Operations Research Forum (2024) 5:32
https://doi.org/10.1007/s43069-024-00310-2

RESEARCH

On a Simple Connection Between Δ-Modular ILP and LP, and a New Bound on the Number of Integer Vertices

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Received: 2 September 2023 / Accepted: 11 March 2024

Abstract
In our note, we present a very simple and short proof of a new interesting fact about the faces of an integer hull of a given rational polyhedron. This fact has a complete analog in linear programming theory and can be useful to establish new constructive upper bounds on the number of vertices in an integer hull of a Δ-modular polyhedron, which are competitive for small values of Δ and can be useful for integer linear maximization problems with a convex or quasiconvex objective function. As an additional corollary, we show that the number of vertices in an integer hull is bounded by $O(n)^n$ for $\Delta = O(1)$. As a part of our method, we introduce the notion of *deep bases* of a linear program. The problem to estimate their number by a non-trivial way seems to be quite challenging.

Keywords Linear programming · Integer linear programming · Number of vertices · Δ-modular

1 Basic Definitions and Notations

Let $A \in \mathbb{Z}^{m \times n}$ be an integer matrix. For sets $\mathcal{I} \subseteq \{1, \ldots, m\}$ and $\mathcal{J} \subseteq \{1, \ldots, n\}$, the symbol $A_{\mathcal{I}\,\mathcal{J}}$ denotes the sub-matrix of A, which is generated by all the rows with indices in $\mathcal{I}$ and all the columns with indices in $\mathcal{J}$. If $\mathcal{I}$ or $\mathcal{J}$ is replaced by $*$, then all the rows or columns are selected, respectively. For the sake of simplicity, we denote $A_{\mathcal{J}} := A_{\mathcal{J}\,*}$, or, in other words, $A_{\mathcal{J}}$ denotes the sub-matrix induced by the rows with indices in $\mathcal{J}$. The maximum absolute value of entries of a matrix A (also known as *the matrix* max-*norm*) is denoted by $\|A\|_{\max} = \max_{i,j} |A_{i\,j}|$. The number of non-zero components of a vector x is denoted by $\|x_0\| = |\{i : x_i \neq 0\}|$. For $v \in \mathbb{R}^n$, by $\operatorname{supp}_\Delta(v)$ and $\operatorname{zeros}_\Delta(v)$, we denote $\{i : |v_i| \geq \Delta\}$ and $\{1, \ldots, n\} \setminus \operatorname{supp}_\Delta(v)$, respectively. Denote $\operatorname{supp}(v) := \operatorname{supp}_0(v)$ and $\operatorname{zeros}(v) := \operatorname{zeros}_0(v)$. Clearly, $\|v_0\| = |\operatorname{supp}(v)|$.

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Published online: 04 April 2024

For a matrix $B \in \mathbb{R}^{m \times n}$, $\operatorname{conv.hull}(B) = \{Bt : t \in \mathbb{R}^n_{\geq 0},\ \sum_{i=1}^n t_i = 1\}$ is the *convex hull spanned by the columns of B*.

Definition 1 *For a matrix $A \in \mathbb{Z}^{m \times n}$, by*

$$\Delta_k(A) = \max\left\{|\det(A_{\mathcal{I}\,\mathcal{J}})| :\ \mathcal{I} \subseteq \{1, \dots, m\},\ \mathcal{J} \subseteq \{1, \dots, n\},\ |\mathcal{I}| = |\mathcal{J}| = k\right\},$$

we denote the maximum absolute value of determinants of all the $k \times k$ sub-matrices of A. Additionally, denote $\Delta(A) = \Delta_{\operatorname{rank}(A)}(A)$. A matrix A with $\Delta(A) \leq \Delta$, for some $\Delta > 0$, is called Δ-modular. Note that $\Delta_1(A) = \|A\|_{\max}$.

