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EIGENSTATES OF QUANTUM PENNING–IOFFE NANOTRAP AT RESONANCE

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We discuss the choice of physical parameters of a quantum Penning nanotrap under the action of a perturbing inhomogeneous Ioffe magnetic field and also the role of resonance frequency operation modes. We present a general scheme for constructing the asymptotic behavior of the eigenstates by the generalized geometric quantization method and obtain the reproducing measure in the integral representation of eigenfunctions.

Keywords: nanotrap, resonance, quantum averaging, symmetry algebra, irreducible representation, reproducing measure

1. Introduction

The ideal Penning trap consists of a hyperbolic-shaped capacitor producing an axially symmetric electric saddle-type potential and a source of a homogeneous magnetic field directed along the saddle axis. The magnetic field must be sufficiently strong to compensate the decrease in the electric potential and to ensure the charge confinement in the transverse directions. Of course, the electric potential in actual traps contains anharmonic corrections and the magnetic field has an inhomogeneity and can deviate from the axial direction (see Sec. 2 below).

The Penning traps find more and more extensive applications as detectors [1]–[5], artificial "atoms" [6], and controlled quantum devices with wide perspectives, for example, in the design of quantum computers [7], [8]. The standard physical presentation of the theory of these traps can be found, for example, in [9]–[11], but there is no discussion of some principal points related to the quantum properties of these devices.

The quantum Penning trap is characterized by three spatial scales:

1. the magnetic length $l_0 = (f/B_0)^{1/2}$, where f is the magnetic flux quantum and B_0 is the magnetic field intensity at the trap center,
2. the characteristic size l of the electric field inhomogeneity (the capacitor size), and
3. the characteristic size L of the magnetic field B inhomogeneity in the trap, which is determined as $L^{-1} = |\nabla B|/B$ (at the trap center).

These physical dimensional parameters define two dimensionless parameters

$$\begin{aligned} &\hbar \overset{\text{def}}{=} (l_0/l)^2, && \text{a semiclassical parameter,} \\ &\varepsilon \overset{\text{def}}{=} (l/L)^{1/2}, && \text{the parameter of perturbation due to the magnetic inhomogeneity.} \end{aligned} \tag{1.1}$$

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We assume that the magnetic field deviation from the axial direction is of the order of ε (see the details below).

We can also introduce a parameter characterizing the anharmonicity of the electric field of the trap, but we do not do this here and assume that this anharmonicity is of the order of ε^3 in dimensionless variables for simplicity.

The maximal value of quantum gaps (distances between neighboring energy levels) in the ideal trap without inhomogeneity and anharmonicity taken into account is equal to

$$\delta \stackrel{\text{def}}{=} \frac{E\lambda^2}{l_0^2}, \tag{1.2}$$

where E is the rest energy of a particle (electron) and λ is its Compton wave length. The trap can be considered a quantum device if the value of δ can be observed sufficiently well in experiments, which implies the requirement that the magnetic scale l_0 must be sufficiently small, i.e., that the magnetic field B_0 must be sufficiently large. We note that this field can be created either by a macrodevice or by a nanostructure with a gigantic magnetic moment (for example, by doped nanofilms [12]). Precisely the inhomogeneity of the trap magnetic field characterized by the scale L is significant in the last case.

For example, we choose $L \sim 150\,\text{nm}$. Nanostructures of such sizes can produce a magnetic field $B_0 \sim 20\,\text{T}$, which corresponds to the magnetic scale $l_0 \sim 5\,\text{nm}$. Without the field inhomogeneity taken into account, the value of the maximum possible quantum gaps in the trap is then sufficiently large: $\delta \sim 3 \cdot 10^{-3}\,\text{eV}$, i.e., the trap can indeed function as a quantum device. Choosing the trap size $l \sim 30\,\text{nm}$, we obtain the values

$$\hbar \sim 3 \cdot 10^{-2}, \qquad \varepsilon \sim 5 \cdot 10^{-1} \tag{1.3}$$

for the small parameters introduced above. With the magnetic field inhomogeneity taken into account, the maximum possible energy gaps in such a trap are of the order of $\varepsilon^2\delta \sim 6 \cdot 10^{-4}\,\text{eV}$, which is quite distinguishable in experiments.

We note that the semiclassical parameter $\hbar$ in (1.1) and (1.3) is necessarily a small quantity, and the Weyl estimate of the asymptotic behavior of the number of eigenvalues (with the multiplicity taken into account) in this energy interval can hence be applied to the leading part of the trap Hamiltonian. This estimate gives an extremely small value of the order of $\delta \cdot \hbar^2 \sim 3 \cdot 10^{-8}\,\text{eV}$ for the spectral gaps of the Hamiltonian of an ideal trap in general position. The energy gaps can be increased to the maximum value because of the resonance between the normal frequencies of the trap. The resonance generates multiplicity of eigenvalues and hence increases the distance between them.

The Penning trap has three normal frequencies, determined by the intensities of the electric and magnetic fields (see the details in Sec. 2). The frequencies can be at resonance for some relations between these controlling fields. The principal resonance corresponds to the arithmetic proportion $2 : (-1) : 2$, the next higher resonance corresponds to the proportion $8 : (-1) : 4$, and so on.

The spectral gaps of an ideal trap take the maximum value δ (discussed above) precisely at the three-frequency resonance, and the system can operate as a quantum device. Moreover, if a perturbation of the order of ε^2 (this occurs due to the magnetic field inhomogeneity in our case) is added to such an ideal harmonic system, then the spectral gaps in general position are determined by a value of the order of $\varepsilon^2 \cdot \delta \cdot \hbar \sim 2 \cdot 10^{-5}\,\text{eV}$, which is too small to be observed.

The gaps can be increased by artificially adding a perturbation of the order of ε to the ideal trap (this can be achieved by the magnetic field deviating from the axial direction in our case) such that its eigenfrequency resonance (we call it the secondary resonance) arises in this additional "azimuthal" perturbing Hamiltonian. The initial physical perturbations of the order of ε^2 then generate gaps of the order of $\varepsilon^2\delta$ rather than $\varepsilon^2\delta\cdot\hbar$, as in the case without the secondary resonance. We can thus use the resonance effect, namely, the principal resonance between the normal frequencies and the secondary resonance between the frequencies of the "azimuthal" Hamiltonian, to obtain distinguishable quantum states and transitions between them in the Penning trap.

We further note that the value of the semiclassical parameter $\hbar$ and the values ε, $\varepsilon^2, \dots$ of perturbations added to the principal Hamiltonian of the ideal trap relate to each other as $\hbar \sim \varepsilon^4$. Moreover, the spectrum degeneration both in the ideal trap Hamiltonian and the "azimuthal" Hamiltonian is infinitely large, and the standard classical perturbation methods hence do not work in this problem, and we must therefore use the contemporary methods of algebraic averaging [13] and the generalized geometric quantization. The central role is then played by the structures of the primary and secondary symmetry algebras of the principal and "azimuthal" Hamiltonians. Because of the frequency resonance, these algebras are noncommutative and have nontrivial (infinite-dimensional) irreducible representations.

The noncommutativity implies a nontrivial symplectic geometry. The continuous geometric objects (such as symplectic leaves of a symmetry algebra) permit modeling and effectively changing the discrete infinite matrix structure. And the integration over a continuous measure permits avoiding the summation of infinite series of matrix elements as in the classical perturbation theory. We can thus use the newest methods of quantum geometry to bypass the "stumbling block" of perturbation theory.

We here encounter an interesting additional fact: the symmetry algebras in the resonance Penning trap are not Lie-type algebras. This means that the algebra cannot be described by linear commutation relations. Even in the simplest case of two-frequency resonance, the symmetry algebra of the Penning trap is described by nonlinear commutation relations [14], [15]. The case of a three-frequency resonance, which we consider in this paper, is much more informative than the case of a two-frequency resonance.

