ISSN 1560-3547, Regular and Chaotic Dynamics, 2023, Vol. 28, No. 6, pp. 865–877. © Pleiades Publishing, Ltd., 2023.

Circular Fleitas Scheme for Gradient-Like Flows on the Surface

Vladislav D. Galkin[1*], **Elena V. Nozdrinova**[1**], **and Olga V. Pochinka**[1***]

[1] *HSE University,*
ul. Bolshaya Pecherckaya 25/12, 603155 Nizhny Novgorod, Russia
Received February 03, 2023; revised August 06, 2023; accepted October 25, 2023

Abstract—In this paper, we obtain a classification of gradient-like flows on arbitrary surfaces by generalizing the circular Fleitas scheme. In 1975 he proved that such a scheme is a complete invariant of topological equivalence for polar flows on 2- and 3-manifolds. In this paper, we generalize the concept of a circular scheme to arbitrary gradient-like flows on surfaces. We prove that the isomorphism class of such schemes is a complete invariant of topological equivalence. We also solve exhaustively the realization problem by describing an abstract circular scheme and the process of realizing a gradient-like flow on the surface. In addition, we construct an efficient algorithm for distinguishing the isomorphism of circular schemes.

MSC2010 numbers: 03C15

DOI: 10.1134/S1560354723060047

Keywords: gradient-like flows, circular scheme, flows on the surface

1. INTRODUCTION

Let M^n, $n \geqslant 2$ be a closed connected n-manifold with a Riemannian metric d.

A *flow* on a manifold M^n is a continuous map $F\colon M^n \times \mathbb{R} \to M^n$ satisfying the group properties:

1) $F(x, 0) = x$, $\forall x \in M^n$;

2) $F(F(x, t), s) = F(x, t + s)$, $\forall x \in M^n$, $\forall s, t \in \mathbb{R}$.

We will use the notation $f^t(x) = F(x, t)$, $x \in M^n$, $t \in \mathbb{R}$. Notice that, for a fixed $t \in \mathbb{R}$, the map $f^t : M^n \to M^n$ is a homeomorphism (see, for example, [7]), so the flow is also called a *one-parameter group of homeomorphisms* acting on the manifold M^n.

The set $\mathcal{O}_x = \{f^t(x), t \in \mathbb{R}\}$ is called the *trajectory* or *flow orbit* of a point $x \in M^n$. Any flow orbit either consists of a unique point (in this case this point is called *fixed*), or is homeomorphic to a circle (in this case the orbit is called *periodic*), or is an injectively immersive line. It is assumed that all trajectories other than a fixed point are oriented in accordance with the increasing parameter t. Two flows $f^t : M^n \to M^n$ and $f'^t : M^n \to M^n$ are called topologically equivalent if there exists a homeomorphism $h : M^n \to M^n$ sending the trajectories of f^t to the trajectory f'^t with orientation preserved. If the homeomorphism h has the property $hf^t(x) = f'^t h(x)$ for any $t \in \mathbb{R}$, then the flows are called *topologically conjugate*.

An *ε-chain of length T* connecting a point x with a point y for a flow f^t is a sequence of points $x = x_0, \ldots, x_n = y$ for which there is a sequence of times $t_1, \ldots, t_n$ such that $d\big(f^{t_i}(x_{i-1}), x_i\big) < \varepsilon$,

[*] E-mail: vgalkin@hse.ru
[**] E-mail: enozdrinova@hse.ru
[***] E-mail: opochinka@hse.ru

$t_i \geqslant 1$ for $1 \leqslant i \leqslant n$ and $t_1 + \dots + t_n = T$ (see Fig. 1). A point $x \in M^n$ is called *chain recurrent* for a flow f^t if for any $\varepsilon > 0$ there exists $T > 0$ depending on $\varepsilon > 0$, and an ε-chain of length T, connecting the point x with itself. The set of all chain recurrent points is called the *chain recurrent set* and is denoted by $\mathcal{R}_{f^t}$. If the chain recurrent set of the flow is finite, then it consists of fixed points.

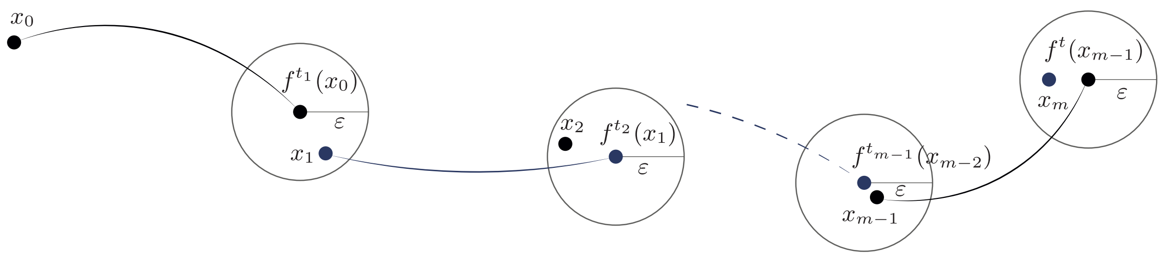

Fig. 1. ε-chain of length T.

Obviously, a fixed point p of a flow f^t is chain recurrent. *The stable and unstable* manifolds of a fixed point p are defined, respectively, as the sets:

$$W^s_p = \{x \in M^n : \lim_{t \to +\infty} d\big(p, f^t(x)\big) = 0\}, \;\; W^u_p = \{x \in M^n : \lim_{t \to -\infty} d\big(p, f^t(x)\big) = 0\}.$$

Following [8], we call a fixed point p of the flow f^t *hyperbolic* if there exists a neighborhood $U_p \subset M^n$ of p, a number $\lambda_p \in \{0, 1, \dots, n\}$ and a homeomorphism $h_p : U_p \to \mathbb{R}^n$, conjugating the flow $f^t|_{U_p}$ with the linear flow $a^t_{\lambda_p} : \mathbb{R}^n \to \mathbb{R}^n$, given by the formula

$$a^t_{\lambda_p}(x_1, \dots, x_{\lambda_p}, x_{\lambda_p+1}, \dots, x_n) = (2^t x_1, \dots, 2^t x_{\lambda_p}, 2^{-t} x_{\lambda_p+1}, \dots, 2^{-t} x_n).$$

The number λ_p is called the *Morse index* of the hyperbolic point p. The index points n and 0 are called a *source* and a *sink*, respectively, otherwise the point p is called a *saddle* (see Fig. 2).