2 A Simple Connection Between Δ-Modular ILP and LP

Let $A \in \mathbb{Z}^{m \times n}$, $\operatorname{rank}(A) = n$, $b \in \mathbb{Z}^m$, and $\mathcal{P}$ be a polyhedron defined by the system $Ax \leq b$. Additionally, we assume that $\dim(\mathcal{P}) = n$, which is justified by the following reasoning. Assume that $\dim(\mathcal{P}) \leq n - 1$, which is equivalent to the existence of an index $j \in \{1, \dots, m\}$ such that $A_j x = b_j$, for all $x \in \mathcal{P}$. Note that such j could be found by a polynomial-time algorithm. W.l.o.g. assume that $j = 1$ and $\gcd(A_1) = 1$, then there exists an unimodular matrix $Q \in \mathbb{Z}^{n \times n}$ such that $A_1 = (1\ \mathbf{0}_{n-1})Q$. After the unimodular map $x \to Q^{-1}x$, the system $Ax \leq b$ transforms to the integrally equivalent[1] system

$$\begin{pmatrix} 1 & \mathbf{0}_{n-1} \\ h & B \end{pmatrix} x \leq b,$$

where $h \in \mathbb{Z}^{m-1}$ and $B \in \mathbb{Z}^{(m-1)\times(n-1)}$. Note that $\Delta(B) = \Delta(A)$. Since the first inequality always holds as an equality on the solutions set, we can just substitute $x_1 = b_1$. As a result, we achieve a new integrally equivalent system with $n - 1$ variables $Bx \leq b'$, where $b' = b_{\{2,\dots,m\}} - b_1 \cdot h$.

Let $\mathcal{F}$ be a t-dimensional face of $\mathcal{P}$. It is a known fact from the theory of linear inequalities that there exist $n - t$ linearly independent inequalities of $Ax \leq b$ that become equalities on $\mathcal{F}$. More precisely, there exists a set of indices $\mathcal{J} \subseteq \{1, \dots, m\}$, such that $|\mathcal{J}| \geq n - t$, $\operatorname{rank}(A_{\mathcal{J}}) = n - t$, and

$$A_{\mathcal{J}} x = b_{\mathcal{J}}, \quad \text{for } x \in \mathcal{F}, \tag{1}$$

and, consequently,

$$|\operatorname{supp}(Ax - b)| \leq m - n + t, \quad \text{for } x \in \mathcal{F}.$$

We are going to prove a similar fact for the polyhedron $\mathcal{P}_I = \operatorname{conv.hull}\big(\mathcal{P} \cap \mathbb{Z}^n\big)$. To help the reader see the close connection of the new result with the fact (1) from LP, we introduce the following notation:

$$\text{we write} \quad x \overset{\Delta}{=} y \quad \Longleftrightarrow \quad \|x - y\|_\infty < \Delta.$$

[1] Saying "integrally equivalent", we mean that the sets of integer solutions of both systems are connected by a bijective unimodular map.

Theorem 1 *Let $\mathcal{F}$ be a t-dimensional face of $\mathcal{P}_I$ and $\Delta = \Delta(A)$. Then, there exists a set of indices $\mathcal{J} \subseteq \{1, \dots, m\}$, such that $|\mathcal{J}| \geq n - t$, $\operatorname{rank}(A_{\mathcal{J}}) = n - t$, and*

$$A_{\mathcal{J}} x \overset{\Delta}{=} b_{\mathcal{J}}, \quad \text{for any} x \in \mathcal{F} \cap \mathbb{Z}^n,$$

and, consequently,

$$|\operatorname{supp}_{\Delta}(Ax - b)| \leq m - n + t, \quad \text{for any} x \in \mathcal{F} \cap \mathbb{Z}^n .$$

Proof Let us consider a point $v \in \mathbb{Z}^n$, lying on a t-dimensional face $\mathcal{F}$ of $\mathcal{P}_I$, and the corresponding slacks vector $u = b - Av$. Let $\mathcal{S} = \operatorname{supp}_{\Delta}(u)$ and $\mathcal{Z} = \operatorname{zeros}_{\Delta}(u)$. Suppose to the contrary that $r := \operatorname{rank}(A_{\mathcal{Z}}) < n - t$. We have

$$\begin{pmatrix} A_{\mathcal{Z}} \\ A_{\mathcal{S}} \end{pmatrix} v + \begin{pmatrix} u_{\mathcal{Z}} \\ u_{\mathcal{S}} \end{pmatrix} = \begin{pmatrix} b_{\mathcal{Z}} \\ b_{\mathcal{S}} \end{pmatrix}.$$