To deal with non-Lie algebras, we must construct an analogue of the entire structure of geometric quantization and the theory of representations for such algebras. Here, we use the methods developed in [15]–[17].

In addition to the algebraic structure, an important role is played by the averaged perturbing Hamiltonians. These Hamiltonians are obtained as a result of applying the general algebraic procedure [13], [15] using the projections on the (primary and secondary) symmetry algebras. These Hamiltonians can be considered systems with a reduced number of degrees of freedom.

As the first perturbing Hamiltonian in the order of ε, we consider the deviation of the homogeneous magnetic field from the axial direction. The Hamiltonian of this perturbation has a very simple form: it is a quadratic form in the canonical phase variables. At first sight, such a perturbation seems trivial, but we unexpectedly see that the projection of such a perturbing Hamiltonian on the (primary) symmetry algebra can again be at resonance with its own noncommutative symmetry algebra. Hence, there is a nonobvious secondary resonance in the "azimuthal" Hamiltonian and the secondary symmetry algebra. This algebra is again not a Lie-type algebra (we present its generators, commutation relations, and irreducible representations below).

We thus obtain a double resonance system in the Penning trap, i.e., the principal resonance oscillator and the perturbing quadratic Hamiltonian at resonance. The classical trajectories of this system are represented as combinations of two cyclic rotations: fast (due to the principal oscillator) and slower (due to the "azimuthal" perturbation). These trajectories are coiled on two-dimensional tori. This double resonance system has a noncommutative secondary symmetry algebra. Its irreducible representations control the spectral degeneration.

On the geometric level, we can say that symplectic leaves of a secondary symmetry algebra determine the reduced phase spaces of a double-resonance system. The effective Hamiltonian on these phase spaces is generated by the projection of the secondary perturbation on the secondary symmetry algebra. The number of degrees of freedom thus decreases, and we obtain one degree of freedom instead of three initial degrees.

As the secondary perturbation in our case, we take the linear inhomogeneity magnetic field, i.e., the so-called Ioffe correction. The linear inhomogeneity of the field generates a quadratic inhomogeneity of the magnetic potential and hence contributes cubically (in the phase coordinates) to the Hamiltonian. Such a cubic correction near the Penning trap center is indeed a small perturbation compared with the coordinate scaling.

The final effective Hamiltonian is obtained by projecting this secondary Ioffe perturbation on the secondary symmetry algebra. It turns out that this Hamiltonian in the irreducible representation is realized by an ordinary second-order differential operator. The nondegenerate spectrum and the eigenfunctions of this final operator are precisely the tools needed for calculating approximate spectral data of the initial Hamiltonian of the Penning trap.

Our algebraic method thus approximately reduces the initial three-dimensional Schrödinger differential operator with double resonance to a one-dimensional differential operator. This procedure can be treated in a sense as an analogue of the method of separation of variables. But instead of seeking such variables, we seek the irreducible representations of the primary and secondary non-Lie symmetry algebras. The calculations of the irreducible representations in the case of non-Lie algebras follow [16].

On the geometric level, this algebraic "separation of variables" presents the classical trajectories of the initial Hamiltonians as a combination of three cyclic rotations: fast, slower, and the slowest (with respect to the averaged Ioffe correction). Such a double resonance reduction using the secondary symmetry algebra in a physical hyperbolic-type system recalls another hyperbolic-type physical example (but elliptic) that arises in the Zeeman–Stark effect [18].

The approximate eigenstates of the initial quantum Penning trap (in the double resonance case) can be represented as follows: the eigenfunctions of the reduced ordinary differential operator together with "squeezed" coherent states for the secondary symmetry algebra are integrated over two-dimensional symplectic leaves with respect to a special reproducing measure. Calculating this measure is one of our main results in this paper.

In the semiclassical approach, the obtained integral can be transformed and written explicitly as an integral of squeezed coherent states over a periodic trajectory of the averaged secondary Ioffe Hamiltonian. Such an integral representation of semiclassical eigenfunctions, which permits avoiding the usual difficulties with focal points, follows the general approach [19] but specifically combines it with quantum geometry and the theory of representations for non-Lie symmetry algebras (a more detailed presentation can be found in [15], [20]).

2. Hamiltonian of the Penning trap

The Hamiltonian of the Penning trap with perturbations has the form

$$\widehat{H} = \widehat{H}_0 + \varepsilon \widehat{H}_1 + \varepsilon^2 \widehat{H}_2 + O(\varepsilon^3). \tag{2.1}$$

Here, $\widehat{H}_0$ is the Hamiltonian of the ideal trap

$$\widehat{H}_0 = \frac{1}{2}[\widehat{p}_1^2 + \widehat{p}_2^2 + \widehat{p}_3^2 + 2\omega(\widehat{p}_1 q_2 - \widehat{p}_2 q_1) + (\omega^2 - \omega_0^2)(q_1^2 + q_2^2) + 2\omega_0^2 q_3^2], \tag{2.2}$$

where $\widehat{p}_j = -i\hbar\,\partial/\partial q_j$ are momentum operators corresponding to the Cartesian coordinates q_j, $j = 1, 2, 3$, and ω and ω_0 are positive parameters satisfying the resonance condition

$$\omega^2 = \frac{9}{8}\omega_0^2. \tag{2.3}$$

The magnetic field of the ideal trap is homogeneous, directed along the third coordinate axis, and equal to 2ω.

The electric potential of the ideal trap has the form

$$U_0 = \frac{\omega_0^2}{2}(2q_3^2 - q_1^2 - q_2^2).$$

It satisfies the Laplace equation $\Delta U_0 = 0$ and represents the model electric field between two hyperbolic cups. The real electric capacitor in the trap does not usually have such a hyperbolic shape but has a cylindrical or cubic shape; its potential U satisfies the Laplace equation $\Delta U = 0$ and is approximated by the function U_0 in a neighborhood of the trap center $q = 0$:

$$U = U_0 + \text{fourth-order terms} + \dots .$$

After the coordinate scale is changed in a small region near the center, all fourth- and higher-order terms become insignificant and can be included in the remainder of the order $O(\varepsilon^3)$ in (2.1).

The term $\widehat{H}_1$ in (2.1) appears because of the homogeneous magnetic field deviation from its "ideal" direction along the third axis. The value of this deviation (i.e., of the parameter ε in (2.1)) is assumed to be small, and the deviation direction is determined by the vector $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$. Hence,

$$\widehat{H}_1 = \frac{1}{2}\widehat{k}[q \times \mathcal{B}],$$

where $k \stackrel{\text{def}}{=} (p_1 + \omega q_2, p_2 - \omega q_1, p_3)$ is the angular momentum.

The term $\widehat{H}_2$ in (2.1) contains an inhomogeneous Ioffe correction to the magnetic field. The value of this correction in (2.1) is of the order $O(\varepsilon^2)$, but we assume this only for simplicity; this value can be greater but must still be less than the correction of the order of ε in (2.1). The analytic expression for $\widehat{H}_2$ has the form

$$\widehat{H}_2 = \widehat{k}_1(\beta_1 q_2 q_3 + \gamma_1(q_2^2 - q_3^2)) + \widehat{k}_2(\beta_2 q_3 q_1 + \gamma_2(q_3^2 - q_1^2)) + \\ + \widehat{k}_3(\beta_3 q_1 q_2 + \gamma_3(q_1^2 - q_2^2)) + \frac{1}{8}|[q \times \mathcal{B}]|^2,$$

where β_j and γ_j are parameters of the Ioffe field.

