvent, and $X^\alpha = H^{2\alpha}$. Note that A has the simple eigenvalue $\lambda_0 = 1$ and the double eigenvalues $\lambda_n = n^2+1$, $n \ge 1$; therefore, the spectrum scatteredness assumption (2.2) does not hold even for the least possible (in this situation) nonlinearity exponent $\theta = 1/2$.

The main result of this paper is as follows.

Theorem. *Under an appropriate choice of the function $f(x,s,p) \in C^\infty$, Eq. (3.1) generates a smooth dissipative semiflow in X^θ, $\theta \in (3/4, 1)$, and the attractor of this equation is not contained in any invariant finite-dimensional C^1 manifold $\mathscr{M} \subset X^\theta$.*

Proof. We begin by constructing the function $f(x,s,p)$ and deriving the announced properties of the solution semiflow. Take any $\theta \in (3/4,1)$. The embeddings $X^\theta \subset C^1(\Gamma) \subset C(\Gamma) \subset X$ are continuous; therefore, for any function $f \in C^\infty(\Gamma \times \mathbb{R}^2)$, the map $u \to f(x,u,u_x)$ and, thereby, the nonlinear component $F(u)$ in (3.3) belong to the class $C^\infty(X^\theta, X)$. Moreover, the function $F\colon X^\theta \to X$ satisfies the Lipschitz condition on bounded subsets of X^θ. Thus, the evolution equation (3.1) satisfies Conditions (H1) and (H2) with nonlinearity exponent θ. We ensure the dissipativity of this equation by choosing a function f of special structure.

We set

$$\begin{aligned} f(x,s,p) &= g(x,s,p) - 2\varepsilon p\cos x + \varepsilon s \sin x, \\ g(x,s,p) &= (\kappa + 2\varepsilon\cos x)\omega(s)w(p) + \varepsilon\gamma(s) + \varepsilon\eta(s)(1-\sin x) + \xi(s), \end{aligned} \tag{3.4}$$

where $\varepsilon, \kappa \in \mathbb{R}$, $\varepsilon > 0$, and $|\kappa| > 1$. We assume that the functions ω, γ, η, and ξ belong to $C^\infty(\mathbb{R})$ and satisfy the conditions

$$\begin{gathered} \omega(z) = z, \quad \gamma(z) = 2z^3 - 3z^2, \quad \eta(z) = 2z^2 - z^3, \quad \xi(z) = 0, \qquad |z| \le 1, \\ \omega(z) = 0, \quad \gamma(z) = 0, \quad \eta(z) = 0, \quad \xi(z) = -z, \qquad |z| \ge 2. \end{gathered} \tag{3.5}$$

Note that $\partial_x B\,\partial_x = J\,\partial_x$ on $X^{1/2}$ and $J\,\partial_x \in \operatorname{End}(X^{1/2}, X)$.

For a while, we represent the right-hand side of (3.1) in the form $F_1(u) - A_\varepsilon u$, where

$$A_\varepsilon = A - J\,\partial_x - \varepsilon D, \qquad D = -2\cos x\,\partial_x + \sin x, \qquad \mathscr{D}(A_\varepsilon) = \mathscr{D}(A) = X^1,$$

and $F_1(u) = u + g(x,u,u_x)$. We regard the non-self-adjoint operator A_ε as a perturbation of the self-adjoint operator $A_0 = A - J\,\partial_x$ with $\mathscr{D}(A_0) = X^1$.

For $u \in X^1$, we have the inequality

$$\|Du\| \le \frac{29}{3}\|u\| + \frac{2}{3}\|A_0 u\|, \tag{3.6}$$

which is easy to obtain from the evident estimates

$$\begin{gathered} \|u_x\|^2 \le 9\|u\|^2 + \frac{1}{16}\|u_{xx}\|^2, \qquad \|u_x\| \le 3\|u\| + \frac{1}{4}\|u_{xx}\|, \qquad \|u_{xx}\| \le \|u\| + \|Au\|, \\ \|Au\| \le \|A_0 u\| + \|Ju_x\|, \qquad \|Du\| \le 2\|u_x\| + \|u\| \end{gathered}$$

and the relation $\|Ju_x\| = \|u_x\|$. In the terminology of [17], inequality (3.6) means the A_0-boundedness of D and (for small ε) ensures the closedness of A_ε.