There exists an unimodular matrix $Q \in \mathbb{Z}^{n \times n}$, such that $A_{\mathcal{Z}} = \begin{pmatrix} H & \mathbf{0} \end{pmatrix} Q$, where $\begin{pmatrix} H & \mathbf{0} \end{pmatrix}$ is the Hermite normal form of $A_{\mathcal{Z}}$ and $H \in \mathbb{Z}^{|\mathcal{Z}| \times r}$. The zero sub-matrix of $\begin{pmatrix} H & \mathbf{0} \end{pmatrix}$ has $n - r > t$ columns. Let $y = Qv$, then

$$\begin{pmatrix} H & \mathbf{0} \\ C & B \end{pmatrix} y + \begin{pmatrix} u_{\mathcal{Z}} \\ u_{\mathcal{S}} \end{pmatrix} = \begin{pmatrix} b_{\mathcal{Z}} \\ b_{\mathcal{S}} \end{pmatrix},$$

where $\begin{pmatrix} C & B \end{pmatrix} = A_{\mathcal{S}} Q^{-1}$ and $B \in \mathbb{Z}^{|\mathcal{S}| \times (n-r)}$. The matrix B has a full column rank $n - r$, has at least t columns, and is Δ-modular. Consider the last $|\mathcal{S}|$ equalities of the previous system. They can be written out as follows:

$$Bz + u_{\mathcal{S}} = b_{\mathcal{S}} - C y_{\{1,\dots,r\}},$$

where $z = y_{\{(r+1),\dots,n\}}$ is composed of last $n - r$ components of y.

From the definition of $\mathcal{S}$, it follows that $(u_{\mathcal{S}})_i \geq \Delta$, for any $i \in \{1, \dots, |\mathcal{S}|\}$. W.l.o.g. assume that B is reduced to the Hermite normal form. Hence, due to Gribanov et al. [1, Lemma 1], $\|B\|_{\max} \leq \Delta$. Let $h_1, h_2, \dots, h_{n-r}$ be the columns of B, and let $e_1, e_2, \dots, e_n$ represent the coordinate vectors of the standard basis in $\mathbb{R}^n$. Consequently, any point of the type $z \pm e_j$, for $j \in \{1, \dots, n - r\}$, with its corresponding slack vector $u_{\mathcal{S}} \pm h_j$ is feasible. Since $n - r > t$, the last fact contradicts the fact that the original point v lies on the t-dimensional face of $\mathcal{P}_I$.

The following corollary describes how our relation looks like for polyhedra defined by systems in the standard form. Let $\mathcal{P}$ be defined by a system $Ax = b,\ x \geq \mathbf{0}$ with $A \in \mathbb{Z}^{k \times n}$, $b \in \mathbb{Z}^k$ and $\operatorname{rank}(A) = k$.

Corollary 2 *Let $\mathcal{F}$ be a t-dimensional face of $\mathcal{P}_I$ and $\Delta = \Delta(A)$. Then, there exists a set of indices $\mathcal{J} \subseteq \{1, \dots, n\}$, such that $|\mathcal{J}| \geq dim(\mathcal{P}) - t = n - k - t$, $\operatorname{rank}(A_{*\overline{\mathcal{J}}})$ $= k + t$ (where $\overline{\mathcal{J}} = \{1, \dots, n\} \setminus \mathcal{J}$), and*

$$x_{\mathcal{J}} \stackrel{\Delta}{=} \mathbf{0}, \quad \text{for any} x \in \mathcal{F} \cap \mathbb{Z}^n .$$

and, consequently,

$$|\operatorname{supp}_\Delta(x)| \le k + t, \quad \text{for any} x \in \mathcal{F} \cap \mathbb{Z}^n .$$

The proof can be directly deduced from Theorem 1 and Lemma 5 of [2].