We can perform a canonical transformation (preserving the commutation relations) of the phase coordinates $(q_1, q_2, q_3; p_1, p_2, p_3) \to (x_+, x_-, x_0; p_+, p_-, p_0)$ by the formulas

$$\begin{aligned} q_1 &= \frac{\sqrt[4]{2}}{\sqrt{\omega_0}}(x_+ + x_-), \qquad & p_1 &= \frac{\sqrt{\omega_0}}{2\sqrt[4]{2}}(p_+ + p_-), \\ q_2 &= \frac{\sqrt[4]{2}}{\sqrt{\omega_0}}(p_+ - p_-), \qquad & p_2 &= \frac{\sqrt{\omega_0}}{2\sqrt[4]{2}}(x_- - x_+), \\ q_3 &= \frac{1}{\sqrt[4]{2}\sqrt{\omega_0}}x_0, \qquad & p_3 &= \sqrt[4]{2}\sqrt{\omega_0}p_0. \end{aligned} \tag{2.4}$$

Hamiltonian (2.2) then takes the normal form

$$\widehat{H}_0 = \frac{\omega_0}{2\sqrt{2}}[2(\widehat{p}_+^2 + x_+^2) - (\widehat{p}_-^2 + x_-^2) + 2(\widehat{p}_0^2 + x_0^2)] = \\ = \frac{\omega_0}{\sqrt{2}}[2\widehat{z}_+^*\widehat{z}_+ - \widehat{z}_-^*\widehat{z}_- + 2\widehat{z}_0^*\widehat{z}_0] + \frac{3\hbar\omega_0}{2\sqrt{2}}, \tag{2.5}$$

where

$$\widehat{z}_\pm = \frac{1}{\sqrt{2}}(x_\pm + i\widehat{p}_\pm), \qquad \widehat{z}_0 = \frac{1}{\sqrt{2}}(x_0 + i\widehat{p}_0). \tag{2.6}$$

Remark 1. Such a change of coordinates corresponds to the transformation in $L^2(\mathbb{R}^3)$

$$\psi(q_1, q_2, q_3) \mapsto \Psi(x_+, x_-, x_0) = F[\psi],$$

$$F[\psi] = \frac{1}{2^{5/8}\sqrt[4]{\omega_0}\sqrt{\pi\hbar}} \int_{\mathbb{R}} \exp\left\{\frac{i\sqrt[4]{\omega_0}}{2^{5/4}\hbar}(x_+ - x_-)q_2\right\}\psi\left(\frac{\sqrt[4]{2}}{\sqrt{\omega_0}}(x_+ + x_-), q_2, \frac{x_0}{\sqrt[4]{2}\sqrt{\omega_0}}\right) dq_2.$$

The inverse transformation is

$$F^{-1}[\Psi] = \frac{\sqrt[4]{\omega_0}\sqrt{\hbar}}{2^{3/8}\sqrt{\pi}} \int_{\mathbb{R}} e^{ip_2 q_2}\Psi\left(\frac{\sqrt{\omega_0}}{2^{5/4}}q_1 - \frac{\sqrt[4]{2}}{\sqrt{\omega_0}}\hbar p_2,\ \frac{\sqrt{\omega_0}}{2^{5/4}}q_1 + \frac{\sqrt[4]{2}}{\sqrt{\omega_0}}\hbar p_2,\ \sqrt[4]{2}\sqrt{\omega_0}q_3\right) dp_2.$$

It follows from formula (2.5) that the Hamiltonian of the ideal Penning trap under condition (2.3) is a linear combination of three oscillators whose frequencies are at resonance $2 : (-1) : 2$. The spectrum of the Hamiltonian $\widehat{H}_0$ is discrete and infinitely degenerate.

We consider the symmetry algebra, i.e., the algebra of all operators commuting with $\widehat{H}_0$. The operators

$$\begin{aligned} &S_\pm = \widehat{z}_\pm^*\widehat{z}_\pm, \qquad S_0 = \widehat{z}_0^*\widehat{z}_0, \\ &A_\rho = \widehat{z}_+^*\widehat{z}_0, \qquad A_\sigma = \widehat{z}_+^*(\widehat{z}_-^*)^2, \qquad A_\theta = (\widehat{z}_-^*)^2\widehat{z}_0^* \end{aligned} \tag{2.7}$$

can be regarded as its generators. Of course, the adjoint operators A_ρ^*, A_σ^*, and A_θ^* must be included in the set of generators of the symmetry algebra.

The commutation relations between the generators are [21]

$$\begin{aligned} &[S_+, A_\rho] = \hbar A_\rho, \qquad [S_0, A_\rho] = -\hbar A_\rho, \\ &[S_+, A_\sigma] = \hbar A_\sigma, \qquad [S_-, A_\sigma] = 2\hbar A_\sigma, \\ &[S_-, A_\theta] = 2\hbar A_\theta, \qquad [S_0, A_\theta] = \hbar A_\theta, \\ &[A_\rho, A_\sigma^*] = -\hbar A_\theta^*, \qquad [A_\rho, A_\theta] = \hbar A_\sigma, \\ &[A_\sigma, A_\theta^*] = -4\hbar\left(S_- + \frac{\hbar}{2}\right)A_\rho, \qquad [A_\rho^*, A_\rho] = \hbar(S_0 - S_+), \\ &[A_\sigma^*, A_\sigma] = \hbar(4S_+S_- + S_-^2 + 2\hbar S_+ + 3\hbar S_- + 2\hbar^2), \\ &[A_\theta^*, A_\theta] = \hbar(S_-^2 + 4S_-S_0 + 3\hbar S_- + 2\hbar S_0 + 2\hbar^2). \end{aligned} \tag{2.8}$$

The other commutators are either adjoint to those listed above or are equal to zero.

Relations (2.8) admit the three Casimir operators (commuting with all generators)

$$\begin{aligned} &C_\rho = A_\rho A_\rho^* - S_+(S_0 + \hbar), \\ &C_\sigma = A_\sigma A_\sigma^* - S_+S_-(S_- - \hbar), \\ &C_\theta = A_\theta A_\theta^* - S_0S_-(S_- - \hbar). \end{aligned} \tag{2.9}$$

These three operators are simply zeros in realization (2.7).

There also are Casimir quasioperators

$$\begin{aligned} C_+ &= A_\rho A_\sigma^* - S_+ A_\theta^*, \\ C_- &= A_\sigma A_\theta^* - S_-(S_- - \hbar)A_\rho, \\ C_0 &= A_\rho A_\theta - (S_0 + \hbar)A_\sigma. \end{aligned} \tag{2.10}$$

The commutators of these operators with all generators are proportional to operators (2.10). Operators (2.10) also vanish in realization (2.7).

The symmetry algebra of resonance oscillator (2.5) can thus be determined as an algebra with nine generators and relations (2.8) factored by the ideal generated by elements (2.9) and (2.10).

Of course, this quotient algebra still has one additional Casimir element

$$C = 2S_+ - S_- + 2S_0, \tag{2.11}$$

which in fact is the operator $\widehat{H}_0$ given by (2.5), namely,

$$\widehat{H}_0 = \frac{\omega_0}{\sqrt{2}}\left(C + \frac{3\hbar}{2}\right).$$

We analyze perturbed operator (2.1) following the general scheme of algebraic averaging [13], [15]. Namely, we perform a unitary transformation that annihilates the part of the perturbation in (2.1) that does not commute with the leading term $\widehat{H}_0$. The new perturbation commutes with $\widehat{H}_0$ and hence belongs to the symmetry algebra described above:

$$V^{-1} \cdot \widehat{H} \cdot V = \widehat{H}_0 + \varepsilon \widehat{H}_{10} + \varepsilon^2 \widehat{H}_{20} + O(\varepsilon^3), \tag{2.12}$$

where

$$[\widehat{H}_0, \widehat{H}_{10}] = [\widehat{H}_0, \widehat{H}_{20}] = 0 \tag{2.13}$$

(see [21] for the details).