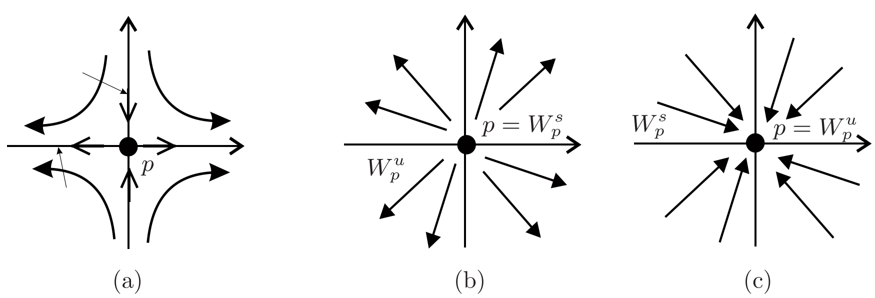

Fig. 2. Dynamics in the neighborhood of a hyperbolic fixed point: (a) saddle point, (b) source point, (c) sink point.

Proposition 1 ([14, Theorem 1]). *Let $f^t : M^n \to M^n$ be a flow with a finite hyperbolic chain recurrent set. Then*

1) $M^n = \bigcup\limits_{p \in \mathcal{R}_{f^t}} W^u_p = \bigcup\limits_{p \in \mathcal{R}_{f^t}} W^s_p$;

2) *the unstable (resp., unstable) manifold W^u_p (resp., W^s_p) of a fixed point p is a topological submanifold of a manifold M^n, homeomorphic to $\mathbb{R}^{\lambda_p}(\mathbb{R}^{n-\lambda_p})$;*

3) $cl(W^u_p) \setminus W^u_p \subset \bigcup\limits_{q \in \mathcal{R}_{f^t} : W^s_q \cap W^u_p \neq \emptyset} W^u_q \; \Big(cl(W^s_p) \setminus W^s_p \subset \bigcup\limits_{q \in \mathcal{R}_{f^t} : W^u_q \cap W^s_p \neq \emptyset} W^s_q\Big)$.

A flow $f^t : M^2 \to M^2$ is called a *gradient-like flow* if its chain recurrent set is finite and hyperbolic and the stable and unstable manifolds of different saddle points do not intersect. In this case *invariant* (stable or unstable) manifolds of every saddle point p of f^t have dimension one, each of the sets $W^s_p \setminus p$, $W^u_p \setminus p$ consists of two connected components named *separatrices*. The flows of the class under consideration have the simplest dynamics, which has inspired many mathematicians to search for invariants of their topological equivalence.

Under the assumptions of different generality for the class under consideration, the following invariants were obtained: the Peixoto graph (M. Peixoto [13]), the equipped Peixoto graph (V. Grines, O. Pochinka [5]), two-color graph (K. Wong [15]), three-color graph (A. Oshemkov, V. Sharko [11]), and circular scheme (G. Fleitas [2]).

In particular, the circular Fleitas scheme was constructed as a complete equivalence invariant for *polar flows* (flows with one sink and one source) on the surface. The scheme consists of a circle around the source point with intersections with saddle stable manifolds marked on it. For every saddle point such an intersection consists of two points, marked by a *spin* which is + (-) if the union of a disk bounded by a circle and a tubular neighborhood of a stable manifold of the saddle point is an annulus (a Möbius band) (see Figs. 3, 4). Two circular schemes are called *isomorphic* if there is a circle homeomorphism preserving the pairs of points and their spins. The isomorphic class of such a scheme is a complete invariant of topological equivalence of a polar flow $f^t : M^2 \to M^2$.

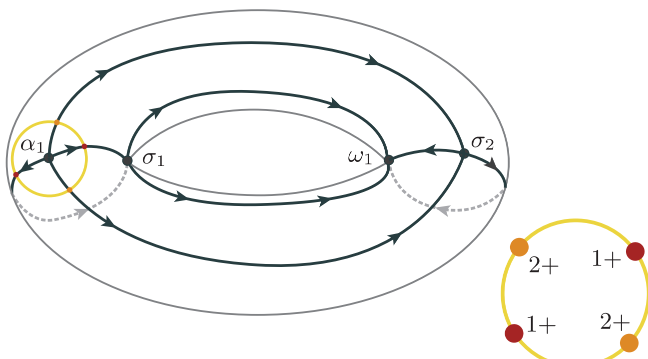

Fig. 3. Polar flow f^t on the torus and its circular scheme.

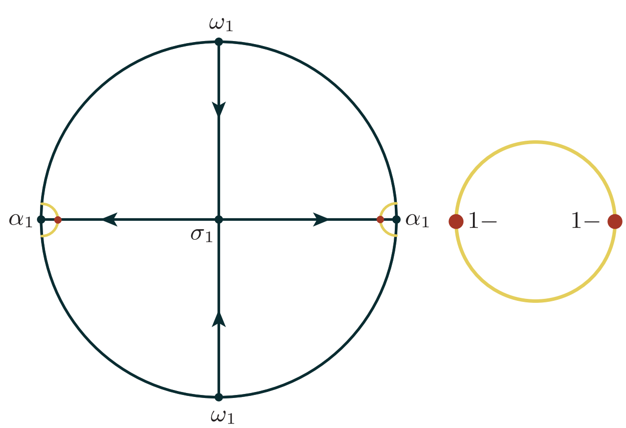

Fig. 4. Polar flow f^t on a projective plane with points of the negative spin.

In this paper we generalize the circular scheme to gradient-like flows. In more detail.

Let $f^t : M^2 \to M^2$ be a gradient-like flow. Denote by $\Omega^\lambda_{f^t}$, $\lambda \in \{0, 1, 2\}$ the set of fixed points of the flow f^t with the Morse index λ. Directly from Proposition 1 we conclude that the sets $\Omega^0_{f^t}$

and $\Omega^2_{f^t}$, the sink and source points, respectively, are not empty for any gradient-like flow f^t. For any subset of $P \subset \mathcal{R}_{f^t}$ we will assume $W^s_P = \bigcup\limits_{p\in P} W^s_p$, $W^u_P = \bigcup\limits_{p\in P} W^u_p$. For any (possibly empty) set $\delta \subset \Omega^1_{f^t}$ of saddle points we put

$$\Omega_\delta = \Omega^0_{f^t} \cup \delta,\ A_\delta = W^u_{\Omega_\delta}.$$

In [4] it is proved that, for any gradient-like flow $f^t : M^2 \to M^2$, the set A_δ is an attractor[1)] of the flow f^t and has a trapping neighborhood U_δ whose boundary Σ_δ intersects every flow trajectory in $f^t|_{W^s_{\Omega_\delta}\setminus A_\delta}$ at exactly one point.