3 The Number of Integer Vertices

Before we present our main result on $|\operatorname{vert}(\mathcal{P}_I)|$, let us make a small survey. Let $\xi(n, m)$ denote the maximum number of vertices in n-dimensional polyhedron with m facets. Due to the seminal paper [3] of P. McMullen, the value of $\xi(n, m)$ attains its maximum on the class of polytopes that are dual to cyclic polytopes with m vertices. Due to the book of Grünbaum [4, Section 4.7], we have

$$\xi(n, m) = \begin{cases} \frac{m}{m-s}\binom{m-s}{s}, \text{for} n = 2s \\ 2\binom{m-s-1}{s}, \text{ for } n = 2s+1 \end{cases} = O\left(\frac{m}{n}\right)^{n/2} .$$

The following bound on $|\operatorname{vert}(\mathcal{P}_I)|$ is due to Chirkov and Veselov [5] (see [6] for the refined analysis; for a survey, see [7–9]):

$$\begin{aligned} &|\operatorname{vert}(\mathcal{P}_I)| \le (n+1)^{n+1} \cdot n! \cdot \xi(n, m) \cdot \log_2^{n-1}(2\sqrt{n+1} \cdot \Delta_{ext}) = \\ &= m^{\frac{n}{2}} \cdot O(n)^{\frac{3}{2}n+1.5} \cdot \log^{n-1}(n \cdot \Delta_{ext}), \end{aligned} \tag{2}$$

Here, $\Delta_{ext} = \Delta(\begin{pmatrix} A \, b \end{pmatrix})$ is the maximal absolute value of $n \times n$ sub-determinants of the augmented matrix $\begin{pmatrix} A \, b \end{pmatrix}$.

Let ϕ be the bit-encoding length of $Ax \le b$. Due to the book of Schrijver [10, Chapter 3.2, Theorem 3.2], we have $\Delta_{ext} = 2^{O(\phi)}$. In notation with ϕ, the last bound (2) becomes

$$m^{\frac{n}{2}} \cdot O(n)^{\frac{3}{2}n+1.5} \cdot (\phi + \log n)^{n-1},$$

which outperforms the more known bound

$$m \cdot \binom{m-1}{n-1} \cdot (5n^2 \cdot \phi + 1)^{n-1} = m^n \cdot O(n)^{n-1} \cdot \phi^{n-1}, \tag{3}$$

due to Cook et al. [11], because $m \ge n$ and (2) depends on m as $m^{n/2}$. Due to Chirkov and Veselov [9], the previous inequality (2) could be combined with the sensitivity result of Cook et al. [12] to construct a bound that depends on Δ instead of Δ_{ext}:

$$\begin{aligned} &|\operatorname{vert}(\mathcal{P}_I)| \le (n+1)^{n+1} \cdot n! \cdot \xi(n, m) \cdot \xi(n, 2m) \cdot \log_2^{n-1}(2 \cdot (n+1)^{2.5} \cdot \Delta^2) = \\ &= m^n \cdot O(n)^{n+1.5} \cdot \log^{n-1}(n \cdot \Delta), \end{aligned} \tag{4}$$

which again is better than the bound (3) due to Cook et al., because (4) depends only from the bit-encoding length of A, while (3) depends on the length of both A and b. In our work, we will prove the bound:

$$\boxed{|\mathrm{vert}(\mathcal{P}_I)| \leq 2 \cdot \binom{m}{n} \cdot \Delta^{n-1}}, \tag{5}$$

which outperforms the state of the art bound (4) for $\Delta = O(n^2)$. The bounds are compared in Table 1:

As an additional corollary, we show that

$$\boxed{\text{for } \Delta = \mathrm{O}(1), \quad |\mathrm{vert}(\mathcal{P}_I)| = O(n)^n}. \tag{6}$$

Note that our bound is constructive, which is a straightforward consequence of our analysis. Theoretically, it can be used in integer convex/quasiconvex maximization problems on polyhedra with $\Delta = O(n^2)$. Fastest algorithms for higher values of Δ are given by the bounds of Chirkov and Veselov.