The operators $\widehat{H}_{10}$ and $\widehat{H}_{20}$ can be expressed in terms of generators of symmetry algebra (2.8) as

$$\widehat{H}_{10} = \mathcal{B}_3\left(2S_+ + S_- + \frac{3\hbar}{2}\right) - \frac{\mathcal{B}_1 + i\mathcal{B}_2}{\sqrt{2}} A_\rho - \frac{\mathcal{B}_1 - i\mathcal{B}_2}{\sqrt{2}} A_\rho^*, \tag{2.14}$$

$$\begin{aligned} \widehat{H}_{20} = {}& f_+ S_+ + f_- S_- + f_0 S_0 + (g_\rho A_\rho + \bar{g}_\rho A_\rho^*) + \\ & + (g_\sigma A_\sigma + \bar{g}_\sigma A_\sigma^*) + (g_\theta A_\theta + \bar{g}_\theta A_\theta^*) + r. \end{aligned} \tag{2.15}$$

The scalar coefficients in the last formula have the forms

$$\begin{aligned} & f_+ = \xi^2(1 - 9\eta^2)\frac{\sqrt{2}}{8\omega_0}, \qquad f_- = \xi^2\left(1 - \frac{7 + 20\eta^2}{3}\right)\frac{\sqrt{2}}{8\omega_0}, \\ & f_0 = -\xi^2(1 - \eta^2)\frac{5\sqrt{2}}{24\omega_0}, \qquad g_\rho = \xi^2\eta\sqrt{1 - \eta^2}\,\frac{e^{i\varphi}}{\sqrt{2}\omega_0}, \\ & g_\sigma = \frac{2^{3/4}}{\sqrt{\omega_0}}(\gamma_2 - i\gamma_1), \qquad g_\theta = \frac{1}{2^{7/4}\sqrt{\omega_0}}(2\beta_3 - \beta_1 - \beta_2 + 4i\gamma_3), \\ & r = -\hbar\xi^2(1 + 7\eta^2)\frac{\sqrt{2}}{8\omega_0}, \end{aligned} \tag{2.16}$$

where we use the notation

$$\xi^2 = \mathcal{B}_1^2 + \mathcal{B}_2^2 + 2\mathcal{B}_3^2, \qquad \eta = \frac{\sqrt{2}\mathcal{B}_3}{\xi}, \tag{2.17}$$

and the angle φ is determined by the relation

$$\mathcal{B}_1 + i\mathcal{B}_2 = \xi\sqrt{1-\eta^2}\, e^{i\varphi}. \tag{2.18}$$

3. Secondary resonance algebra

The perturbing terms $\widehat{H}_{10}, \widehat{H}_{20}, \dots$ in (2.12) commute with the leading term $\widehat{H}_0$. It therefore suffices to consider the Hamiltonian

$$\widehat{H}_{10} + \varepsilon\widehat{H}_{20} + O(\varepsilon^2) \tag{3.1}$$

on eigenspaces of the resonance oscillator $\widehat{H}_0$. We can determine these eigenspaces by calculating the irreducible representations of algebra (2.8).

We note that the generators A_ρ and A_ρ^* in (2.14) commute with S_-, and hence $[\widehat{H}_{10}, S_-] = 0$. Hamiltonian (3.1) is hence a perturbation of the integrable system defined by the Hamiltonian $\widehat{H}_{10}$.

The symmetry algebra of the operator $\widehat{H}_{10}$ given by (2.14) is trivial in general position (commutative and generated by the operator S_- and by $\widehat{H}_{10}$ itself), and its spectrum is nondegenerate. But under a special resonance condition on the components of the perturbing magnetic field $\mathcal{B}$, this algebra becomes noncommutative, and the spectral degeneration appears. The condition for such a secondary resonance has the form [21]

$$(k-l)^2(\mathcal{B}_1^2 + \mathcal{B}_2^2) = 16\left((k+l)^2 + \frac{kl}{2}\right)\mathcal{B}_3^2, \tag{3.2}$$

where k and l are some positive coprime numbers.

For simplicity, we consider only the case of the simplest secondary resonance $k : l = 1 : 0$ in (3.2), i.e., we assume that

$$\mathcal{B}_1^2 + \mathcal{B}_2^2 = 16\mathcal{B}_3^2. \tag{3.3}$$

This condition means that the angle between the trap axis and the direction of the perturbing magnetic field $\mathcal{B}$ is approximately equal to 76°.

Under condition (3.2), the spectrum of Hamiltonian (2.1) or (2.12) has the form

$$\frac{\omega_0}{\sqrt{2}}\left(n + \frac{3}{2}\right)\hbar + \varepsilon\left(6m - n + \frac{3}{2}\right)\hbar\mathcal{B}_3 + O(\varepsilon^2), \tag{3.4}$$

where $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_+$. We describe the corresponding eigensubspace of the Hamiltonian $\widehat{H}_0 + \varepsilon\widehat{H}_{10}$ in $L^2(\mathbb{R}^3)$ in Appendix A. The correction of the order $O(\varepsilon^2)$ in this spectrum is determined by the secondary perturbation $\widehat{H}_{20}$ given by (2.15). To take this correction into account, we must perform the averaging as in Sec. 2. We apply a unitary transformation to obtain the new Hamiltonian

$$\widehat{H}_{10} + \varepsilon\widehat{H}_{200} + O(\varepsilon^2) \tag{3.5}$$

with a new "azimuthal" perturbation that commutes with the leading part: $[\widehat{H}_{10}, \widehat{H}_{200}] = 0$.

If condition (3.3) is satisfied, then the symmetry algebra of the operator $\widehat{H}_{10}$ given by (2.14) is noncommutative. The generators of this secondary resonance algebra are

$$A_0 = S_-, \tag{3.6}$$

$$A_\pm \stackrel{\text{def}}{=} (1\pm\eta)S_+ + (1\mp\eta)S_0 \mp \sqrt{1-\eta^2}(e^{i\varphi}A_\rho + e^{-i\varphi}A_\rho^*), \tag{3.7}$$

$$B = \sqrt{\frac{2}{3}}e^{i\varphi/2}A_\sigma + \frac{2}{\sqrt{3}}e^{-i\varphi/2}A_\theta, \tag{3.8}$$

where $\eta = 1/3$ and the angle φ is given by (2.18).

The commutation relations between generators (3.6), (3.7), and (3.8) have the forms [21]

$$
\begin{aligned}
&[A_0, B] = 2\hbar B, \qquad [A_-, B] = 2\hbar B, \qquad [A_+, B] = 0,\\
&[B^*, B] = 2\hbar(A_0^2 + 2A_0A_- + 3\hbar A_0 + \hbar A_- + 2\hbar^2).
\end{aligned}
\tag{3.9}
$$

There are the Casimir elements

$$
M = A_+ - A_- + A_0, \qquad C = A_+ + A_- - A_0, \qquad K = BB^* - A_0(A_0 - \hbar)A_-. \tag{3.10}
$$

Theorem 1. *Under secondary resonance conditions* (3.3), *the "azimuthal" Hamiltonian* $\widehat{H}_{200}$ *in* (3.5) *is expressed in terms of generators of secondary symmetry algebra* (3.9) *as*

$$
\widehat{H}_{200} = \varkappa B + \bar{\varkappa} B^* + \mu_0 A_0 + \mu_+ A_+ + \mu_- A_- + \nu, \tag{3.11}
$$

where the scalar coefficients are

$$
\begin{aligned}
&\varkappa = \frac{1}{2^{7/4}\sqrt{3\omega_0}}[(2\beta_3 - \beta_1 - \beta_2 + 4i\gamma_3)e^{i\varphi/2} + 4(\gamma_2 - i\gamma_1)e^{-i\varphi/2}],\\
&\mu_0 = -\frac{14\sqrt{2}}{3\omega_0}\mathcal{B}_3^2, \qquad \mu_+ = -\frac{17\sqrt{2}}{9\omega_0}\mathcal{B}_3^2, \qquad \mu_- = \frac{2\sqrt{2}}{9\omega_0}\mathcal{B}_3^2, \qquad \nu = -\hbar\frac{97\sqrt{2}}{16\omega_0}\mathcal{B}_3^2
\end{aligned}
\tag{3.12}
$$

and the angle φ *is determined by the relations* $\cos\varphi = \mathcal{B}_1/4\mathcal{B}_3$ *and* $\sin\varphi = \mathcal{B}_2/4\mathcal{B}_3$.