Proposition 2 ([4, Theorem 1]). *For any gradient-like flow $f^t\colon M^2 \to M^2$ there is a set $\delta_* \subset \Omega^1_{f^t}$ consisting of $|\Omega^0_{f^t}| - 1$ points and such that $U_{\delta_*} \cong \mathbb{D}^2$.*

Let $\Sigma_{\delta_*} \cong \mathbb{S}^1$, $L^s_{\delta_*} = \{W^s_\sigma \cap \Sigma_{\delta_*},\ \sigma \in \delta_*\}$, $L^u_{\delta_*} = \{W^u_\sigma \cap \Sigma_{\delta_*},\ \sigma \in (\Omega^1_{f^t} \setminus \delta_*)\}$. Every element of the set $L^s_{\delta_*}$ ($L^u_{\delta_*}$) is a pair of intersection points of the circle Σ_{δ_*} with the stable (unstable) saddle manifold W^s_σ (W^u_σ) of a saddle point $\sigma \in \delta_*$ ($\sigma \in \Omega^1_{f^t} \setminus \delta_*$). Pairs of points of the set $L^u_{\delta_*}$ are marked by the spin $+$ ($-$) if the union of the disk U_{δ_*} with the tubular neighborhood of the unstable manifold W^u_σ of the corresponding saddle σ is homeomorphic to the annulus (Möbius band). The set

$$S_{\delta_*} = (\Sigma_{\delta_*},\ L^s_{\delta_*},\ L^u_{\delta_*})$$

is called *a circular scheme* of a gradient-like flow $f^t : M^2 \to M^2$ (see Figs. 5, 6). Two circular schemes S_{δ_*} and $S_{\delta'_*}$ of gradient-like flows $f^t : M^2 \to M^2$ and $f'^t : M'^2 \to M'^2$ are called *equivalent* if there is a homeomorphism $\psi : \Sigma_{\delta_*} \to \Sigma_{\delta'_*}$, sending pairs of points of sets $L^s_{\delta_*}$, $L^u_{\delta_*}$ into pairs of points of sets $L^s_{\delta'_*}$, $L^u_{\delta'_*}$, respectively, with the spins being preserved[2)].

Since there is no unique way to choose A_{δ_*} (see Figs. 5, 6), we denote by S_{f^t} the set of all possible different circular schemes of gradient-like flow $f^t : M^2 \to M^2$. The sets S_{f^t} and $S_{f'^t}$ of gradient-like flows f^t and f'^t are called *equivalent* if they contain equivalent circular schemes $S_{\delta_*} \in S_{f^t}$ and $S_{\delta'_*} \in S_{f'^t}$.

Next, the main result of this paper follows.

Theorem 1. *Let f^t, f'^t be gradient-like flows which are topologically equivalent if and only if the sets of their circular schemes S_{f^t}, $S_{f'^t}$ are equivalent.*

Corollary 1. *It follows from Theorem 1 that, if S_{f^t} and $S_{f'^t}$ are equivalent, then all their circular schemes are pairwise equivalent.*

To solve the realization problem, we introduce the concept of an abstract circular scheme. Let $\Sigma = \mathbb{S}^1$ and L^s, $L^u \subset \Sigma$ be sets of pairs of pairwise distinct points having the following properties:

1) the paired points in L^s are arranged so that the chords joining them are pairwise disjoint;

2) the paired points in L^u are marked by $+$ or $-$.

[1)] An invariant set $A \subset M^n$ of a flow $f^t : M^n \to M^n$ is called an *attractor* if it has a closed neighborhood U_A, which is called *trapping*, such that $f^t(U_A) \subset \operatorname{int} U_A$ for $t > 0$ and $\bigcap\limits_{t>0} f^t(U_A) = A$.

[2)] Notice that an invariant similar to the circular scheme was used in [6] for a description of the connected components of gradient-like vector fields on closed surfaces.

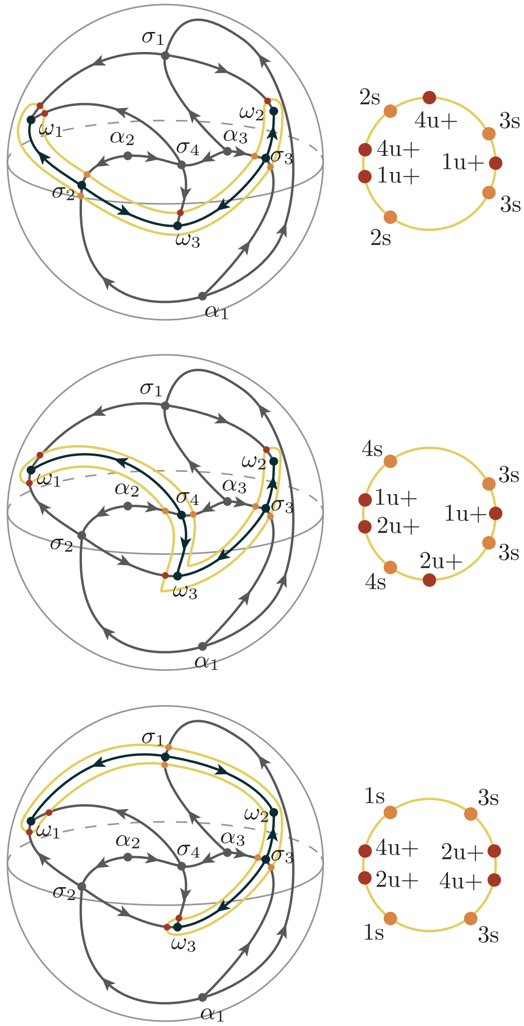

Fig. 5. Gradient-like flow on a sphere with a set of all possible circular schemes.

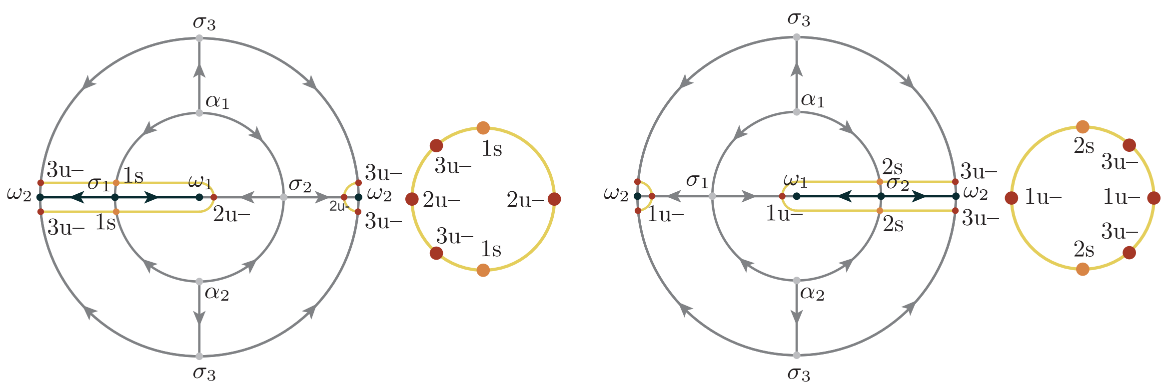

Fig. 6. Gradient-like flow on a projective plane with a set of all possible circular schemes.