4 Other Related Work

Assume that $\mathcal{P}$ is defined by a system in the standard form

$$\begin{cases} Ax = b \\ x \in \mathbb{R}^n_{\geq 0}, \end{cases}$$

where $A \in \mathbb{Z}^{k \times n}$, $b \in \mathbb{Z}^k$ and $\mathrm{rank}(A) = k$. It is natural to call the value of k as the *co-dimension* of A or $\mathcal{P}$. The next bounds on $|\,\mathrm{vert}(\mathcal{P}_I)|$ assume that the co-dimension of $\mathcal{P}$ is bounded. Let $\Delta_1 = \Delta_1(A)$, then, due to Aliev et al. [13]:

$$|\mathrm{vert}(\mathcal{P}_I)| = (n \cdot k \cdot \Delta_1)^{O(k^2 \cdot \log(\sqrt{k} \cdot \Delta_1))}. \tag{7}$$

Table 1 Bounds on $|\mathrm{vert}(\mathcal{P}_I)|$

$m^n \cdot O(n)^{n-1} \cdot \phi^{n-1}$	Due to Cook et al. [11]
$m^{\frac{n}{2}} \cdot O(n)^{\frac{3}{2}n+1.5} \cdot \log^{n-1}(n \cdot \Delta_{ext}) =$	
$= m^{\frac{n}{2}} \cdot O(n)^{\frac{3}{2}n+1.5} \cdot (\phi + \log n)^{n-1}$	Due to Chirkov and Veselov [5]
$m^n \cdot O(n)^{n+1.5} \cdot \log^{n-1}(n \cdot \Delta)$	Due to Chirkov and Veselov [9]
$2 \cdot \binom{m}{n} \cdot \Delta^{n-1} =$	
$m^n \cdot \Omega(n)^{-n} \cdot \Delta^{n-1}$	This work

It is possible to improve the last bound. Let $s = \max\bigl\{\|v\|_0 \colon v \in \operatorname{vert}(\mathcal{P}_I)\bigr\}$ be the *sparsity parameter* of $\mathcal{P}_I$. Due to Berndt et al. [14], we have

$$|\operatorname{vert}(\mathcal{P}_I)| = n^{k+s} \cdot s \cdot O(k)^{s-k} \cdot \log^s(k \cdot \Delta_1). \tag{8}$$

The following improvement of (8) was proposed in the work [2], due to Gribanov et al.:

$$|\operatorname{vert}(\mathcal{P}_I)| = n^s \cdot O(s)^{s+1} \cdot O(k)^{s-1} \cdot \log^{s-1}(k \cdot \Delta_1). \tag{9}$$

Since $s = O\bigl(k \cdot \log(k\Delta_1)\bigr)$, due to Aliev et al. [13], we substitute s to both bounds (8) and (9), and get

$$|\operatorname{vert}(\mathcal{P}_I)| = \bigl(n \cdot k \cdot \log(k\Delta_1)\bigr)^{O\left(k \cdot \log(k\Delta_1)\right)},$$

which outperforms the bound (7), due to [13]. The last equality was proposed in Berndt et al. [14]. Due to Gribanov et al. [2], it holds $s = O(k + \log(\Delta))$, where $\Delta = \Delta(A)$. Consequently, the bound (9) could be used to estimate $|\operatorname{vert}(\mathcal{P}_I)|$ with respect to the Δ parameter instead of Δ_1:

$$|\operatorname{vert}(\mathcal{P}_I)| = \bigl(n \cdot k \cdot \log(\Delta)\bigr)^{O\left(k+\log(\Delta)\right)}. \tag{10}$$

Note that, due to [2], the bounds (9) and (10) can be used to work with the systems $Ax \le b$ having $m = n + k$ rows. Therefore, for the case when $\mathcal{P}$ is defined by $Ax \le b$, it is also convenient to call k as the co-dimension of $\mathcal{P}$. The bounds with respect to the co-dimension are compared in Table 2.

5 Proof of the Bound (5)

First of all, let us formulate some definitions.

