OPEN r-SPIN THEORY III: A PREDICTION FOR HIGHER GENUS

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Abstract. In our previous two papers, we constructed an r-spin theory in genus zero for Riemann surfaces with boundary and fully determined the corresponding intersection numbers, providing an analogue of Witten's r-spin conjecture in genus zero in the open setting. In particular, we proved that the generating series of open r-spin intersection numbers is determined by the genus-zero part of a special solution of a certain extension of the Gelfand–Dickey hierarchy, and we conjectured that the whole solution controls the open r-spin intersection numbers in all genera, which do not yet have a geometric definition. In this paper, we provide geometric and algebraic evidence for the correctness of this conjecture.

1. Introduction

One of the most important results in the study of the intersection theory on the moduli spaces of stable curves $\overline{\mathcal{M}}_{g,n}$ is Witten's conjecture [19], proved by Kontsevich [13], saying that the generating series of intersection numbers

$$\mathcal{F}^c(t_0,t_1,\ldots,\varepsilon)=\sum_{g\geq 0}\varepsilon^{2g-2}\mathcal{F}^c_g(t_0,t_1,\ldots):=\sum_{\substack{g\geq 0,\,n\geq 1\\ 2g-2+n>0}}\sum_{d_1,\ldots,d_n\geq 0}\frac{\varepsilon^{2g-2}}{n!}\left(\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\right)t_{d_1}\cdots t_{d_n}$$

is the logarithm of a tau-function of the KdV hierarchy. Here, $\psi_i\in H^2(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ is the first Chern class of the cotangent line bundle corresponding to the i-th marked point, $t_0,t_1,\ldots$ and ε are formal variables, and the superscript "c," which stands for "closed," is to contrast with the open theory discussed below.

Witten also proposed a much more general conjecture, the so-called r-spin Witten conjecture [20], which considers the moduli space of stable curves with r-spin structure. On a smooth marked curve $(C;z_1,\ldots,z_n)$, an *r-spin structure* is a line bundle S together with an isomorphism $S^{\otimes r}\cong\omega_C\left(-\sum_{i=1}^n a_i[z_i]\right)$, where $a_i\in\{0,1,\ldots,r-1\}$ and ω_C denotes the canonical bundle. There is a natural compactification $\overline{\mathcal{M}}^{1/r}_{g,(a_1,\ldots,a_n)}$ of the moduli space of smooth curves with r-spin structure, and this space admits a virtual fundamental class $c_W\in H^*(\overline{\mathcal{M}}^{1/r}_{g,(a_1,\ldots,a_n)},\mathbb{Q})$ known as *Witten's class*. In genus zero, c_W is the Euler class of the derived pushforward $(R^1\pi_*\mathcal{S})^\vee$, where $\pi\colon\mathcal{C}\to\overline{\mathcal{M}}^{1/r}_{0,(a_1,\ldots,a_n)}$ is the universal curve and $\mathcal{S}$ is the universal r-spin structure. In higher genus, the sheaf $R^1\pi_*\mathcal{S}$ may not be a vector bundle, and the definition of Witten's class is much more intricate; see [16, 9, 14, 12, 8] for various constructions.

Witten's r-spin conjecture, proved by Faber–Shadrin–Zvonkine [11], states that if t^a_d are formal variables indexed by $0\leq a\leq r-1$ and $d\geq 0$, then the generating series

$$\begin{aligned}\mathcal{F}^{\frac{1}{r},c}(t^*_*,\varepsilon)&=\sum_{g\geq 0}\varepsilon^{2g-2}\mathcal{F}^{\frac{1}{r},c}_g(t^*_*)\\ &:=\sum_{\substack{g\geq 0,\,n\geq 1\\ 2g-2+n>0}}\sum_{\substack{0\leq a_1,\ldots,a_n\leq r-1\\ d_1,\ldots,d_n\geq 0}}\frac{\varepsilon^{2g-2}}{n!}r^{1-g}\left(\int_{\overline{\mathcal{M}}^{1/r}_{g,(a_1,\ldots,a_n)}}c_W\cdot\psi_1^{d_1}\cdots\psi_n^{d_n}\right)t^{a_1}_{d_1}\cdots t^{a_n}_{d_n}\end{aligned}$$

is, after a simple change of variables, the logarithm of a tau-function of the r-th Gelfand–Dickey hierarchy.

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While curves with r-spin structure generalize the study of $\overline{\mathcal{M}}_{g,n}$ in one direction, a different direction was taken up by Pandharipande, Solomon, and the third author in [15], in which the study of intersection theory on the moduli space $\overline{\mathcal{M}}_{g,k,l}$ of Riemann surfaces with boundary was initiated. Here, the genus g of a Riemann surface with boundary $(C, \partial C)$ is defined as the genus of the closed surface obtained by gluing two copies of C along the boundary ∂C, and the numbers k and l are the numbers of boundary and internal marked points, respectively. In [15], intersection numbers on $\overline{\mathcal{M}}_{0,k,l}$, called *open intersection numbers* and denoted by

$$\left\langle \tau_{d_1} \cdots \tau_{d_l} \sigma^k \right\rangle_0^o, \quad d_i \geq 0,$$

were defined and explicitly computed. Moreover, in [15] the authors proposed a conjectural description of open intersection numbers in all genera, which can be viewed as an open analogue of Witten's conjecture on $\overline{\mathcal{M}}_{g,n}$. A geometric construction of open intersection numbers in all genera, denoted by $\left\langle \tau_{d_1} \cdots \tau_{d_l} \sigma^k \right\rangle_g^o$, was given by Solomon and the third author, hence the generating function

$$\mathcal{F}^o(t_0, t_1, \ldots, s, \varepsilon) = \sum_{g \geq 0} \varepsilon^{g-1} \mathcal{F}_g^o(t_0, t_1, \ldots, s) := \sum_{\substack{g,k,l \geq 0 \\ 2g-2+k+2l>0}} \frac{\varepsilon^{g-1}}{k!l!} \left\langle \tau_{d_1} \cdots \tau_{d_l} \sigma^k \right\rangle_g^o t_{d_1} \cdots t_{d_l} s^k$$

was defined as a direct generalization of $\mathcal{F}^c$; the new formal variable s tracks the number of boundary marked points. The definition of the all-genus open intersection numbers can be found in [18], which summarizes the construction of [17]. A combinatorial formula for the open intersection numbers in all genera was given in [18] and then used in [7] to prove the open analogue of Witten's conjecture. By [2], the generating series $\mathcal{F}^o$ gives a solution of a certain extension of the KdV hierarchy and is related to the wave function of the KdV hierarchy (corresponding to the tau-function $\exp(\mathcal{F}^c)$) by an explicit formula [3].