The process of calculating the averaged Hamiltonian $\widehat{H}_{200}$ is described in detail in Appendix B.

Corollary 1. *Under principal and secondary resonance conditions* (2.3) *and* (3.3), *the spectrum of Hamiltonian* (2.1) *of the perturbed Penning trap has the asymptotic expansion*

$$
\frac{\omega_0}{\sqrt{2}}\left(n + \frac{3}{2}\right)\hbar + \varepsilon\left(6m - n + \frac{3}{2}\right)\hbar\mathcal{B}_3 + \varepsilon^2\lambda_{n,m,k} + O(\varepsilon^3), \tag{3.13}
$$

where $n \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, *and* $\lambda_{n,m,k}$ *are eigenvalues of operator* (3.11) *over algebra* (3.9) *in its irreducible representation with Casimir elements* (3.10) *equal to*

$$
K = 0, \qquad C = n\hbar, \qquad M = (4m - n)\hbar. \tag{3.14}
$$

4. Irreducible representations of the secondary resonance algebra

We recall that we consider the resonance case $k : l = 1 : 0$. We now describe the irreducible representations of algebra (3.9).

In the space of antiholomorphic functions over the field $\mathbb{C}$, we define the inner product

$$
(u, v) = \sum_{j \geq 0} s_{n,m}(j) u_j \bar{v}_j. \tag{4.1}
$$

Here, $u(\bar{z}) = \sum_{j\geq 0} u_j \bar{z}^j$, $v(\bar{z}) = \sum_{j\geq 0} v_j \bar{z}^j$, and the numbers $s_{n,m}(j)$ for fixed n and m and each $j = 0, 1, 2, \ldots$ are given by the formulas

$$
s_{n,m}(j) = (2\hbar)^j j!\,\frac{\big(m - n/2 + 3/4 + (-1)^n/4\big)_j}{\big(m - n/2 + 3/4 - (-1)^n/4\big)_j}, \quad n \leq 2m,
$$

and

$$s_{n,m}(j) = (2\hbar)^j j! \frac{\big([(n+1)/2] - m + 1\big)_j}{\big(1 - (-1)^n/2\big)_j}, \qquad n > 2m.$$

The reproducing kernel in the space with inner product (4.1) has the form

$$\mathcal{K}_{n,m}(\overline{w}, z) \stackrel{\text{def}}{=} \sum_{j \geq 0} \frac{(\overline{w}z)^j}{s_{n,m}(j)}. \tag{4.2}$$

We here use the term "reproducing" because of the property that if we take inner product (4.1) of two functions $\mathcal{K}_{n,m}(\,\cdot\,, z)$ and $\mathcal{K}_{n,m}(\,\cdot\,, w)$, then the result is equal to $\mathcal{K}_{n,m}(\overline{w}, z)$.

The reproducing kernel $\mathcal{K}_{n,m}$ given by (4.2) can be expressed in terms of known special functions. For this, we introduce numbers a and c as

$$\begin{aligned} a &\stackrel{\text{def}}{=} m - \frac{n}{2} + \frac{3}{4} - \frac{(-1)^n}{4}, \qquad & c &\stackrel{\text{def}}{=} m - \frac{n}{2} + \frac{3}{4} + \frac{(-1)^n}{4}, \quad & n \leq 2m, \\ a &\stackrel{\text{def}}{=} 1 - \frac{(-1)^n}{2}, & c &\stackrel{\text{def}}{=} \left[\frac{n+1}{2}\right] - m + 1 & n > 2m. \end{aligned} \tag{4.3}$$

Then the reproducing kernel $\mathcal{K}_{n,m}$ is given by the hypergeometric series (Vol. 1 of [22])

$$\mathcal{K}_{n,m}(\overline{w}, z) = {}_1F_1\left(a; c; \frac{z\overline{w}}{2\hbar}\right).$$

We now can define the family of coherent states $\mathfrak{p}_{n,m}(z)$ for our algebra such that

$$(\mathfrak{p}_{n,m}(z), \mathfrak{p}_{n,m}(w))_{L^2} = \mathcal{K}_{n,m}(\overline{w}, z). \tag{4.4}$$

The inner product in the left-hand side of (4.4) is taken in the space $L^2 = L^2(\mathbb{R}^3)$, i.e., in the original Hilbert space for Schrödinger operator (2.1).

The family of coherent states satisfying (4.4) can be defined by the formula

$$\mathfrak{p}_{n,m}(z) = F_{n,m}(z, B)\mathfrak{p}_{n,m}(0), \tag{4.5}$$

where

$$F_{n,m}(z, \overline{w}) = \sum_{j=0}^{\infty} \frac{\Gamma(c)}{j!\Gamma(j+c)} \left(\frac{z\overline{w}}{2\hbar^2}\right)^j = \Gamma(c) \left(\frac{2\hbar}{\sqrt{z\overline{w}}}\right)^{c-1} I_{c-1}\left(\frac{\sqrt{z\overline{w}}}{\hbar}\right)$$

and I_ν is the Bessel function of an imaginary argument (see Vol. 2 of [22]), Γ is the gamma function, and the number c is defined in (4.3).

The vacuum vector $\mathfrak{p}_{n,m}(0)$ in (4.5) is an eigenvector of all commuting generators A_0, A_+, and A_- of algebra (3.9) annihilated by the generator B^*, i.e.,

$$\begin{aligned} &B^*\mathfrak{p}_{n,m}(0) = 0, \qquad A_0\mathfrak{p}_{n,m}(0) = \hbar(2(m + m_-) - n)\mathfrak{p}_{n,m}(0), \\ &A_+\mathfrak{p}_{n,m}(0) = 2\hbar m\mathfrak{p}_{n,m}(0), \qquad A_-\mathfrak{p}_{n,m}(0) = 2\hbar m_-\mathfrak{p}_{n,m}(0). \end{aligned} \tag{4.6}$$

The normalized solution of the system of Eqs. (4.6) has the form

$$\mathfrak{p}_{n,m}(0) = \chi_{2(m+m_-)-n,m,m_-}.$$

Here, the vectors χ_{t_0,t_+,t_-} are determined by formula (A.1) (see Appendix A), and the number m_- is equal to

$$m_- = \left[\frac{1+\max\{n-2m,0\}}{2}\right], \tag{4.7}$$

where the symbol $[\,\cdot\,]$ denotes the integer part.