A collection

$$S = (\Sigma, L^s, L^u)$$

with properties 1, 2 above will be called *an abstract circular scheme*. Obviously, the circular scheme of any gradient-like flow $f^t : M^2 \to M^2$ is equivalent to some abstract scheme.

Denote by k^s the number of paired points in L^s, by k^u_+ (k^u_-) the number of the paired points with spin + (-) in L^u. Let $k^u = k^u_- + k^u_+$ and $k = k^s + k^u$. We assume that the circle Σ is counterclockwise oriented and the paired points in L^u are numbered: $(z_1, y_1), \ldots, (z_{k^u}, y_{k^u})$. Select the arcs $[a_i, b_i]$, $[c_i, d_i] \subset \Sigma$, $i \in \{1, \ldots, k^u\}$ (oriented consistent with the orientation of the circle Σ) so that $[a_i, b_i] \cap (L^u \cup L^s) = z_i$, $[c_i, d_i] \cap (L^u \cup L^s) = y_i$. Now, choose pairwise disjoint arcs $\gamma_{a_1}, \gamma_{b_1}, \ldots, \gamma_{a_{k^u}}, \gamma_{b_{k^u}}$ whose interiors avoid Σ, as follows:

- $\partial\gamma_{a_i} = a_i \sqcup d_i$, $\partial\gamma_{b_i} = b_i \sqcup c_i$ if a pair of points z_i, y_i has a spin +;
- $\partial\gamma_{a_i} = a_i \sqcup c_i$, $\partial\gamma_{b_i} = b_i \sqcup d_i$ if a pair of points z_i, y_i has a spin −.

Let $C^u = \left(\Sigma \setminus \bigcup\limits_{i=1}^{k^u} \big((a_i, b_i) \cup (c_i, d_i)\big)\right) \cup \bigcup\limits_{i=1}^{k^u} (\gamma_{a_i} \cup \gamma_{b_i})$ (see Fig. 7). Denote by m^u the number of the connected components of the set C^u.

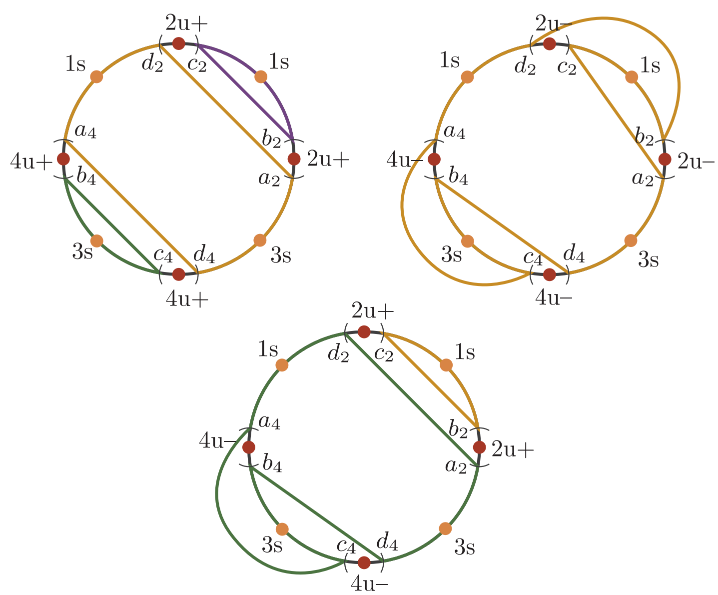

Fig. 7. Building a set C^u.

Theorem 2. *For every abstract circular scheme S there is a gradient-like flow $f^t : M^2 \to M^2$ with a circular scheme $S_{\delta*}$ equivalent to S. Also, the surface M^2 is orientable (nonorientable) if and only if the scheme S does not contain (contains) a point with negative spin, and its genus g is calculated by the formula*

$$g = \frac{k^u - m^u + 1}{2} \quad (g = k^u - m^u + 1).$$

Theorem 3. *Let $S = (\Sigma, L^s, L^u)$ and $S' = (\Sigma, L'^s, L'^u)$ be abstract circular schemes such that $k^s = k'^s$, $k^u_- = k'^u_-$, $k^u_+ = k'^u_+$. Then there is an efficient (polynomial-dependent on $k = k^s + k^u_- + k^u_+$) algorithm for distinguishing their isomorphism.*

2. THE CIRCULAR SCHEME IS A COMPLETE EQUIVALENCE INVARIANT OF GRADIENT-LIKE FLOWS ON SURFACES

In this section we prove Theorem 1: gradient-like flows f^t and f'^t on surfaces are topologically equivalent if and only if the sets of their circular schemes S_{f^t} and $S_{f'^t}$ are equivalent.

Proof.

Necessity. Let gradient-like flows $f^t : M^2 \to M^2$, $f'^t : M'^2 \to M'^2$ be topologically equivalent by means of a homeomorphism $h : M^2 \to M'^2$. Let us show that the sets of circular schemes S_{f^t} and $S_{f'^t}$ are equivalent.

Let $S_{\delta_*} = (\Sigma_{\delta_*}, L^s_{\delta_*}, L^u_{\delta_*})$ be a circular scheme of flow f^t and $\delta'_* = h(\delta_*)$. Let us show that $S_{\delta'_*}$ is a circular scheme for f'^t which is equivalent to S_{δ_*}, which completes the proof.

Indeed, $h(A_{\delta_*}) = A_{\delta'_*}$ and the disk $h(U_{\delta_*})$ is a trapping neighborhood of the attractor $A_{\delta'_*}$. The homeomorphism h sends the circle Σ_{δ_*} into the circle $h(\Sigma_{\delta_*})$ which intersects every trajectory in $f'^t|_{W^s_{\delta'_*} \setminus A_{\delta'_*}}$ at the unique point, similar to the circle $\Sigma_{\delta'_*}$. Define homeomorphism $\psi : \Sigma_{\delta_*} \to \Sigma_{\delta'_*}$ by the formula

$$\psi(y) = f'^{\tau_y}\big(h(y)\big),$$

where $\tau_y \in \mathbb{R}$, $y \in \Sigma_{\delta_*}$ is a value for which $f'^{\tau_y}\big(h(y)\big) \in \Sigma_{\delta'_*}$.