It is natural to ask whether these two generalizations of Witten's conjecture can be combined, producing an open analogue of Witten's r-spin conjecture. Toward this end, in [6] we defined a moduli space of *graded r-spin disks* $\overline{\mathcal{M}}^{1/r}_{0,k,(a_1,\ldots,a_l)}$ as well as an open Witten bundle $\mathcal{W}$ and cotangent line bundles $\mathbb{L}_1, \ldots, \mathbb{L}_l$ at the internal marked points, and then in [4] we defined the corresponding *open r-spin intersection numbers*[1]

$$\left\langle \tau^{a_1}_{d_1} \cdots \tau^{a_l}_{d_l} \sigma^k \right\rangle_0^{\frac{1}{r},o}, \quad 0 \leq a_i \leq r-1, \quad d_i \geq 0.$$

Equipped with these numbers, we defined an *open r-spin potential* in genus zero by

$$\mathcal{F}_0^{\frac{1}{r},o}(t^*_*, s) := \sum_{\substack{k,l \geq 0 \\ k+2l>2}} \sum_{\substack{0 \leq a_1,\ldots,a_l \leq r-1 \\ d_1,\ldots,d_l \geq 0}} \frac{1}{k!l!} \left\langle \tau^{a_1}_{d_1} \cdots \tau^{a_l}_{d_l} \sigma^k \right\rangle_0^{\frac{1}{r},o} t^{a_1}_{d_1} \cdots t^{a_l}_{d_l} s^k.$$

We then considered a special solution

$$\phi = \sum_{g \geq 0} \varepsilon^{g-1} \phi_g(t^*_*), \quad \phi_g \in \mathbb{C}[[t^*_*]]$$

of a certain extension of the Gelfand–Dickey hierarchy and proved a formula for the generating series $\mathcal{F}_0^{\frac{1}{r},o}$ in terms of the formal power series ϕ_0. We will recall the details in Section 2.

While these constructions and results are limited to genus zero, in [4] we conjectured that for any genus $g \geq 1$ there is a geometric construction of open r-spin intersection numbers $\left\langle \tau^{a_1}_{d_1} \cdots \tau^{a_l}_{d_l} \sigma^k \right\rangle_g^{\frac{1}{r},o}$ generalizing our construction in genus zero. Given such intersection numbers,

[1] The construction from [15] is recovered as a special case, when $r = 2$ and all a_i are zero.

we defined a generating series $\mathcal{F}_g^{\frac{1}{r},o}(t^*_*,s)$ by

$$\mathcal{F}_g^{\frac{1}{r},o}(t^*_*,s):=\sum_{l,k\geq 0}\frac{1}{l!k!}\sum_{\substack{0\leq a_1,\ldots,a_l\leq r-1\\ d_1,\ldots,d_l\geq 0}}\left\langle\tau^{a_1}_{d_1}\cdots\tau^{a_l}_{d_l}\sigma^k\right\rangle^{\frac{1}{r},o}_g t^{a_1}_{d_1}\cdots t^{a_l}_{d_l}s^k,$$

and conjectured an explicit formula for it in terms of the formal power series ϕ_g. See Conjecture 1 below for the explicit statement.

In this paper, we study the main conjecture from [4] in more detail. One can expect that the numbers $\left\langle\tau^{a_1}_{d_1}\cdots\tau^{a_l}_{d_l}\sigma^k\right\rangle^{\frac{1}{r},o}_g$ satisfy a series of natural properties:

(1) In the case $r=2$ and $a_1=\cdots=a_l=0$, it is natural to identify the corresponding open r-spin intersection numbers with the intersection numbers on $\overline{\mathcal{M}}_{g,k,l}$.
(2) From the dimension of the moduli space of genus-g r-spin surfaces with boundary and an expected formula for the degree of an open analogue of Witten's class, one can see that the open r-spin intersection numbers should vanish unless a dimension constraint is satisfied.
(3) From a natural expectation for the behavior of an open analogue of Witten's class under the map forgetting a marked point of twist 0, one can see that the generating series $\mathcal{F}_g^{\frac{1}{r},o}$ should satisfy open string and open dilaton equations. The genus-zero part of these equations was proved in [4], while analogous equations for the intersection numbers $\left\langle\tau_{d_1}\cdots\tau_{d_l}\sigma^k\right\rangle^o_g$ were conjectured in [15] and proved in [7] (a geometric proof will appear in [17]).
(4) Based on the genus-one topological recursion relations for the intersection numbers $\left\langle\tau_{d_1}\ldots\tau_{d_l}\sigma^k\right\rangle^o_1$ conjectured by the authors of [15] and proved by Solomon and the third author (a geometric proof will appear in [17]), we expect a natural generalization of these relations to be satisfied by the open r-spin intersection numbers in genus one.

The main result of this paper, proved in Section 3, is that all of these expected properties of the open r-spin intersection numbers agree with Conjecture 1.

Convention. We use the standard convention of sum over repeated Greek indices.

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2. The Gelfand–Dickey hierarchy and the main conjecture

Consider formal variables T_i for $i\geq 1$. A *pseudo-differential operator* A is a Laurent series

$$A=\sum_{n=-\infty}^{m}a_n\partial_x^n,\quad a_n\in\mathbb{C}[\varepsilon,\varepsilon^{-1}][[T_1,T_2,\ldots]],$$

where m is an integer and ∂_x is a formal variable. Denote

$$A_+:=\sum_{n=0}^{m}a_n\partial_x^n,\qquad A_-:=A-A_+,\qquad \operatorname{res}A:=a_{-1}.$$

The space of such operators is endowed with the structure of a noncommutative associative algebra, in which the multiplication, denoted by $\circ$, is defined by the formula

$$\partial_x^k\circ f:=\sum_{l=0}^{\infty}\frac{k(k-1)\cdots(k-l+1)}{l!}\frac{\partial^l f}{\partial T_1^l}\partial_x^{k-l},\quad f\in\mathbb{C}[\varepsilon,\varepsilon^{-1}][[T_*]],\quad k\in\mathbb{Z}.$$

We identify $x = T_1$, and in the case $A_- = 0$ we interpret A as a differential operator acting in the space of formal power series $\mathbb{C}[[T_*]]$ in the obvious way.