Coherent states (4.5) can be expanded in the orthonormal basis $\chi_{2(m+m_-)-n+2j,m,m_-+j}$ in the common eigensubspace (in $L^2(\mathbb{R}^3)$) of the operators C and M:

$$\mathfrak{p}_{n,m}(z) = \sum_{j\geq 0} \bar{h}_{n,m,j}(\bar{z})\,\chi_{2(m+m_-)-n+2j,m,m_-+j},$$

where

$$h_{n,m,j}(\bar{z}) = \frac{\bar{z}^j}{\sqrt{s_{n,m}(j)}}$$

is an orthonormal basis in the space of antiholomorphic functions with inner product (4.1).

The coherent states permit intertwining the initial representation of algebra (3.9) with its irreducible representations. The intertwining map has the form

$$g \to I_{n,m}(g) \stackrel{\text{def}}{=} \frac{1}{2\pi\hbar}\int_{\mathbb{C}} g(\bar{z})\mathfrak{p}_{n,m}(z)l(|z|^2)\,d\bar{z}\,dz. \tag{4.8}$$

Here, l is the density of the reproducing measure with respect to which the reproducing property

$$\int_{\mathbb{C}} \mathcal{K}_{n,m}(\overline{w},z)\mathcal{K}_{n,m}(\bar{z},w)l(|z|^2)\,d\bar{z}\,dz = \mathcal{K}(\overline{w},w). \tag{4.9}$$

is satisfied. Inner product (4.1) can be expressed in terms of the measure as

$$(u,v) = \int_{\mathbb{C}} u(\bar{z})\bar{v}(\bar{z})l(|z|^2)\,d\bar{z}\,dz. \tag{4.10}$$

Map (4.8) has the intertwining property

$$\begin{aligned} A_{0,\pm}I_{n,m}(g) &= I_{n,m}(\mathbb{A}_{0,\pm}(g)),\\ BI_{n,m}(g) &= I_{n,m}(\mathbb{B}(g)),\\ B^*I_{n,m}(g) &= I_{n,m}(\mathbb{B}^*(g)), \end{aligned} \tag{4.11}$$

where A_0, A_+, A_-, B, and B^* are initial generators of the algebra and $\mathbb{A}_0$, $\mathbb{A}_+$, $\mathbb{A}_-$, $\mathbb{B}$, and $\mathbb{B}^*$ are generators of its irreducible representation.

The operators $\mathbb{B}$ and $\mathbb{B}^*$ in the irreducible representation are given by the second-order differential operators [21] as

$$\begin{aligned} &\left.\begin{aligned} \mathbb{B} &= 2\hbar\bar{z}\left(\bar{z}\frac{d}{d\bar{z}} + m - \frac{n}{2} + \frac{3}{4} - \frac{(-1)^n}{4}\right),\\ \mathbb{B}^* &= 4\hbar^2\left(\bar{z}\frac{d}{d\bar{z}} + m - \frac{n}{2} + \frac{3}{4} + \frac{(-1)^n}{4}\right)\frac{d}{d\bar{z}}, \end{aligned}\right\} \quad n \leq 2m,\\ &\left.\begin{aligned} \mathbb{B} &= 2\hbar\bar{z}\left(\bar{z}\frac{d}{d\bar{z}} + 1 - \frac{(-1)^n}{2}\right),\\ \mathbb{B}^* &= 4\hbar^2\left(\bar{z}\frac{d}{d\bar{z}} - m + \left[\frac{n+1}{2}\right] + 1\right)\frac{d}{d\bar{z}}, \end{aligned}\right\} \quad n > 2m. \end{aligned} \tag{4.12}$$

The operators $\mathbb{A}_\pm$ and $\mathbb{A}_0$ are given by the first-order differential operators as

$$\left.\begin{aligned} &\mathbb{A}_0 = 2\hbar\Big(m - \frac{n}{2} + \bar{z}\frac{d}{d\bar{z}}\Big), \\ &\mathbb{A}_+ = 2\hbar m, \qquad \mathbb{A}_- = 2\hbar\bar{z}\frac{d}{d\bar{z}}, \end{aligned}\right\} \quad n \le 2m,$$
$$\left.\begin{aligned} &\mathbb{A}_0 = 2\hbar\Big(\frac{1-(-1)^n}{4} + \bar{z}\frac{d}{d\bar{z}}\Big), \\ &\mathbb{A}_+ = 2\hbar m, \qquad \mathbb{A}_- = 2\hbar\Big(\Big[\frac{n+1}{2}\Big] - m + \bar{z}\frac{d}{d\bar{z}}\Big), \end{aligned}\right\} \quad n > 2m. \tag{4.13}$$

Theorem 2. *The density of the reproducing measure l is given by the formula*

$$l(|z|^2) = \frac{\Gamma(a)}{2\Gamma(c)} e^{-|z|^2/2\hbar} \Big(\frac{|z|^2}{2\hbar}\Big)^{c-1} \Psi\Big(a-1; c; \frac{|z|^2}{2\hbar}\Big).$$

Here, Ψ is the Tricomi function (see Vol. 1 of [22])

$$\begin{aligned} \Psi(\alpha, N+1; x) = \frac{(-1)^{N-1}}{N!\Gamma(\alpha - N)} \Big\{ &{}_1F_1(\alpha, N+1; x)\log x + \\ &+ \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(N+1)_r} [\psi(\alpha + r) - \psi(1+r) - \psi(1+N+r)] \frac{x^r}{r!} \Big\} + \\ &+ \frac{(N-1)!}{\Gamma(\alpha)} \sum_{r=0}^{N-1} \frac{(\alpha - N)_r}{(1-N)_r} \frac{x^{r-N}}{r!} \end{aligned}$$

(the last sum is omitted if $N = 0$), where $\psi(z) = d\log\Gamma(z)/dz$ is the logarithmic derivative of the Γ function.

Remark 2. The function $y(x) \stackrel{\text{def}}{=} l(2\hbar x)$ is a solution of the degenerate hypergeometric equation

$$xy''(x) + (x + 2 - c)y'(x) + (2 - a)y(x) = 0,$$

where a and c are defined in (4.3). We note that $c \in \mathbb{N}$. This equation has two linearly independent solutions (see Vol. 1 of [22])

$$y_1(x) = e^{-x}x^{c-1}{}_1F_1(a-1, c; x), \qquad y_2(x) = e^{-x}x^{c-1}\Psi(a-1, c; x).$$

These solutions have the asymptotic behavior as $x \to 0$

$$\begin{aligned} &y_1(x) \sim x^{c-1}, \\ &y_2(x) \sim \begin{cases} \dfrac{\Gamma(c-1)}{\Gamma(a-1)}, & c > 1, \\ -\dfrac{1}{\Gamma(a-1)}[\log x + \psi(a-1) - 2\gamma], & c = 1. \end{cases} \end{aligned}$$

Here, $\gamma = \lim_{k\to\infty}\big(\sum_{n=1}^{k} 1/n - \log k\big) = 0.5772156649\ldots$ is the Euler–Mascheroni constant.

The asymptotic behavior of y_1 and y_2 as $x \to +\infty$ has the forms

$$y_1(x) \sim \frac{\Gamma(c)}{\Gamma(a-1)} x^{a-2}, \qquad y_2(x) \sim e^{-x} x^{c-a}.$$

The density of the reproducing measure must satisfy the condition that the integral

$$\int_0^\infty x^k l(2\hbar x)\, dx$$

converges for all $k \in \mathbb{Z}_+$. Therefore, $l(2\hbar x) = \mathrm{const} \cdot y_2(x)$, where const is determined by the normalization condition

$$\int_0^\infty l(2\hbar x)\, dx = \frac{1}{2}.$$

We can easily calculate the integral in the left-hand side of this normalization condition by integrating the equation for l.