Sufficiency. Let $S_{\delta*}$, $S_{\delta'*}$ be circular schemes of the flows $f^t : M^2 \to M^2$, $f'^t : M'^2 \to M'^2$ equivalent by means of a homeomorphism $\psi : \Sigma_{\delta_*} \to \Sigma_{\delta'_*}$. Let us construct, step-by-step, a homeomorphism $h : M^2 \to M'^2$ sending the trajectories of $f^t : M^2 \to M^2$ to the trajectories of $f'^t : M'^2 \to M'^2$ with the saving orientation on the trajectories.

Step 1. For a point $x \in M^2$ ($x' \in M'^2$) denote by $\mathcal{O}_x$ ($\mathcal{O}'_{x'}$) the trajectory of f^t (f'^t) passing through the point x (x'). Let $N = \bigcup\limits_{x \in \Sigma_{\delta_*}} \mathcal{O}_x$ and $N' = \bigcup\limits_{x' \in \Sigma_{\delta'_*}} \mathcal{O}'_{x'}$. Define a homeomorphism $h_1 : N \to N'$ realizing the equivalence of flows $f^t|_N, f'^t|_{N'}$ by the formula

$$h_1(y) = f'^{-\tau_y} \circ \psi \circ f^{\tau_y}(x),$$

where $\tau_y \in \mathbb{R}$ is a value for which $f^{\tau_y}(y) \in \Sigma_{\delta_*}$ for $y \in N$. From the definition of the section Σ_{δ_*} ($\Sigma_{\delta'_*}$) it follows that it intersects exactly one invariant manifold of each saddle point $p \in \Omega^1_{f^t}$ ($p' \in \Omega^1_{f'^t}$) exactly at a pair of points. Since the homeomorphism ψ sends the paired points into the paired points preserving their stability, the homeomorphism h_1 sends an invariant manifold of a point p without a point p into an invariant manifold of the same stability of a point p' without a point p'. That is, the homeomorphism h_1 uniquely extends to the set $\Omega^1_{f^t}$ so that $h_1(p) = p'$.

Step 2. Consider the linear flow $a^t : \mathbb{R}^2 \to \mathbb{R}^2$, given by the formula $a^t(x_1, x_2) = (2^t x_1, 2^{-t} x_2)$. Let $U_0 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 1\}$. The definition of a hyperbolic point implies the existence of a neighborhood U_p of the point $p \in \Omega^1_{f^t}$ and a homeomorphism $h_p \colon U_p \to U_0$ conjugating $f^t|_{U_p}$ and $a^t|_{U_0}$. Let $V_p = \bigcup\limits_{x \in \mathrm{U}_p} \mathcal{O}_x$ and $V_0 = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_1 x_2| \leqslant \frac{1}{2}\right\}$ (see Fig. 8). Define a homeomorphism $h_{V_p} : V_p \to V_0$ realizing the equivalence of the flows $f^t|_{V_p}, a^t|_{V_0}$ by the formula

$$h_{V_p}(y) = a^{-\tau_y} \circ h_p \circ f^{\tau_y}(y),$$

where $\tau_y \in \mathbb{R}$ is a value for which $f^{\tau_y}(y) \in a_p$ for $y \in V_p$.

Then we construct a homeomorphism $h_{V_{p'}}(y) : V_{p'} \to V_0$ for points $p' \in \Omega^1_{f'^t}$ analogous to homeomorphism h_{V_p}. At the same time, we choose the homeomorphism $h_{p'}$ so that for a connected component v of the set $V_0 \setminus (Ox_1 \cup Ox_2)$ the intersection $h_{V_{p'}}(v) \cap h_1\big(h_{V_p}(v)\big)$ is not empty. Let $\tilde{h}_{V_p} = h^{-1}_{V_{p'}} \circ h_{V_p} : V_p \to V_{p'}$, $V = \bigcup\limits_{p \in \Omega^1_{f^t}} V_p$, $V' = \bigcup\limits_{p' \in \Omega^1_{f'^t}} V_p$ and denote by $h_0 : V \to V'$ a homeomorphism composed by the homeomorphisms $\tilde{h}_{V_p}$ for all $p \in \Omega^1_{f^t}$.

Step 3. Denote by $\tilde{V}_{p'}$ a subset of M^2 containing the invariant manifolds of saddle p' and bounded by curves $h_1(\partial V_p)$. For $\rho > 0$ let $V_0^\rho = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 x_2| \leqslant \rho\}$. Choose $0 < \rho_p < \frac{1}{2}$ such that $h_0(V_p^{\rho_p}) \subset \operatorname{int} \tilde{V}_{p'}$. Let $W_p = V_p^{\rho_p}$, $\tilde{W}_{p'} = h_0(W_p)$. Let T be a connected component of the

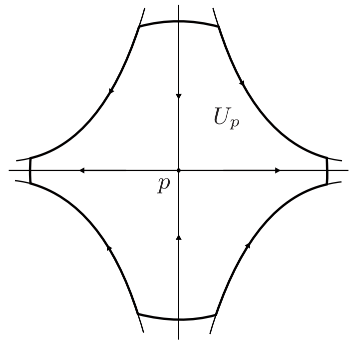

Fig. 8. Neighborhood of U_0.

set $V_p \setminus \operatorname{int} W_p$. Then one of its boundary components (denote it by T_0) belongs to ∂W_p, and the other (denote it by T_1) belongs to ∂V_p. Denote by $\tilde{T}$ the connected component of the set $\tilde{V}_{p'} \setminus \operatorname{int} \tilde{W}_{p'}$ having boundary components $\tilde{T}_0 = h_0(T_0)$ and $\tilde{T}_1 = h_1(T_1)$.