Definition 2 *Let $\mathcal{P} = \mathcal{P}(A, b)$ be a polyhedron as in the definition of Theorem 1. The set of indices $\mathcal{B} \subseteq \{1, \dots, m\}$ is a Δ-deep base if*

1. $|\mathcal{B}| = n$ and $\det(A_{\mathcal{B}}) \neq 0$;

Table 2 Bounds for $|\operatorname{vert}(\mathcal{P}_I)|$ with dependence on the co-dimension k

$(n \cdot k \cdot \Delta_1)^{O(k^2 \cdot \log(\sqrt{k} \cdot \Delta_1))}$	Due to Aliev et al. [13]
$n^{k+s} \cdot s \cdot O(k)^{s-k} \cdot \log^s(k \cdot \Delta_1) =$	
$= \bigl(n \cdot k \cdot \log(k\Delta_1)\bigr)^{O\left(k \cdot \log(k\Delta_1)\right)}$	Due to Berndt et al. [14]
$n^s \cdot O(s)^{s+1} \cdot O(k)^{s-1} \cdot \log^{s-1}(k \cdot \Delta_1) =$	Due to Gribanov et al. [2] plus
$= \bigl(n \cdot k \cdot \log(k\Delta_1)\bigr)^{O\left(k \cdot \log(k\Delta_1)\right)}$	Aliev et al. [13]
$\bigl(n \cdot k \cdot \log(\Delta)\bigr)^{O\left(k+\log(\Delta)\right)}$	Due to Gribanov et al. [2]

2. *the following system is feasible:*

$$\begin{cases} b_{\mathcal{B}} - (\Delta - 1) \cdot \mathbf{1}_n \le A_{\mathcal{B}}\, x \le b_{\mathcal{B}} \\ A_{\overline{\mathcal{B}}}\, x \le b_{\overline{\mathcal{B}}} \\ x \in \mathbb{R}^n, \end{cases}$$

$\overline{\mathcal{B}} = \{1, \ldots, m\} \setminus \mathcal{B}$.

Let us denote the number of Δ-deep bases of $\mathcal{P}$ by $\beta_\Delta(\mathcal{P})$. Note that any vertex of $\mathcal{P}$ corresponds to some trivial Δ-deep base, so $\beta_\Delta(\mathcal{P}) \ge |\operatorname{vert}(\mathcal{P})|$.

Definition 3 *Let $\mathcal{M} \subseteq \{0, \ldots, \Delta - 1\}^n$ be a convex-independent set, i.e., any point of $\mathcal{M}$ can not be expressed as a convex combination of other points from $\mathcal{M}$. Let us denote the maximal possible cardinality of $\mathcal{M}$ by $\gamma(n, \Delta)$.*

Trivially, $\gamma(n, \Delta) \le \Delta^n$. We will use a different, simple bound mentioned by Brass [15]. It follows by the pigeonhole principle that $\gamma(n, \Delta) \le \Delta \cdot \gamma(n-1, \Delta)$. Together with $\gamma(1, \Delta) = 2$, it gives

$$\gamma(n, \Delta) \le 2 \cdot \Delta^{n-1}. \tag{11}$$

The lower bound $\gamma(n, \Delta) \ge \frac{4}{n} \cdot \Delta^{n-2}$ is proposed by Erdős et al. [16].

Lemma 1 *Let $\mathcal{P} = \mathcal{P}(A, b)$ be a polyhedron as in the definition of Theorem 1. Then, we have:*

$$|\operatorname{vert}(\mathcal{P}_I)| \le \beta_\Delta(\mathcal{P}) \cdot \gamma(n, \Delta).$$

Proof Let us consider the family $\mathscr{B}$ of all possible Δ-deep bases of $\mathcal{P}$. For $\mathcal{B} \in \mathscr{B}$, we use the following notation:

$$\mathcal{V}_{\mathcal{B}} = \{v \in \operatorname{vert}(\mathcal{P}_I) \colon b_{\mathcal{B}} - A_{\mathcal{B}} v < \Delta \cdot \mathbf{1}\}.$$