For any $r \geq 2$ and any pseudo-differential operator A of the form

$$A = \partial_x^r + \sum_{n=1}^{\infty} a_n \partial_x^{r-n},$$

there exists a unique pseudo-differential operator $A^{\frac{1}{r}}$ of the form

$$A^{\frac{1}{r}} = \partial_x + \sum_{n=0}^{\infty} \widetilde{a}_n \partial_x^{-n}$$

such that $\left(A^{\frac{1}{r}}\right)^r = A$.

Let $r \geq 2$, and consider the pseudo-differential operator

$$L := \partial_x^r + \sum_{i=0}^{r-2} f_i \partial_x^i, \quad f_i \in \mathbb{C}[\varepsilon, \varepsilon^{-1}][[T_*]].$$

For any $n \geq 1$, the commutator $[(L^{n/r})_+, L]$ has the form $\sum_{i=0}^{r-2} h_i \partial_x^i$ with $h_i \in \mathbb{C}[\varepsilon, \varepsilon^{-1}][[T_*]]$. The *$r$-th Gelfand–Dickey hierarchy* is the following system of partial differential equations for the formal power series $f_0, f_1, \ldots, f_{r-2}$:

$$\frac{\partial L}{\partial T_n} = \varepsilon^{n-1}[(L^{n/r})_+, L], \quad n \geq 1.$$

Consider the solution L of the Gelfand–Dickey hierarchy specified by the initial condition

$$L|_{T_{\geq 2}=0} = \partial_x^r + \varepsilon^{-r} r x. \tag{2.1}$$

The r-spin Witten conjecture states that $\frac{\partial \mathcal{F}^{\frac{1}{r},c}}{\partial t_d^{r-1}} = 0$ for $d \geq 0$, and under the change of variables

$$T_k = \frac{1}{(-r)^{\frac{3k}{2(r+1)} - \frac{1}{2} - d} k!_r} t_d^a, \quad 0 \leq a \leq r-2, \quad d \geq 0,$$

where $k = a + 1 + rd$ and $k!_r := \prod_{i=0}^{d}(a+1+ri)$, we have

$$\operatorname{res} L^{n/r} = \varepsilon^{1-n} \frac{\partial^2 \mathcal{F}^{\frac{1}{r},c}}{\partial T_1 \partial T_n}, \quad n \geq 1, \quad r \nmid n.$$

With L as above, let $\Phi(T_*, \varepsilon) \in \mathbb{C}[\varepsilon, \varepsilon^{-1}][[T_*]]$ be the solution of the system of equations

$$\frac{\partial \Phi}{\partial T_n} = \varepsilon^{n-1}(L^{n/r})_+ \Phi, \quad n \geq 1, \tag{2.2}$$

that satisfies the initial condition $\Phi|_{T_{\geq 2}=0} = 1$. Consider the expansion

$$\phi := \log \Phi = \sum_{g \in \mathbb{Z}} \varepsilon^{g-1} \phi_g, \quad \phi_g \in \mathbb{C}[[T_*]].$$

Note that by [5, Lemma 4.4] $\phi_g = 0$ for $g < 0$. Comparing to the formal power series $\mathcal{F}_0^{\frac{1}{r},c}$, which depends only on the variables $t_d^0, \ldots, t_d^{r-2}$, the formal power series $\mathcal{F}_0^{\frac{1}{r},o}$ depends also on t_d^{r-1} and s. We relate the variables T_{mr} and t_{m-1}^{r-1} as follows:

$$T_{mr} = \frac{1}{(-r)^{\frac{m(r-2)}{2(r+1)}} m! r^m} t_{m-1}^{r-1}, \quad m \geq 1.$$

In [4] we proved the following result:

$$\mathcal{F}_0^{\frac{1}{r},o} = \frac{1}{\sqrt{-r}}\phi_0\Big|_{t_d^{r-1}\mapsto\frac{1}{\sqrt{-r}}(t_d^{r-1}-r\delta_{d,0}s)} - \frac{1}{\sqrt{-r}}\phi_0\Big|_{t_d^{r-1}\mapsto\frac{1}{\sqrt{-r}}t_d^{r-1}}.$$

Regarding the open r-spin intersection numbers in higher genera, we proposed the following conjecture.

Conjecture 1 ([4])**.** *For any $g\geq 1$ we have*

$$\mathcal{F}_g^{\frac{1}{r},o} = (-r)^{\frac{g-1}{2}}\phi_g\Big|_{t_d^{r-1}\mapsto\frac{1}{\sqrt{-r}}(t_d^{r-1}-\delta_{d,0}rs)}.$$

3. Evidence for the main conjecture

3.1. **The case $r=2$.** Suppose $r=2$. In [4] we proved that

$$\left\langle\tau_{d_1}^0\cdots\tau_{d_l}^0\sigma^k\right\rangle_0^{\frac{1}{2},o} = (-2)^{\frac{k-1}{2}}\left\langle\tau_{d_1}\cdots\tau_{d_l}\sigma^k\right\rangle_0^o, \tag{3.1}$$

where we recall that the right-hand side refers to the intersection numbers on $\overline{\mathcal{M}}_{0,k,l}$. Regarding higher genus, in [7] the authors proved that

$$\mathcal{F}_g^o(t_0,t_1,\ldots,s) = \phi_g\Big|_{\substack{t_d^0=t_d\\ t_d^1=\delta_{d,0}s}}. \tag{3.2}$$

Thus, it is natural to identify

$$\left\langle\tau_{d_1}^0\cdots\tau_{d_l}^0\sigma^k\right\rangle_g^{\frac{1}{2},o} := (-2)^{\frac{g+k-1}{2}}\left\langle\tau_{d_1}\cdots\tau_{d_l}\sigma^k\right\rangle_g^o. \tag{3.3}$$

Indeed, note that the factor $2^{-\frac{g+k-1}{2}}$ is forcibly included in the definition of the intersection numbers $\left\langle\tau_{d_1}\cdots\tau_{d_l}\sigma^k\right\rangle_g^o$. Also, note that the intersection number $\left\langle\tau_{d_1}\cdots\tau_{d_l}\sigma^k\right\rangle_g^o$ is zero unless $g+k$ is odd. Thus, changing the orientation of $\overline{\mathcal{M}}_{g,k,l}$ by $(-1)^{\frac{g+k-1}{2}}$ when $g+k$ is odd gives the additional factor $(-1)^{\frac{g+k-1}{2}}$. This confirms Conjecture 1 in the case $r=2$ after setting $t_d^1=0$.