Let $\phi : T \to [0,1] \times \mathbb{R}$ ($\tilde{\phi} : \tilde{T} \to [0,1] \times \mathbb{R}$) be a homeomorphism such that $\phi(\mathcal{O}_x) = \{s\} \times \mathbb{R}$ ($\tilde{\phi}(\mathcal{O}'_{\tilde{x}}) = \{\tilde{s}\} \times \mathbb{R}$) for $x \in T$, $s \in [0,1]$ ($\tilde{x} \in \tilde{T}$, $\tilde{s} \in [0,1]$) and $\phi(T_0) = \{0\} \times \mathbb{R}$, $\phi(T_1) = \{1\} \times \mathbb{R}$ ($\tilde{\phi}(\tilde{T}_0) = \{0\} \times \mathbb{R}$, $\tilde{\phi}(\tilde{T}_1) = \{1\} \times \mathbb{R}$). Then the homeomorphism $\zeta_0 = \tilde{\phi} \circ h_0 \circ (\phi|_{T_0})^{-1} : \{0\} \times \mathbb{R} \to \{0\} \times \mathbb{R}$ has a form $\zeta_0(0,r) = (0, \eta_0(r))$. Similarly, the homeomorphism $\zeta_1 = \tilde{\phi} \circ h_1 \circ (\phi|_{T_1})^{-1} : \{1\} \times \mathbb{R} \to \{1\} \times \mathbb{R}$ has a form $\zeta_1(1,r) = \big(1, \eta_1(r)\big)$. For $s \in [0,1]$ denote by $\eta_s : \mathbb{R} \to \mathbb{R}$ a homeomorphism given by the formula

$$\eta_s(r) = s\eta_1(r) + (1-s)\eta_0(r)$$

and let $\zeta : [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$ be a homeomorphism given by the formula

$$\zeta(s,r) = \big(s, \eta_s(r)\big).$$

By construction, the homeomorphism $\zeta_T = \tilde{\phi} \circ \zeta \circ \phi^{-1} : T \to T'$ realizes the equivalence of flows $f^t|_T, f'^t|_{T'}$, it coincides with h_0 on T_0 and with h_1 on T_1. Similarly, we construct a homeomorphism on all connected components of the set $V_p \setminus \operatorname{int} W_p$ and get a homeomorphism $\tilde{h}_{V_p} : V_p \to \tilde{V}_{p'}$. Let $\tilde{V}' = \bigcup\limits_{p' \in \Omega^1_{f't}} \tilde{V}_{p'}$ and denote by $\tilde{h}_0 : V \to \tilde{V}'$ a homeomorphism composed by the homeomorphisms $\tilde{h}_{V_p}$ for all $p \in \Omega^1_{f^t}$.

Step 4. Let $\dot{M} = M^2 \setminus (\Omega^0_{f^t} \cup \Omega^2_{f^t})$, $\dot{M}' = M'^2 \setminus (\Omega^0_{f't} \cup \Omega^2_{f't})$ and define a homeomorphism $h : \dot{M} \to \dot{M}'$ coinciding with $\tilde{h}_0$ on V and with h_1 on $\dot{M} \setminus V$. We show that the homeomorphism h uniquely extends to M^2, which completes the proof.

Assume that the circle Σ_{δ_*} is oriented and the paired points of the sets $L^u_{\delta_*}$ are numbered: $(z_1, y_1), \ldots, (z_{k^u}, y_{k^u})$ and belong to unstable manifolds of saddle points $p_1, \ldots, p_{k^u}$, respectively. Then the set $V^{\rho_{p_i}}_{p_i} \cap \Sigma_{\delta_*}$ consists of two arcs $[a_i, b_i] \sqcup [c_i, d_i]$ (oriented in accordance with the orientation of the circle),which are the neighborhoods of the points z_i, y_i, respectively. Let $A_i = h_{V_{p_i}}(a_i), B_i = h_{V_{p_i}}(b_i), C_i = h_{V_{p_i}}(c_i), D_i = h_{V_{p_i}}(d_i)$. Without loss of generality, we assume that the points A_i and B_i belong to the fourth and first quadrants of $\mathbb{R}^2$ (in other cases, the reasoning is similar). Then for a pair of points z_i, y_i with spin $+$ $(-)$, the points C_i and D_i belong to the second and third (third and second) quadrants, respectively. Next, we construct the section $\gamma_{A_i}, \gamma_{B_i}$ for trajectories in $a^t|_{V^{\rho_i}_0 \setminus W^u_O}$ as follows (we will construct for spin $+$, for spin $-$ it is similar).

Let $A_i = a^{T_{A_i}}(1, -\rho_i)$, $D_i = a^{T_{D_i}}(-1, -\rho_i)$. For $x_1 \in [-1,1]$ let

$$t_i(x_1) = 0,5(x_1+1)(T_{A_i} - T_{D_i}) + T_{D_i},\ \gamma_{A_i} = \bigcup_{x_1 \in [-1,1]} f^{t_i(x_1)}(x_1, -\rho_i).$$

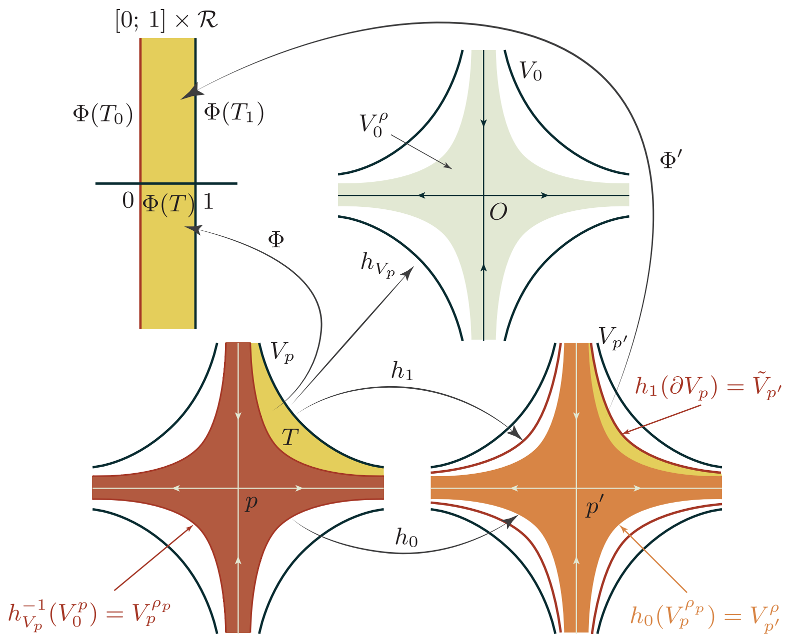

Fig. 9. Illustration of step 3.

Similarly, we construct a section γ_{B_i} with boundary points B_i, C_i. Let $\gamma_{a_i} = h_{V_p}^{-1}(\gamma_{A_i})$, $\gamma_{b_i} = h_{V_p}^{-1}(\gamma_{B_i})$ (see Fig. 10) and

$$C^u = \left(\Sigma_{\delta_*} \setminus \bigcup_{i=1}^{k^u} \big((a_i, b_i) \cup (c_i, d_i)\big)\right) \cup \bigcup_{i=1}^{k^u} (\gamma_{a_i} \cup \gamma_{b_i}).$$

By construction, every connected component c_j^g of the set C^g is a section for trajectories of f^t in the basin of some source α_j. Then $h(c_j^u)$ is also a section for the trajectories of f'^t in the basin of some source α'_j. Hence, $|\Omega^2_{f^t}| = |\Omega^2_{f'^t}|$ and h can be continuously extended to the set $\Omega^2_{f^t}$.