Since $A_0 1 = 1$, it follows from the relations (3.2)

$$A_0\colon\ \cos nx \to (1+n+n^2)\cos nx, \qquad A_0\colon\ \sin nx \to (1-n+n^2)\sin nx \tag{3.7}$$

for $n \ge 1$ that the spectrum of A_0 consists of the double eigenvalues $\lambda_n^{(0)} = 1 + n + n^2$ with eigenspaces

$$X_n = \{\cos nx, \sin(n+1)x\}, \qquad n \ge 0,$$

which form an orthogonal basis in X. Let Q_n be the corresponding two-dimensional spectral projections on X. The operator A_0 has compact resolvent $R_\lambda(A_0)$. It follows from Theorem 3.17 in [17, Chap. 4] that inequality (3.6) ensures the compactness of the resolvent $R_\lambda(A_\varepsilon)$ and the continuity of the resolvent set $\rho(A_\varepsilon)$ as $\varepsilon \to 0$. Moreover, standard estimates for the resolvents of closed operators ensure the convergence in norm

$$R_\lambda(A_\varepsilon) \xrightarrow{\varepsilon \to 0} R_\lambda(A_0)$$

uniform on compact sets in $\rho(A_0)$. According to formulas of perturbation theory [17, Chap. 8, Theorem 2.6], the eigenvalues of the operator $A_\varepsilon = A_0 - \varepsilon D$ have the form

$$\lambda_n^{\pm} = \lambda_n^{(0)} + \varepsilon\mu_n^{\pm} + o(\varepsilon), \qquad \varepsilon \to 0,$$

where the $\mu_n^\pm$ are the eigenvalues of the two-dimensional operators Q_nDQ_n. Straightforward calculations yield the matrix representations (in the bases of the subspaces X_n specified above)

$$Q_0DQ_0 = \begin{pmatrix} 0 & 1 \\ -\dfrac{1}{2} & 0 \end{pmatrix}, \quad Q_nDQ_n = \begin{pmatrix} 0 & \dfrac{2n+1}{2} \\ -\dfrac{2n+1}{2} & 0 \end{pmatrix}, \qquad n \geq 1,$$

whence

$$\lambda_0^\pm = 1 \pm i\frac{\varepsilon}{\sqrt{2}} + o(\varepsilon), \qquad \lambda_n^\pm = (1+n+n^2) \pm i\varepsilon\frac{2n+1}{2} + o(\varepsilon). \tag{3.8}$$

For small ε, the spectrum of A_ε lies in the half-plane $\operatorname{Re}\lambda > \delta > 0$, and $(A - A_\varepsilon)A^{-1/2} \in \operatorname{End} X$. It follows [2, Sec. 1.4] from the last property that A_ε is sectorial in X and $\mathscr{D}(A_\varepsilon^\alpha) = X^\alpha$ for all $\alpha \geq 0$.

Since $|s + g(x,s,p)| \leq \text{const}$ on $\Gamma \times \mathbb{R}^2$, it follows that $F_1\colon X^{1/2} \to X$ and $\|F_1(u)\| \leq \text{const}$ on $X^{1/2}$. The construction of $g(x,s,p)$ chosen above ensures that $|g_s| \leq \text{const}$ and $|g_p| \leq \text{const}$ on $\Gamma \times \mathbb{R}^2$ and, therefore, the fulfillment of the global Lipschitz condition

$$\|F_1(u) - F_1(v)\| \leq L\|u - v\|_{1/2} \qquad \text{for} \quad u, v \in X^{1/2}.$$

Since $\theta > 1/2$ and the embedding $X^\theta \subset X^{1/2}$ is continuous, it follows that $\|F_1(u)\| \leq \text{const}$ on X^θ and $F_1 \in \operatorname{Lip}(X^\theta, X)$. Lemma 1.1 guarantees the dissipativity of Eq. (3.1) in the state space X^θ, so that this equation satisfies assumptions (H1)–(H3) with nonlinearity exponent θ.