3.2. **Dimension constraint.** It is natural to expect that the open r-spin intersection number

$$\left\langle\tau_{d_1}^{\alpha_1}\cdots\tau_{d_l}^{\alpha_l}\sigma^k\right\rangle_g^{\frac{1}{r},o}$$

is zero unless

$$\frac{(g+k-1)(r-2)+2\sum\alpha_i}{r}+2\sum d_i = 3g-3+2l+k.$$

Indeed, the right-hand side is the dimension of the moduli space of r-spin genus-g surfaces with boundary, while the left-hand side is the virtual dimension of the purported open analogue of the virtual fundamental cycle that should correspond to the intersection problem. This is equivalent to the constraint

$$\sum\left(\frac{\alpha_i}{r}+d_i-1\right)-\frac{k}{r} = \frac{(r+1)(g-1)}{r}. \tag{3.4}$$

On the other hand, in [5, Lemma 4.4] we proved that the derivative $\left.\frac{\partial^n\phi_g}{\partial t_{d_1}^{\alpha_1}\cdots\partial t_{d_n}^{\alpha_n}}\right|_{t_*^*=0}$ is zero unless $\sum\left(\frac{\alpha_i}{r}+d_i-1\right)=\frac{(r+1)(g-1)}{r}$. We see that if Conjecture 1 is true then this gives exactly the expected constraint for the open r-spin intersection numbers.

3.3. **Open string and open dilaton equations.** One would expect the formal power series $\mathcal{F}_g^{\frac{1}{r},o}$ to satisfy the open string and the open dilaton equations

$$\frac{\partial \mathcal{F}_g^{\frac{1}{r},o}}{\partial t_0^0} = \sum_{n\geq 0} t_{n+1}^\alpha \frac{\partial \mathcal{F}_g^{\frac{1}{r},o}}{\partial t_n^\alpha} + \delta_{g,0}s, \tag{3.5}$$

$$\frac{\partial \mathcal{F}_g^{\frac{1}{r},o}}{\partial t_1^0} = (g-1)\mathcal{F}_g^{\frac{1}{r},o} + \sum_{n\geq 0} t_n^\alpha \frac{\partial \mathcal{F}_g^{\frac{1}{r},o}}{\partial t_n^\alpha} + s\frac{\partial \mathcal{F}_g^{\frac{1}{r},o}}{\partial s} + \delta_{g,1}\frac{1}{2}. \tag{3.6}$$

Indeed, in the case $r=2$ and $t_d^1=0$, using the identification (3.3), these equations become exactly the open string and the open dilaton equations for the generating series $\mathcal{F}_g^o$, which where conjectured in [15], proved there in genus 0 using geometric technique, and proved in all genera in [7] using a matrix model (a geometric proof will appear in [17]).

In [4, Proposition 5.2], we proved equations (3.5) and (3.6) in genus zero for any $r\geq 2$. Although this was done using an open-closed correspondence and open string and open dilaton equations for the closed extended r-spin intersection numbers, we could also prove them geometrically, imitating the proofs in [15]. In particular, we expect the geometric proof to work in all genera and all $r\geq 2$, given a construction of an open virtual fundamental cycle for the higher-genus Witten bundle that satisfies certain expected properties (being pulled back from the moduli without markings of twist zero, for example, and boundary behavior similar to that of canonical sections in genus zero).

By [1, Theorem 1.2] (see also [5, Lemmas 4.5]) we have

$$\frac{\partial \phi_g}{\partial t_0^0} = \sum_{n\geq 0} t_{n+1}^\alpha \frac{\partial \phi_g}{\partial t_n^\alpha} + \delta_{g,0} t_0^{r-1}.$$

If Conjecture 1 is true, then this implies the expected equation (3.5). The expected equation (3.6), on the other hand, is implied by Conjecture 1 by way of the following proposition.

Proposition 3.1. *We have*

$$\frac{\partial \phi}{\partial t_1^0} = \sum_{n\geq 0} t_n^\alpha \frac{\partial \phi}{\partial t_n^\alpha} + \varepsilon\frac{\partial \phi}{\partial \varepsilon} + \frac{1}{2}. \tag{3.7}$$

Proof. In the variables T_i, equation (3.7) looks as follows:

$$\left(\frac{1}{r+1}\frac{\partial}{\partial T_{r+1}} - \varepsilon\frac{\partial}{\partial\varepsilon} - \sum_{i\geq 1} T_i \frac{\partial}{\partial T_i}\right)\phi = \frac{1}{2}. \tag{3.8}$$

Before we proceed with the proof, let us recall more facts from the theory of the Gelfand–Dickey hierarchy (see, e.g., [10]).

Consider a pseudo-differential operator $A=\sum_{n=-\infty}^m a_n(T_*,\varepsilon)\partial_x^n$. The Laurent series

$$\widehat{A}(T_*,\varepsilon,z) := \sum_{n=-\infty}^m a_n(T_*,\varepsilon) z^n,$$

in which z is a formal variable, is called the *symbol* of the operator A. Suppose an operator L is a solution of the Gelfand–Dickey hierarchy. Then there exists a pseudo-differential operator P of the form

$$P = 1 + \sum_{n\geq 1} p_n(T_*,\varepsilon)\partial_x^{-n},$$

satisfying $L = P \circ \partial_x^r \circ P^{-1}$ and

$$\frac{\partial P}{\partial T_n} = -\varepsilon^{n-1} \left(L^{n/r}\right)_- \circ P, \quad n \geq 1.$$

The operator P is called a *dressing operator* of the operator L.