It follows from the definition of the circular scheme that the circle $\Sigma_{\delta*}$ bounds a two-dimensional disk $U_{\delta*}$ on the surface M^2. Also, the arcs $W^s_{\delta*}$ divide this disk into $|\Omega^0_{f^t}|$ two-dimensional disks, whose the interiors belong to basins of pairwise different sinks of f^t. It follows from the construction of the homeomorphism h that the circle $h(\Sigma_{\delta*})$ bounds the two-dimensional disk $h(U_{\delta*})$ on the

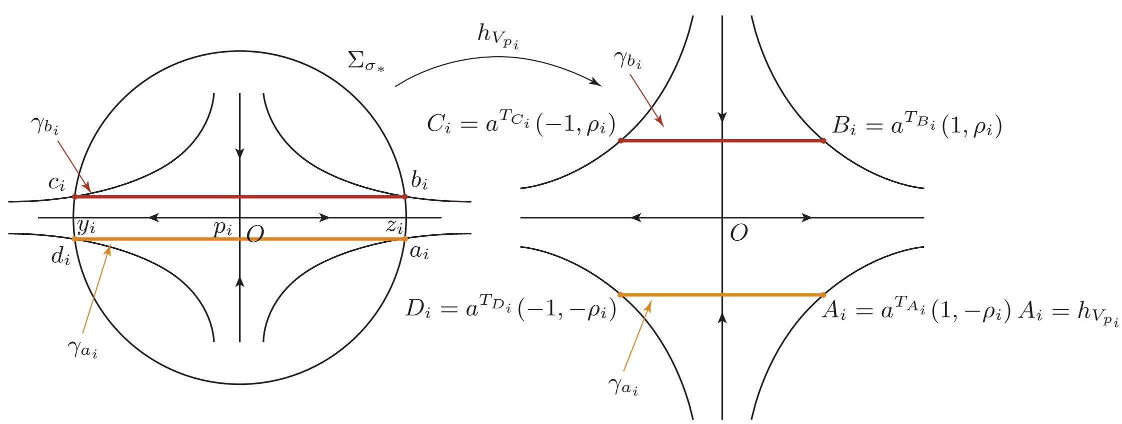

Fig. 10. Illustration of step 4.

surface M'^2 and the arcs $W^s_{\delta' *} = h(W^s_{\delta *})$ divide it into $|\Omega^0_{f't}|$ two-dimensional disks. It implies that $|\Omega^0_{f^t}| = |\Omega^0_{f't}|$ and the homeomorphism h continuously extends to the set $\Omega^0_{f^t}$.

3. REALIZATION

The proof of the realization theorem consists of the construction of a gradient-like flow $f^t : M^2 \to M^2$ having a circular scheme $S_{\delta_*} = (\Sigma_{\delta_*}, L^s_{\delta_*}, L^u_{\delta_*})$ equivalent to the given abstract scheme $S = (\Sigma, L^s, L^u)$.

Step 1. Consider an abstract circular scheme $S = (\Sigma, L^s, L^u)$. Set the flow q^t on the manifold $\mathbb{R} \times \Sigma$ by the formula $q^t(s, r) = (s, r + t)$. Assume that the circle Σ is counterclockwise oriented and the paired points of the set L^u are numbered: $(z_1, y_1), \ldots, (z_{k^u}, y_{k^u})$. Choose pairwise disjoint arcs oriented in accordance with the circle orientation $[a_i, b_i]$, $[c_i, d_i] \subset \Sigma$, $i \in \{1, \ldots, k^u\}$ such that $[a_i, b_i] \cap (L^u \cup L^s) = z_i$, $[c_i, d_i] \cap (L^u \cup L^s) = y_i$.

Let $\ell^u_{z_i} = \{z_i\} \times \mathbb{R}$, $\ell^u_{y_i} = \{y_i\} \times \mathbb{R}$, $N^u_{z_i} = [a_i, b_i] \times \mathbb{R}$, $N^u_{y_i} = [c_i, d_i] \times \mathbb{R}$, $\ell^u_i = \ell^u_{z_i} \cup \ell^u_{y_i}$ and $N^u_i = N^u_{z_i} \cup N^u_{y_i}$. On the set $\mathcal{N} = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 x_2| \leqslant 1\}$ we define the flow a^t by the formula $a^t(x_1, x_2) = (2^t x_1, 2^{-t} x_2)$. Let $\mathcal{N}^u = \mathcal{N} \setminus Ox_2$. Define a diffeomorphism $\mu^u_i : N^u_i \to \mathcal{N}^u$, realizing the equivalence of flows q^t, a^t, by the formulas

$$\mu^u_i|_{N^u_{z_i}}(s, r) = \left(2^r, 2^{-r}\left(\frac{2s - b_i - a_i}{b_i - a_i}\right)\right),$$

$$\mu^u_i|_{N^u_{y_i}}(s, r) = \left(-2^r, \delta_i 2^{-r}\left(\frac{2s - d_i - c_i}{d_i - c_i}\right)\right),$$

where $\delta_i \in \{-, +\}$ is the spin of the pair (z_i, y_i). It is directly checked that $a^t \mu^u_i = \mu^u_i q^t$. Let $\Lambda^u = \bigcup\limits_{i=1}^{k^u} \ell^u_i$ and $N_{\Lambda^u} = \bigcup\limits_{i=1}^{k^u} N^u_i$. Denote by $\mu^u : N_{\Lambda^u} \to \mathcal{N}^u \times \mathbb{Z}_{k^u}$ a diffeomorphism composed by the diffeomorphisms $\mu^u_1, \ldots, \mu^u_{k^u}$.