5. Integral formula for the eigenstates

We apply (4.8) and (4.11) to the operator $\widehat{H}_{200}$ given by (3.11) and obtain

$$\widehat{H}_{200} I_{n,m}(g) = I_{n,m}(\mathbb{H}(g)),$$

where

$$\mathbb{H} \stackrel{\mathrm{def}}{=} \varkappa\mathbb{B} + \bar{\varkappa}\mathbb{B}^* + \mu_0\mathbb{A}_0 + \mu_+\mathbb{A}_+ + \mu_-\mathbb{A}_- + \nu. \tag{5.1}$$

Hence, to calculate the spectrum and the eigenvectors of the operator $\widehat{H}_{200}$ on an eigenspace of the operators $\widehat{H}_0$ and $\widehat{H}_{10}$, we must consider the eigenvalue problem in the space of antiholomorphic functions with inner product (4.10) for the operator $\mathbb{H}$ given by (5.1):

$$\mathbb{H}g = \lambda g. \tag{5.2}$$

It follows from (4.12) or (4.13) that Eq. (5.2) is an ordinary second-order differential equation with coefficients linear and quadratic in $\bar{z}$.

We let $\lambda = \lambda_{n,m,k}$ and $g = g_{n,m,k}$, where $k = 0, 1, 2, \dots$ is a new quantum number enumerating the eigenvalues, denote the eigenvalues and eigenfunctions of operator (5.2). Then the vectors $I_{n,m}(g_{n,m,k})$ are eigenvectors of $\widehat{H}_{200}$.

We can now reconstruct the eigenstates of the initial Hamiltonian $\widehat{H}$ by applying the unitary operator V in (2.12) and another unitary operator V_1 transforming (3.1) into (3.5). A method for calculating V was described in [21], and the formula for V_1 is given in Appendix B.

Theorem 3. *If the frequencies ω and ω_0 satisfy resonance condition* (2.3) *and the components of the perturbing homogeneous magnetic field $\mathcal{B}$ satisfy secondary resonance condition* (3.3), *then the eigenstates of the Penning trap Hamiltonian $\widehat{H}$ given by* (2.1) *are determined up to $O(\varepsilon)$ by the integral formula*

$$\psi_{n,m,k} = \frac{1}{2\pi\hbar} \int_{\mathbb{C}} g_{n,m,k}(\bar{z}) V V_1 \mathfrak{p}_{n,m}(z) l(|z|^2)\, d\bar{z}\, dz, \tag{5.3}$$

where $g_{n,m,k}$ are eigenfunctions of the ordinary second-order differential operator $\mathbb{H}$ given by (5.1) *with the generators $\mathbb{A}_{0,\pm}$, $\mathbb{B}$, and $\mathbb{B}^*$ given by formulas* (4.13) *or* (4.12).

The corresponding eigenvalues of $\widehat{H}$ have the form

$$\frac{\omega_0}{\sqrt{2}}\left(n + \frac{3}{2}\right)\hbar + \varepsilon\left(6m - n + \frac{3}{2}\right)\hbar\mathcal{B}_3 + \varepsilon^2\lambda_{n,m,k} + O(\varepsilon^3),$$

where $n \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, and $\lambda_{n,m,k}$ are eigenvalues of $\mathbb{H}$.

Remark 3. The family of vectors $VV_1\mathfrak{p}_{n,m}(z)$ in (5.3) can be considered some "squeezed" coherent states of the secondary resonance algebra of the Penning trap. The squeezing is performed by unitary transformations of the operators V and V_1 reducing initial Hamiltonian (2.1) to form (3.11). The coherent ground states $\mathfrak{p}_{n,m}(z)$ in (5.3) correspond to irreducible representations of the non-Lie secondary resonance algebras (3.9).

We note that some coherent states of the Penning trap were studied in [23], [24].

Remark 4. We must still calculate the solutions $g_{n,m,k}$ of ordinary differential equation (5.2) in integral (5.3). But in the semiclassical approximation, this can be done explicitly, and formula (5.3) reduces to an integral of the form

$$\psi_{n,m,k} \simeq \frac{1}{\sqrt{2\pi\hbar}} \int_{\Lambda_{n,m,k}} e^{(i/\hbar)\{\text{Action}\}} \sqrt{\{\text{Jacobian}\}} \cdot VV_1\mathfrak{p}_{n,m} \cdot \{\text{Measure}\} + O(\hbar),$$

where $\Lambda_{n,m,k}$ are level lines of the Wick symbol of the operator $\mathbb{H}$ satisfying the quantization condition, and $\{\text{Action}\}$, $\{\text{Jacobian}\}$, and $\{\text{Measure}\}$ are introduced on $\Lambda_{n,m,k}$ by a canonical procedure [19], [20]. The eigenvalues $\lambda_{n,m,k}$ are determined up to $O(\hbar^2)$ by the values of the Wick symbol on the level lines $\Lambda_{n,m,k}$.

Appendix A

We introduce the commuting operators

$$b_0 \stackrel{\text{def}}{=} \hat{z}_-^*, \qquad b_\pm \stackrel{\text{def}}{=} \sqrt{1 \pm \eta} e^{i\varphi/2} \hat{z}_+^* \mp \sqrt{1 \mp \eta} e^{-i\varphi/2} \hat{z}_3^*,$$

where $\eta = 1/3$. The operators b_0 and $b_\pm$ and the operators adjoint to them in the space $L^2(\mathbb{R}^3)$ satisfy the commutation relations

$$[b_0^*, b_0] = \hbar, \qquad [b_\pm^*, b_\pm] = 2\hbar, \qquad [b_\pm^*, b_\mp] = 0.$$

The operators b_0^* and $b_\pm^*$ annihilate the exponential

$$\chi_0(x_+, x_-, x_0) \stackrel{\text{def}}{=} \frac{1}{(\pi\hbar)^{3/4}} e^{(x_+^2 + x_-^2 + x_0^2)/2\hbar}, \qquad \|\chi_0\| = 1.$$

We consider the vectors

$$\chi_t(x_+, x_-, x_0) \stackrel{\text{def}}{=} \gamma_t b^t \chi_0, \tag{A.1}$$

where

$$\gamma_t = \frac{1}{\sqrt{2^{t_+ + t_-} \hbar^{|t|} t!}} \tag{A.2}$$

and $t = (t_0, t_+, t_-)$ such that $t_j \in \mathbb{Z}_+$, $|t| = t_0 + t_+ + t_-$, and $t! = t_0! t_+! t_-!$.

Proposition 1. *Vectors* (A.1) *satisfy the formula*

$$\chi_t = \frac{1}{\sqrt{2^{t_0} \hbar^{t_+ + t_-} t!}} H_{t_0}\left(\frac{x_-}{\sqrt{\hbar}}\right) \mathcal{H}^{\nu,\varphi}_{t_+,t_-}(x_+, x_0) \chi_0.$$

Here, H_{t_0} are Hermite polynomials, and $\mathcal{H}^{\nu,\varphi}_{t_+,t_-}$ denote the polynomials

$$\mathcal{H}^{\nu,\varphi}_{t_+,t_-}(x_+, x_0) \stackrel{\text{def}}{=} d_+^{t_+} d_-^{t_-}(1),$$

where

$$d_\pm \stackrel{\text{def}}{=} \sqrt{1 \pm \eta} e^{i\varphi/2} \left(x_+ - \frac{\hbar}{2} \frac{\partial}{\partial x_+}\right) \mp \sqrt{1 \mp \eta} e^{-i\varphi/2} \left(x_0 - \frac{\hbar}{2} \frac{\partial}{\partial x_0}\right).$$