Due to Theorem 1, we have $\operatorname{vert}(\mathcal{P}_I) = \bigcup_{\mathcal{B} \in \mathscr{B}} \mathcal{V}_{\mathcal{B}}$. Now, we are going to estimate $|\mathcal{V}_{\mathcal{B}}|$. Let $\mathcal{U}_{\mathcal{B}} = \{b_{\mathcal{B}} - A_{\mathcal{B}} v \colon v \in \mathcal{V}_{\mathcal{B}}\}$. Clearly, there exists a bijection between $\mathcal{U}_{\mathcal{B}}$ and $\mathcal{V}_{\mathcal{B}}$. Since $\mathcal{V}_{\mathcal{B}}$ is a convex-independent set, the same is true for $\mathcal{U}_{\mathcal{B}}$. Moreover, $\mathbf{0} \le u < \Delta \cdot \mathbf{1}$, for $u \in \mathcal{U}_{\mathcal{B}}$. Consequently, $|\mathcal{V}_{\mathcal{B}}| = |\mathcal{U}_{\mathcal{B}}| \le \gamma(n, \Delta)$, and $|\operatorname{vert}(\mathcal{P}_I)| \le \beta_\Delta(\mathcal{P}) \cdot \gamma(n, \Delta)$.

Corollary 3 *In the assumptions of Theorem 1, the following statements hold:*

1. *For any $v \in \operatorname{vert}(\mathcal{P}_I)$, there exists a Δ-deep base $\mathcal{B}$ such that $A_{\mathcal{B}} v \overset{\Delta}{=} b_{\mathcal{B}}$.*
2. $|\operatorname{supp}_\Delta(b - Av)| \le m - n$.
3. *The inequality $|\operatorname{vert}(\mathcal{P}_I)| \le 2 \cdot \binom{m}{n} \cdot \Delta^{n-1}$ holds.*

Proof Propositions 1 and 2 are straightforward consequences of Theorem 1. Proposition 3 is a straight consequence of the trivial inequality $\beta_\Delta(\mathcal{P}) \le \binom{m}{n}$, inequality (11) and Lemma 1.

Using results of the papers [17] and [18] due to Averkov and Schymura and Lee et al., we can give bounds that are independent on m:

Corollary 4 *In the assumptions of Theorem 1, the following statements hold:*

1. $|\text{vert}(\mathcal{P}_I)| = O(n)^n \cdot \Delta^{3n-1}$.
2. $|\text{vert}(\mathcal{P}_I)| = O(n)^{3n} \cdot \Delta^{2n-1}$.

Proof Clearly, $\binom{m}{n} = O\left(\frac{m}{n}\right)^n$. Due to [17] and [18], we can assume that $m = O(n^4 \cdot \Delta)$ or $m = O(n^2 \cdot \Delta^2)$ respectively.

Consequently, for constant values of Δ, we have $|\text{vert}(\mathcal{P}_I)| = O(n)^n$, which is equivalent of (6).

6 Conclusions and Directions for Future Research

Due to Lemma 1, we estimate the integer vertices number of $\mathcal{P}_I$ by $\beta_\Delta(\mathcal{P}) \cdot \gamma(n, \Delta)$. Due to Erdös et al. [16], we have $\gamma(n, \Delta) \geq \frac{4}{n} \cdot \Delta^{n-2}$, so our bound (5) on $|\operatorname{vert}(\mathcal{P}_I)|$ cannot be significantly improved, using only improvements on $\gamma(n, \Delta)$. On the other hand, we do not know any upper or lower bounds for the number $\beta_\Delta(\mathcal{P})$ of Δ-deep bases with respect to $\mathcal{P}$ except trivial ones: $|\operatorname{vert}(\mathcal{P})| \leq \beta_\Delta(\mathcal{P}) \leq \binom{m}{n}$. We believe that some significant improvements can be obtained using accurate analysis of $\beta_\Delta(\mathcal{P})$, which seems to be a quite challenging task.

Author Contribution All authors contributed equally.

Funding This work was prepared under financial support of Russian Science Foundation grant No 21-11-00194.

Data Availability The manuscript has no associated data.

Code Availability Not applicable.

Declarations

Ethics Approval Not applicable.

Consent to Participate Not applicable.

Consent for Publication Not applicable.

Competing Interest The authors declare no competing interests.

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