Denote by G_z the shift operator that acts on a formal power series $f \in \mathbb{C}[\varepsilon,\varepsilon^{-1}][[T_1,T_2,\ldots]]$ as follows:

$$G_z(f)(T_1,T_2,T_3,\ldots) := f\left(T_1 - \frac{1}{z}, T_2 - \frac{1}{2\varepsilon z^2}, T_3 - \frac{1}{3\varepsilon^2 z^3}, \ldots\right).$$

Let $P = 1+\sum_{n\geq 1} p_n(T_*,\varepsilon)\partial_x^{-n}$ be a dressing operator of some operator L satisfying the Gelfand–Dickey hierarchy. Then there exists a series $\tau \in \mathbb{C}[\varepsilon,\varepsilon^{-1}][[T_1,T_2,T_3,\ldots]]$ with constant term $\tau|_{T_i=0} = 1$ for which

$$\widehat{P} = \frac{G_z(\tau)}{\tau}.$$

The series τ is called a *tau-function* of the Gelfand–Dickey hierarchy. The operator L can be reconstructed from the tau-function τ by the following formula:

$$\operatorname{res} L^{n/r} = \varepsilon^{1-n}\frac{\partial^2 \log \tau}{\partial T_1 \partial T_n}, \quad n \geq 1.$$

Denote the linear differential operator in the brackets on the left-hand side of equation (3.8) by O. Let L be the solution of the Gelfand–Dickey hierarchy specified by the initial condition (2.1). Let us show that

$$\left(z\frac{\partial}{\partial z} + O\right)\widehat{L} = r\widehat{L}. \tag{3.9}$$

Witten's r-spin conjecture [20], proved by Faber–Shadrin–Zvonkine [11], says that the formal power series $\tau^{\frac{1}{r},c} := \exp(\mathcal{F}^{\frac{1}{r},c})$ is a tau-function of the Gelfand–Dickey hierarchy corresponding to the operator L. Therefore, a dressing operator P of the operator L is given by

$$\widehat{P} = \frac{G_z(\tau^{\frac{1}{r},c})}{\tau^{\frac{1}{r},c}}.$$

The function $\tau^{\frac{1}{r},c}$ satisfies the dilaton equation

$$O\tau^{\frac{1}{r},c} = \frac{r-1}{24}\tau^{\frac{1}{r},c}.$$

We compute

$$\left(z\frac{\partial}{\partial z} + O\right)G_z(\tau^{\frac{1}{r},c}) = G_z(O\tau^{\frac{1}{r},c}) = \frac{r-1}{24}G_z(\tau^{\frac{1}{r},c}),$$

and, thus,

$$\left(z\frac{\partial}{\partial z} + O\right)\widehat{P} = \left(z\frac{\partial}{\partial z} + O\right)\frac{G_z(\tau^{\frac{1}{r},c})}{\tau^{\frac{1}{r},c}} = 0.$$

Note that the commutation relation $\left[O, \frac{\partial}{\partial T_1}\right] = \frac{\partial}{\partial T_1}$ implies that if $\left(z\frac{\partial}{\partial z} + O\right)\widehat{A} = a\widehat{A}$ and $\left(z\frac{\partial}{\partial z} + O\right)\widehat{B} = b\widehat{B}$ for some pseudo-differential operators A, B and $a, b \in \mathbb{Z}$, then $\left(z\frac{\partial}{\partial z} + O\right)\widehat{A \circ B} = (a+b)\widehat{A \circ B}$. Since $L = P \circ \partial_x^r \circ P^{-1}$, we conclude that equation (3.9) is true.

Note also that we have

$$\left(z\frac{\partial}{\partial z} + O\right)\widehat{L^{n/r}} = n\widehat{L^{n/r}}, \quad n \geq 1, \tag{3.10}$$

and that for any pseudo-differential operator A and $f\in\mathbb{C}[[T_*]]$ we have

$$O(A_+f)=A_+(Of)+\left(\left(z\frac{\partial}{\partial z}+O\right)\widehat{A}_+\Big|_{z\mapsto\partial_x}\right)f. \tag{3.11}$$

Let us finally prove equation (3.8). Equivalently, we have to prove that $O\Phi=\frac{1}{2}\Phi$. We compute

$$\begin{aligned} L^{\frac{1}{r}}\Big|_{T_{\ge 2}=0}&=\partial_x+\varepsilon^{-r}x\partial_x^{-r+1}-\frac{r-1}{2}\varepsilon^{-r}\partial_x^{-r}+\ldots,\\ (L^{\frac{r+1}{r}})_+\Big|_{T_{\ge 2}=0}&=\partial_x^{r+1}+(r+1)\varepsilon^{-r}x\partial_x+\frac{r+1}{2}\varepsilon^{-r}. \end{aligned}$$

Taking into account that $\Phi|_{T_{\ge 2}=0}=1$, we obtain

$$O\Phi|_{T_{\ge 2}=0}=\frac{1}{r+1}\frac{\partial\Phi}{\partial T_{r+1}}\Big|_{T_{\ge 2}=0}=\frac{\varepsilon^r}{r+1}(\widehat{L^{\frac{r+1}{r}}})_+\Big|_{z=T_{\ge 2}=0}=\frac{1}{2}. \tag{3.12}$$

We also have

$$\frac{\partial}{\partial T_n}(O\Phi)=(O-1)(\varepsilon^{n-1}(L^{n/r})_+\Phi)\overset{\text{eqs. (3.10),(3.11)}}{=}\varepsilon^{n-1}(L^{n/r})_+(O\Phi).$$

We see that the formal power series $O\Phi$ satisfies the same system of PDEs (2.2) as the formal power series Φ, with the initial condition (3.12). Thus, $O\Phi=\frac{1}{2}\Phi$ and the proposition is proved. □

Clearly, if Conjecture 1 is true, then the proposition implies the expected equation (3.6).

3.4. Open topological recursion relations in genus one.

Theorem 1. *We have*

$$\frac{\partial\phi_1}{\partial t^\alpha_{p+1}}=\sum_{\mu+\nu=r-2}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial t^\alpha_p\partial t^\mu_0}\frac{\partial\phi_1}{\partial t^\nu_0}+\frac{\partial\phi_0}{\partial t^\alpha_p}\frac{\partial\phi_1}{\partial t^{r-1}_0}+\frac{1}{2}\frac{\partial^2\phi_0}{\partial t^\alpha_p\partial t^{r-1}_0},\quad 0\le\alpha\le r-1,\quad p\ge 0.$$

Before proving the theorem, let us discuss some consequences.

Corollary 3.2. *The generating series of intersection numbers on $\overline{\mathcal{M}}_{1,k,l}$ satisfies the relation*

$$\frac{\partial\mathcal{F}_1^o}{\partial t_{p+1}}=\frac{\partial^2\mathcal{F}_0^c}{\partial t_p\partial t_0}\frac{\partial\mathcal{F}_1^o}{\partial t_0}+\frac{\partial\mathcal{F}_0^o}{\partial t_p}\frac{\partial\mathcal{F}_1^o}{\partial s}+\frac{1}{2}\frac{\partial^2\mathcal{F}_0^o}{\partial t_p\partial s},\quad p\ge 0. \tag{3.13}$$

Proof. This follows from the theorem and equation (3.2). □

The system of relations (3.13) was conjectured by the authors of [15] and called the *open topological recursion relations in genus one*. A geometric proof will appear in [17].