We proceed to write Eq. (3.1) in which the nonlinear component $F(x)$ is of the form (3.3) and the function $f(x,s,p)$ has structure (3.4), (3.5). The linear operator $J\,\partial_x + \varepsilon D$ is continuous as an operator from $X^{1/2}$ to X and, hence, as an operator from X^θ to X; therefore, $F \in \operatorname{Lip}(X^{1/2}, X)$ and $F \in \operatorname{Lip}(X^\theta, X)$.

Let us prove the second part of the theorem. The linearization $T(u) = F'(u) - A$ of the vector field $F(u) - Au$ of Eq. (3.1) at a point $u \in X^\theta$ is a closed unbounded linear operator on the Hilbert space X with compact resolvent and dense domain X^1 (see Sec. 2 above). This operator acts on functions $h \in X^1$ by the rule

$$T(u)h = h_{xx} + Jh_x + g_s(x,u,u_x)h + g_p(x,u,u_x)h_x + \varepsilon Dh.$$

Relations (3.5) imply

$$g(x,0,0) = 0, \qquad g(x,1,0) = -\varepsilon\sin x, \qquad g_s(x,0,0) = g_p(x,0,0) = 0,$$
$$g_s(x,1,0) = \varepsilon(1 - \sin x), \qquad g_p(x,1,0) = \kappa + 2\varepsilon\cos x.$$

Since $D1 = \sin x$, it follows that $u_0 = 0$ and $u_1 = 1$ are stationary solutions of Eq. (3.1). Moreover,

$$T(u_0) = \partial_{xx} + J\,\partial_x + \varepsilon D, \qquad T(u_1) = \partial_{xx} + J\,\partial_x + \kappa\,\partial_x + \varepsilon.$$

Relations $T(u_0) = I - A_\varepsilon$ and (3.8) imply that the spectrum $\sigma(T(u_0))$ is purely nonreal. On the other hand, as seen from (3.7), the operator $T(u_1) = I - A_0 + \kappa\,\partial_x + \varepsilon$ leaves invariant the mutually orthogonal subspaces

$$Y_n = \{\cos nx, \sin nx\}, \qquad n \geq 1,$$

in X; on each of these subspaces, it is described by a matrix of the form

$$\begin{pmatrix} -n^2 - n + \varepsilon & -\kappa n \\ \kappa n & -n^2 + n + \varepsilon \end{pmatrix},$$

with eigenvalues

$$-n^2 + \varepsilon \pm i\,dn, \qquad d = (\kappa^2 - 1)^{1/2} > 0.$$

Since $T(u_1) = \varepsilon$, it follows that the real part of the spectrum $\sigma(T(u_1))$ consists of a single simple eigenvalue $\varepsilon > 0$.

Thus, for a function $f(x, s, p)$ of the form (3.4), (3.5) and sufficiently small ε, the spectra σ_0 and σ_1 of the linearizations $T(u)$ of the vector field $F(u) - Au$ of Eq. (3.1) at the stationary points $u_0, u_1 \in X^\theta$ have the properties

$$\sigma_0 \cap \mathbb{R} = \varnothing, \qquad \sigma_1 \cap \mathbb{R} = \{\varepsilon\},$$

where ε is a simple positive eigenvalue. Let $l(u)$ be the number of positive eigenvalues (with multiplicities taken into account) in the spectrum of $T(u)$ for $u \in X^\theta$; then $l(u_0) = 0$ and $l(u_1) = 1$. Thus, according to Lemma 2.1, the attractor of the semilinear parabolic equation (3.1) is not contained in any smooth invariant finite-dimensional manifold $\mathscr{M} \subset X^\theta$. This completes the proof of the theorem. $\square$

The theorem remains valid under the replacement of the state space X^θ, $\theta \in (3/4, 1)$, by $X^{1/2} = H^1$, provided that the weakened version of the notion of differentiability of nonlinear maps given in [10, p. 813] (see also [12, Definition 1.1]) is used.

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