Corollary 2. *In the initial coordinates, basis* (A.1) *is given by the formulas*

$$F^{-1}[\chi_t] = \sqrt{\frac{2^{t_0}}{t!}}\, P^{\nu,\varphi}_{t_0,t_+,t_-}\biggl(\frac{\sqrt{\omega_0}(q_1+iq_2)}{2^{5/4}\sqrt{\hbar}}, \frac{\sqrt{\omega_0}(q_1-iq_2)}{2^{5/4}\sqrt{\hbar}}, \frac{\sqrt[4]{2}\sqrt{\omega_0}q_3}{\sqrt{\hbar}}\biggr)F^{-1}[\chi_0],$$

$$F^{-1}[\chi_0] = \frac{\omega_0^{3/4}}{2^{5/8}(\pi\hbar)^{3/4}}\exp\biggl\{-\frac{\omega_0}{\sqrt{2}\hbar}\biggl(\frac{q_1^2+q_2^2}{4}+q_3^2\biggr)\biggr\},$$

where $P^{\nu,\varphi}_{t_0,t_+,t_-}$ are polynomials in three variables. They are defined by the formula

$$P^{\nu,\varphi}_{t_0,t_+,t_-} \stackrel{\text{def}}{=} D_+^{t_+}D_-^{t_-}(y^{t_0}),$$

where $D_\pm$ are the mutually commuting differential operators

$$D_\pm = \sqrt{1\pm\eta}\,e^{i\varphi/2}\biggl(\bar{y}-\frac{1}{2}\frac{\partial}{\partial y}\biggr) \mp \sqrt{1\mp\eta}\,e^{-i\varphi/2}\biggl(y_0-\frac{1}{2}\frac{\partial}{\partial y_0}\biggr).$$

Lemma 1. *The family $\{\chi_t \mid t\in\mathbb{Z}^3_+\}$ forms an orthonormal basis in $L^2(\mathbb{R}^3)$.*

This is an eigenbasis for the operators S_- and $S_+ + S_0$:

$$S_-\chi_t = \hbar t_0\chi_t, \qquad (S_+ + S_0)\chi_t = \hbar(t_+ + t_-)\chi_t,$$

because S_- and $S_+ + S_0$ are related to b_0, $b_\pm$, b_0^*, and $b_\pm^*$ as

$$S_- = b_0b_0^*, \qquad S_+ + S_0 = \frac{1}{2}(b_+b_+^* + b_-b_-^*).$$

Therefore, $\{\chi_t\}$ is also an eigenbasis for the Casimir element C given by (2.11):

$$C\chi_t = \hbar(2t_+ + 2t_- - t_0)\chi_t.$$

Lemma 2. *The eigensubspace of the Hamiltonian $\widehat{H}_0$ given by* (2.5)*, where the element C takes the value $n\hbar$, is spanned by the vectors*

$$\{\chi_{2(t_++t_-)-n,t_+,t_-} \mid 2(t_+ + t_-) - n \ge 0\}. \tag{A.3}$$

The formulas

$$A_0 = b_0b_0^*, \qquad A_\pm = b_\pm b_\pm^*$$

for generators (3.6) and (3.7) similarly imply

$$A_0\chi_t = \hbar t_0\chi_t, \qquad A_\pm\chi_t = 2\hbar t_\pm\chi_t,$$

and hence $\{\chi_t\}$ is also an eigenbasis for the Casimir element M given by (3.10):

$$M = \hbar(2t_+ - 2t_- + t_0)\chi_t.$$

We then obtain

$$M\chi_t = \hbar(4t_+ - n)\chi_t$$

on eigensubspace (A.3) of the Hamiltonian $\widehat{H}_0$, where C takes the value $n\hbar$.

Theorem 4. *The common eigensubspace of the Casimir elements C and M in realization* (2.7) *is spanned by the vectors*

$$\{\chi_{2(m+t_-)-n,m,t_-} \mid 2(m+t_-)-n \geq 0\} = \{\chi_{2(m+m_-)-n+2j,m,m_-+j} \mid j \geq 0\}, \tag{A.4}$$

where m_- is defined in (4.7).

We note that the operator B given by (3.8) can be represented as

$$B = b_0^2 b_-$$

in realization (2.7).

Corollary 3. *Eigenbasis* (A.4) *satisfies the formula*

$$\chi_{2(m+m_-)-n+2j,m,m_-+j} = \frac{\gamma_{2(m+m_-)-n+2j,m,m_-+j}}{\gamma_{2(m+m_-)-n,m,m_-}} B^j \chi_{2(m+m_-)-n,m,m_-},$$

where γ_t are coefficients (A.2). *The vector $\chi_{2(m+m_-)-n,m,m_-}$ is here annihilated by the operator B^*.*

Appendix B

We introduce the operators

$$\begin{aligned} B_+ &\stackrel{\text{def}}{=} \frac{2}{\sqrt{3}} e^{i\varphi/2} A_\sigma - \frac{\sqrt{2}}{\sqrt{3}} e^{-i\varphi/2} A_\theta, \\ F &\stackrel{\text{def}}{=} \frac{2}{3}\big(\sqrt{2}(S_+ - S_0) + 2e^{i\varphi} A_\rho - e^{-i\varphi} A_\rho^*\big). \end{aligned} \tag{B.1}$$

Lemma 3. *Generators* (2.7) *and the adjoint generators can be expressed in terms of generators* (3.6)–(3.8), *and* (B.1) *and in terms of the adjoint operators by the formulas*

$$\begin{aligned} &S_- = A_0, \qquad S_+ + S_0 = \frac{1}{2}(A_+ + A_-), \\ &S_+ - S_0 = \frac{1}{6}\big[(A_+ - A_-) + 2\sqrt{2}(F + F^*)\big], \\ &e^{i\varphi} A_\rho + e^{-i\varphi} A_\rho^* = \frac{1}{6}\big[(F + F^*) - 2\sqrt{2}(A_+ - A_-)\big], \\ &e^{i\varphi} A_\rho - e^{-i\varphi} A_\rho^* = \frac{1}{2}(F - F^*), \\ &A_\sigma = \frac{1}{\sqrt{6}} e^{-i\varphi/2}(\sqrt{2} B_+ + B), \qquad A_\theta = \frac{1}{\sqrt{6}} e^{i\varphi/2}(\sqrt{2} B - B_+). \end{aligned}$$

Lemma 4. *The commutation relations*

$$\begin{aligned} &[M, A_0] = [M, A_\pm] = [M, B] = 0, \\ &MF = F(M + 4\hbar), \qquad MB_+ = B_+(M + 4\hbar) \end{aligned}$$

hold.

We use Lemma 3 to write the Hamiltonian $\widehat{H}_{20}$ as a linear combination of the operators A_0, $A_\pm$, B, B^*, B_+, B_+^*, F, and F^*. Further using Lemma 4 allows easily "commuting" the group of the operator M with the Hamiltonian $\widehat{H}_{20}$ and calculating "azimuthal" Hamiltonian (3.11) and the averaging operator V_1.

Proposition 2. *The operator* V_1 *transforming* (3.1) *into* (3.5) *is given by*

$$V_1 = \exp\biggl\{\frac{\varepsilon}{\hbar}(\rho F^* + \sigma B_+^* - \bar{\rho} F - \bar{\sigma} B_+)\biggr\},$$

where

$$\rho = \frac{\sqrt{2}}{12\xi}\bigl(\sqrt{2}(f_+ - f_0) - e^{-i\varphi} g_\rho + 2e^{i\varphi}\bar{g}_\rho\bigr),$$

$$\sigma = \frac{\sqrt{3}}{6\xi}\bigl(\sqrt{2}e^{i\varphi/2}\bar{g}_\sigma - e^{-i\varphi/2}\bar{g}_\theta\bigr),$$

the numbers f_+, f_0, g_ρ, g_σ, *and* g_θ *are defined in* (2.16), *and* $\eta = 1/3$.

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