We expect the following topological recursion relations for the genus-one open r-spin intersection numbers for any $0\le\alpha\le r-1$ and $p\ge 0$:

$$\frac{\partial\mathcal{F}_1^{\frac{1}{r},o}}{\partial t^\alpha_{p+1}}=\sum_{\mu+\nu=r-2}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial t^\alpha_p\partial t^\mu_0}\frac{\partial\mathcal{F}_1^{\frac{1}{r},o}}{\partial t^\nu_0}+\frac{\partial\mathcal{F}_0^{\frac{1}{r},\text{ext}}}{\partial t^\alpha_p}\frac{\partial\mathcal{F}_1^{\frac{1}{r},o}}{\partial t^{r-1}_0}+\frac{\partial\mathcal{F}_0^{\frac{1}{r},o}}{\partial t^\alpha_p}\frac{\partial\mathcal{F}_1^{\frac{1}{r},o}}{\partial s}+\frac{1}{2}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},o}}{\partial t^\alpha_p\partial s}, \tag{3.14}$$

where $\mathcal{F}_0^{\frac{1}{r},\text{ext}}(t^0_*,\ldots,t^{r-1}_*)$ is the generating series of genus-zero closed extended r-spin intersection numbers defined in the same way as the usual genus-zero r-spin intersection numbers, but where exactly one of a_i-s is equal to -1. In [5, Theorem 4.6], we proved that

$$\mathcal{F}_0^{\frac{1}{r},\text{ext}}=\sqrt{-r}\phi_0\Big|_{t^{r-1}_d\mapsto\frac{1}{\sqrt{-r}}t^{r-1}_d}.$$

The geometric proof of (3.13) from [17] cannot work for the new system (3.14), due to the lack of a rigorous construction of the open virtual fundamental cycle. Still, as in the case of the open string and dilaton equations, the claim is expected to be true, and with a similar proof, under some mild assumptions on the open virtual fundamental cycle.

It is easy to see that if Conjecture 1 is true, then Theorem 1 implies the system of relations (3.14).

Proof of Theorem 1. Equivalently, we have to prove that

$$\frac{\partial\phi_1}{\partial T_{a+r}}=\sum_{b=1}^{r-1}\frac{a+r}{b(r-b)}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}\frac{\partial\phi_1}{\partial T_{r-b}}+\frac{a+r}{r}\frac{\partial\phi_0}{\partial T_a}\frac{\partial\phi_1}{\partial T_r}+\frac{a+r}{2r}\frac{\partial^2\phi_0}{\partial T_a\partial T_r},\quad a\geq 1. \tag{3.15}$$

In [5, Lemma 4.2], we proved that the operator L has the form

$$L=\partial_x^r+\sum_{i=0}^{r-2}\sum_{j\geq 0}\varepsilon^{i-r+j}f_i^{[j]}\partial_x^i,\quad f_i^{[j]}\in\mathbb{C}[[T_*]].$$

Denote

$$L_0:=\partial_x^r+\sum_{i=0}^{r-2}f_i^{[0]}\partial_x^i,\qquad L_1:=\sum_{i=0}^{r-2}f_i^{[1]}\partial_x^i,\qquad (f_i^{[0]})^{(k)}:=\partial_x^k f_i^{[0]}.$$

For any $a\geq 1$, the Laurent series $\widehat{L_0^{\frac{a}{r}}}$ has the form

$$\widehat{L_0^{\frac{a}{r}}}=\sum_{i=-\infty}^{a}P_i((f_*^{[0]})^{(*)})z^i,$$

where P_i are polynomials in $(f_j^{[0]})^{(k)}$ for $0\leq j\leq r-2$ and $k\geq 0$. Let us assign to $(f_j^{[0]})^{(k)}$ differential degree k. Then we can decompose the polynomials $P_i((f_*^{[0]})^{(*)})$ as $P_i((f_*^{[0]})^{(*)})=\sum_{m\geq 0}P_{i,m}((f_*^{[0]})^{(*)})$, where a polynomial $P_{i,m}$ has differential degree m. Introduce the following notations:

$$\left(\widehat{L_0^{\frac{a}{r}}}\right)_m:=\sum_{i=-\infty}^{a}P_{i,m}((f_*^{[0]})^{(*)})z^i,\qquad \left(\widehat{L_0^{\frac{a}{r}}}\right)_{+,m}:=\sum_{i=0}^{a}P_{i,m}((f_*^{[0]})^{(*)})z^i.$$

Lemma 3.3. *For any $a\geq 1$ we have*

$$\frac{\partial\phi_1}{\partial T_a}=\left[(\phi_1)_x\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_++\frac{(\phi_0)_{xx}}{2}\partial_z^2\left(\widehat{L}_0^{\frac{a}{r}}\right)_++\left(\widehat{L_0^{\frac{a}{r}}}\right)_{+,1}+\frac{a}{r}\left(\widehat{L}_0^{\frac{a}{r}-1}\widehat{L}_1\right)_+\right]\Bigg|_{z=(\phi_0)_x}, \tag{3.16}$$

where ∂_z denotes the partial derivative $\frac{\partial}{\partial z}$.

Proof. We have $\frac{\partial\phi}{\partial T_a}=\varepsilon^{a-1}\frac{(L^{a/r})_+e^\phi}{e^\phi}$. The operator $L^{\frac{a}{r}}$ has the form $L^{\frac{a}{r}}=\sum_{i=-\infty}^{a}R_i\partial_x^i$, where $R_i=\sum_{j\geq 0}\varepsilon^{i-a+j}R_{i,j}$, $R_{i,j}\in\mathbb{C}[[T_*]]$. We have to check that

$$\begin{aligned}\operatorname{Coef}_{\varepsilon^0}\left[\varepsilon^{a-1}\sum_{i=0}^{a}R_i\frac{\partial_x^ie^\phi}{e^\phi}\right]=&\\ =\left[(\phi_1)_x\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_++\frac{(\phi_0)_{xx}}{2}\partial_z^2\left(\widehat{L}_0^{\frac{a}{r}}\right)_++\left(\widehat{L_0^{\frac{a}{r}}}\right)_{+,1}+\frac{a}{r}\left(\widehat{L}_0^{\frac{a}{r}-1}\widehat{L}_1\right)_+\right]\Bigg|_{z=(\phi_0)_x}&.\end{aligned} \tag{3.17}$$

By the induction on i, it is easy to prove that

$$\frac{\partial_x^i e^\phi}{e^\phi}=i!\sum_{\substack{m_1,m_2,\ldots\ge 0\\ \sum jm_j=i}}\prod_{j\ge 1}\frac{(\partial_x^j\phi)^{m_j}}{(j!)^{m_j}m_j!},\quad i\ge 0.$$

We have

$$\prod_{j\ge 1}(\partial_x^j\phi)^{m_j}=\begin{cases}\varepsilon^{-i}(\phi_0)_x^i+\varepsilon^{-i+1}i(\phi_0)_x^{i-1}(\phi_1)_x+O(\varepsilon^{-i+2}), & \text{if } m_1=i \text{ and } m_{\ge 2}=0,\\ \varepsilon^{-i+1}(\phi_0)_x^{i-2}(\phi_0)_{xx}+O(\varepsilon^{-i+2}), & \text{if } m_1=i-2,\ m_2=1 \text{ and } m_{\ge 3}=0,\\ O(\varepsilon^{-i+2}), & \text{otherwise.}\end{cases}$$

As a result,

$$\frac{\partial_x^i e^\phi}{e^\phi}=\varepsilon^{-i}(\phi_0)_x^i+\varepsilon^{-i+1}\left(i(\phi_0)_x^{i-1}(\phi_1)_x+\frac{i(i-1)}{2}(\phi_0)_x^{i-2}(\phi_0)_{xx}\right)+O(\varepsilon^{-i+2}),$$

and

(3.18)
$$\mathrm{Coef}_{\varepsilon^0}\left[\varepsilon^{a-1}\sum_{i=0}^a R_i\frac{\partial_x^i e^\phi}{e^\phi}\right]=\sum_{i=0}^a R_{i,0}\left(i(\phi_0)_x^{i-1}(\phi_1)_x+\frac{i(i-1)}{2}(\phi_0)_x^{i-2}(\phi_0)_{xx}\right)+\sum_{i=0}^a R_{i,1}(\phi_0)_x^i.$$

Note that

$$\sum_{i=0}^a R_{i,0}z^i=\left(\widehat{L}_0^{\frac{a}{r}}\right)_+,\qquad \sum_{i=0}^a R_{i,1}z^i=\left(\widehat{L_0^{\frac{a}{r}}}\right)_{+,1}+\frac{a}{r}\left(\widehat{L}_0^{\frac{a}{r}-1}\widehat{L}_1\right)_+.$$

We can see now that the first sum on the right-hand side of (3.18) gives the first and the second terms on the right-hand side of (3.17). The second sum on the right-hand side of (3.18) gives the third and the fourth terms on the right-hand side of (3.17). This completes the proof of the lemma. □

In [5, Lemma 4.7], we proved that

$$\frac{\partial\phi_0}{\partial T_a}=\left.\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\right|_{z=(\phi_0)_x},\tag{3.19}$$

which implies

$$\frac{\partial^2\phi_0}{\partial T_a\partial T_r}=\frac{\partial}{\partial T_r}\left[\left.\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\right|_{z=(\phi_0)_x}\right]=\left.\left((\phi_0)_{xx}\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_z\widehat{L}_0+\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_x\widehat{L}_0\right)\right|_{z=(\phi_0)_x}.$$

Therefore, equation (3.15) is equivalent to

$$\begin{aligned}\frac{\partial\phi_1}{\partial T_{a+r}}=&\sum_{b=1}^{r-1}\frac{a+r}{b(r-b)}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}\frac{\partial\phi_1}{\partial T_{r-b}}+\frac{a+r}{r}\frac{\partial\phi_0}{\partial T_a}\frac{\partial\phi_1}{\partial T_r}+\\&+\frac{a+r}{2r}\left.\left((\phi_0)_{xx}\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_z\widehat{L}_0+\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_x\widehat{L}_0\right)\right|_{z=(\phi_0)_x}.\end{aligned}$$

By formulas (3.16) and (3.19), this is equivalent to the equation

$$\underline{(\phi_1)_x\partial_z\left(\widehat{L}_0^{\frac{a+r}{r}}\right)_+}+\underline{\underline{\frac{(\phi_0)_{xx}}{2}\partial_z^2\left(\widehat{L}_0^{\frac{a+r}{r}}\right)_+}}+\boxed{\left(\widehat{L_0^{\frac{a+r}{r}}}\right)_{+,1}}+\frac{a+r}{r}\left(\widehat{L}_0^{\frac{a}{r}}\widehat{L}_1\right)_+=$$

$$=\sum_{b=1}^{r-1}\frac{a+r}{b(r-b)}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}\left[\underline{(\phi_1)_x\partial_z\left(\widehat{L}_0^{\frac{r-b}{r}}\right)_+}+\underline{\underline{\frac{(\phi_0)_{xx}}{2}\partial_z^2\left(\widehat{L}_0^{\frac{r-b}{r}}\right)_+}}+\boxed{\left(\widehat{L_0^{\frac{r-b}{r}}}\right)_{+,1}}+\frac{r-b}{r}\left(\widehat{L}_0^{-\frac{b}{r}}\widehat{L}_1\right)_+\right]$$

$$+\frac{a+r}{r}\widehat{L}_0^{\frac{a}{r}}\left[\underline{(\phi_1)_x\partial_z\widehat{L}_0}+\underline{\underline{\frac{(\phi_0)_{xx}}{2}\partial_z^2\widehat{L}_0}}+\widehat{L}_1\right]$$

$$+\frac{a+r}{2r}\left(\underline{\underline{(\phi_0)_{xx}\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_z\widehat{L}_0}}+\boxed{\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_x\widehat{L}_0}\right)$$

where we should substitute $z=(\phi_0)_x$. Collecting together the terms marked in the same way, we see that this equation is a consequence of the following four equations:

$$\partial_z\left(\widehat{L}_0^{\frac{a+r}{r}}\right)_+=\sum_{b=1}^{r-1}\frac{a+r}{b(r-b)}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}\partial_z\left(\widehat{L}_0^{\frac{r-b}{r}}\right)_++\frac{a+r}{r}\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_z\widehat{L}_0, \tag{3.20}$$

$$\begin{aligned}\partial_z^2\left(\widehat{L}_0^{\frac{a+r}{r}}\right)_+=&\sum_{b=1}^{r-1}\frac{a+r}{b(r-b)}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}\partial_z^2\left(\widehat{L}_0^{\frac{r-b}{r}}\right)_++\frac{a+r}{r}\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_z^2\widehat{L}_0+\\&+\frac{a+r}{r}\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_z\widehat{L}_0,\end{aligned} \tag{3.21}$$

$$\left(\widehat{L_0^{\frac{a+r}{r}}}\right)_{+,1}=\sum_{b=1}^{r-1}\frac{a+r}{b(r-b)}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}\left(\widehat{L_0^{\frac{r-b}{r}}}\right)_{+,1}+\frac{a+r}{2r}\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_x\widehat{L}_0, \tag{3.22}$$

$$\frac{a+r}{r}\left(\widehat{L}_0^{\frac{a}{r}}\widehat{L}_1\right)_+=\sum_{b=1}^{r-1}\frac{a+r}{b(r-b)}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}\frac{r-b}{r}\left(\widehat{L}_0^{-\frac{b}{r}}\widehat{L}_1\right)_++\frac{a+r}{r}\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\widehat{L}_1. \tag{3.23}$$

In [5, equation (4.19)], we proved that

$$d\left(\widehat{L}_0^{\frac{a+r}{r}}\right)_+=\sum_{b=1}^{r-1}\frac{a+r}{b(r-b)}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}d\left(\widehat{L}_0^{\frac{r-b}{r}}\right)_++\frac{a+r}{r}\left(\widehat{L}_0^{\frac{a}{r}}\right)_+d\widehat{L}_0,$$

from which equation (3.20) clearly follows. Applying the derivative ∂_z to both sides of (3.20), we get equation (3.21). Equation (3.23) follows from the property [5, equation (4.22)]

$$\left(\widehat{L}_0^{\frac{a}{r}}\right)_--\sum_{b=1}^{r-1}\frac{1}{b}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}\widehat{L}_0^{-\frac{b}{r}}\in z^{-r-1}\mathbb{C}[f_*^{[0]}][[z^{-1}]]. \tag{3.24}$$

It remains to prove equation (3.22). Let us first prove that for any $a\ge 1$ we have

$$\left(\widehat{L_0^{\frac{a}{r}}}\right)_1=\frac{a(a-r)}{2r^2}\widehat{L}_0^{\frac{a}{r}-2}\partial_z\widehat{L}_0\partial_x\widehat{L}_0. \tag{3.25}$$

It is easy to see that

$$\begin{aligned}\left(\widehat{\left(L_0^{\frac{1}{r}}\right)^r}\right)_1 =& r\widehat{L}_0^{\frac{r-1}{r}}\left(\widehat{L_0^{\frac{1}{r}}}\right)_1+\sum_{j=0}^{r-1}\widehat{L}_0^{\frac{r-j-1}{r}}\partial_z\widehat{L}_0^{\frac{1}{r}}\partial_x\widehat{L}_0^{\frac{j}{r}}=\\ =& r\widehat{L}_0^{\frac{r-1}{r}}\left(\widehat{L_0^{\frac{1}{r}}}\right)_1+\sum_{j=0}^{r-1}\frac{j}{r^2}\widehat{L}_0^{-1}\partial_z\widehat{L}_0\partial_x\widehat{L}_0=\\ =& r\widehat{L}_0^{\frac{r-1}{r}}\left(\widehat{L_0^{\frac{1}{r}}}\right)_1+\frac{r-1}{2r}\widehat{L}_0^{-1}\partial_z\widehat{L}_0\partial_x\widehat{L}_0.\end{aligned}$$

Since we obviously have $\left(\widehat{\left(L^{\frac{1}{r}}\right)^r}\right)_1=0$, equation (3.25) is proved for $a=1$. For an arbitrary $a\geq 1$, we compute

$$\begin{aligned}\left(\widehat{\left(L_0^{\frac{1}{r}}\right)^a}\right)_1 =& a\widehat{L}_0^{\frac{a-1}{r}}\left(\widehat{L_0^{\frac{1}{r}}}\right)_1+\sum_{j=0}^{a-1}\widehat{L}_0^{\frac{a-1-j}{r}}\partial_z\widehat{L}_0^{\frac{1}{r}}\partial_x\widehat{L}_0^{\frac{j}{r}}=\\ =& \frac{a(1-r)}{2r^2}\widehat{L}_0^{\frac{a}{r}-2}\partial_z\widehat{L}_0\partial_x\widehat{L}_0+\sum_{j=0}^{a-1}\frac{j}{r^2}\widehat{L}_0^{\frac{a}{r}-2}\partial_z\widehat{L}_0\partial_x\widehat{L}_0=\\ =& \frac{a(a-r)}{2r^2}\widehat{L}_0^{\frac{a}{r}-2}\partial_z\widehat{L}_0\partial_x\widehat{L}_0.\end{aligned}$$

Thus, equation (3.25) is proved.

Let us rewrite formula (3.25) in the following way: $\left(\widehat{L_0^{\frac{a}{r}}}\right)_1=\frac{a}{2r}\partial_z\widehat{L}_0^{\frac{a}{r}-1}\partial_x\widehat{L}_0$. Then we see that equation (3.22) is equivalent to

$$\frac{a+r}{2r}\left(\partial_z\widehat{L}_0^{\frac{a}{r}}\partial_x\widehat{L}_0\right)_+=\sum_{b=1}^{r-1}\frac{a+r}{2rb}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}\left(\partial_z\widehat{L}_0^{-\frac{b}{r}}\partial_x\widehat{L}_0\right)_+ +\underline{\frac{a+r}{2r}\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_x\widehat{L}_0}. \tag{3.26}$$

We obviously have

$$\frac{a+r}{2r}\left(\partial_z\widehat{L}_0^{\frac{a}{r}}\partial_x\widehat{L}_0\right)_+=\underline{\frac{a+r}{2r}\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_+\partial_x\widehat{L}_0}+\frac{a+r}{2r}\left(\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_-\partial_x\widehat{L}_0\right)_+.$$

Note that the underlined term here cancels the underlined term on the right-hand side of (3.26). Therefore, equation (3.26) is equivalent to the identity

$$\left(\partial_z\left(\widehat{L}_0^{\frac{a}{r}}\right)_-\partial_x\widehat{L}_0\right)_+=\sum_{b=1}^{r-1}\frac{1}{b}\frac{\partial^2\mathcal{F}_0^{\frac{1}{r},c}}{\partial T_a\partial T_b}\left(\partial_z\widehat{L}_0^{-\frac{b}{r}}\partial_x\widehat{L}_0\right)_+,$$

which follows from equation (3.24). The theorem is proved. □

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