Then, we number the points of the set L^s: $(\bar{z}_1, \bar{y}_1), \ldots, (\bar{z}_{k^s}, \bar{y}_{k^s})$ and choose pairwise disjoint arcs $[\bar{a}_i, \bar{b}_i]$, $[\bar{c}_i, \bar{d}_i] \subset \Sigma$, $i \in \{1, \ldots, k^s\}$, avoiding Σ, such that $[\bar{a}_i, \bar{b}_i] \cap (N_{\Lambda^u} \cup L^s) = \bar{z}_i$, $[\bar{c}_i, \bar{d}_i] \cap (N_{\Lambda^u} \cup L^s) = \bar{y}_i$. Let $\ell^s_{\bar{z}_i} = \{\bar{z}_i\} \times \mathbb{R}$, $\ell^s_{\bar{y}_i} = \{\bar{y}_i\} \times \mathbb{R}$, $N^s_{\bar{z}_i} = [\bar{a}_i, \bar{b}_i] \times \mathbb{R}$, $N^s_{\bar{y}_i} = [\bar{c}_i, \bar{d}_i] \times \mathbb{R}$, $\ell^s_i = \ell^s_{\bar{z}_i} \cup \ell^s_{\bar{y}_i}$ and $N^s_i = N^s_{\bar{z}_i} \cup N^s_{\bar{y}_i}$. Let $\mathcal{N}^s = \mathcal{N} \setminus Ox_1$. Define a diffeomorphism $\mu^s_i : N^s_i \to \mathcal{N}^s$, realizing the equivalence of the flows q^t, a^t, by the formulas

$$\mu^s_i|_{N^s_{\bar{z}_i}}(w, r) = \left(2^r\left(\frac{2w - \bar{b}_i - \bar{a}_i}{\bar{b}_i - \bar{a}_i}\right), 2^{-r}\right),$$

$$\mu^s_i|_{N^s_{\bar{y}_i}}(w, r) = \left(2^r\left(\frac{2w - \bar{d}_i - \bar{c}_i}{\bar{d}_i - \bar{c}_i}\right), -2^{-r}\right).$$

It is directly checked that $a^t \mu^s_i = \mu^s_i q^t$. Let $\Lambda^s = \bigcup\limits_{i=1}^{k^s} \ell^s_i$ and $N_{\Lambda^s} = \bigcup\limits_{i=1}^{k^s} N^s_i$. Denote by $\mu^s : N_{\Lambda^s} \to \mathcal{N}^s \times \mathbb{Z}_{k^s}$ a diffeomorphism composed by the diffeomorphisms $\mu^s_1, \ldots, \mu^s_{k^s}$. Let $Q = (\mathbb{R} \times \Sigma) \cup_{\mu^u} (\mathcal{N} \times \mathbb{Z}_{k^u}) \cup_{\mu^s} (\mathcal{N} \times \mathbb{Z}_{k^s})$, $\overline{Q} = (\mathbb{R} \times \Sigma) \sqcup (\mathcal{N} \times \mathbb{Z}_{k^u}) \sqcup (\mathcal{N} \times \mathbb{Z}_{k^s})$ and denote by $p : \overline{Q} \to Q$ the natural projection. Let $p_1 = p|_{(\mathbb{R} \times \Sigma)}, p_2 = p|_{\mathcal{N} \times \mathbb{Z}_{k^u}}, p_3 = p|_{\mathcal{N} \times \mathbb{Z}_{k^s}}$. Let us define on the manifold Q a flow $Y^t : Q \to Q$ by the formula

$$Y^t = \begin{cases} p_1\Big(q^t\big(p_1^{-1}(x)\big)\Big) & x \in p_1(\mathbb{R} \times \Sigma) \\ p_2\Big(a^t\big(p_2^{-1}(x)\big)\Big) & x \in p_2(\mathcal{N} \times \{i\}), i \in \mathbb{Z}_{k^u} \\ p_3\Big(a^t\big(p_3^{-1}(x)\big)\Big) & x \in p_3(\mathcal{N} \times \{i\}), i \in \mathbb{Z}_{k^s} \end{cases}$$

By construction, the nonwandering set of the flow Y^t consists of $k^u + k^s$ saddle fixed hyperbolic points.

Step 2. Let $R^s = Q \setminus W^s_{\mathcal{R}_{Y^t}}$ and denote by $\varrho^s_1, \ldots, \varrho^s_{k^s+1}$ the connected components of the set R^s. Consider the linear flow $b^t(x_1, x_2) = (2^{-t}x_1, 2^{-t}x_2)$. From property 1 of the definition of the abstract scheme it follows that the flow $Y^t|_{\varphi^s_i}$ is conjugated to the flow $b^t|_{\mathbb{R}^2\setminus O}$ by some diffeomorphism ν^s_i. Denote by $\nu^s : R^s \to (\mathbb{R}^2 \setminus O) \times \mathbb{Z}_{k^s+1}$ a diffeomorphism composed by the diffeomorphisms $\nu^s_1, \ldots, \nu^s_{k^s+1}$. Let $M_s = Q \cup_{\nu^s} (\mathbb{R}^2 \times \mathbb{Z}_{k^s+1})$, $\bar{M}_s = Q \sqcup (\mathbb{R}^2 \times \mathbb{Z}_{k^s+1})$ and denote by $p_s : \bar{M}_s \to M_s$ the natural projection. Let $p_{s,1} = p_s|_Q, p_{s,2} = p_s|_{\mathbb{R}^2\times\mathbb{Z}_{k^s+1}}$. Then the flow X^t_s on the manifold M_s is defined by the formula

$$X^t_s = \begin{cases} p_{s,1}\Big(Y^t\big(p^{-1}_{s,1}(x)\big)\Big) & x \in p_{s,1}(Q) \\ p_{s,2}\Big(b^t\big(p^{-1}_{s,2}(x)\big)\Big) & x \in p_{s,2}(\mathbb{R}^2 \times \{i\}), i \in \mathbb{Z}_{k^s+1}. \end{cases}$$

By construction, the nonwandering set of the flow X^t_s consists of $k^u + k^s$ saddle and $k^s + 1$ sink fixed hyperbolic points.

Step 3. Let $R^u = M_s \setminus W^u_{\mathcal{R}_{X^t_s}}$ and denote by $\varrho^u_1, \ldots, \varrho^u_{m^u}$ the connected components of the set R^u. Similarly to step 4 of the proof of Theorem 1, a set of circles $C^u = \{c^u_1, \ldots, c^u_{m^u}\}$ can be constructed, which are sections to the trajectories of flow in the components. Then the flow $X^t_s|_{\varrho^u_i}$ is conjugated with the flow $b^{-t}|_{\mathbb{R}^2\setminus O}$ by means of some diffeomorphism ν^u_i. Denote by $\nu^u : R^u \to (\mathbb{R}^2 \setminus O) \times \mathbb{Z}_{m^u}$ a diffeomorphism composed by the diffeomorphisms $\nu^u_1, \ldots, \nu^u_{m^u}$. Let $M^2 = M_s \cup_{\nu^u} (\mathbb{R}^2 \times \mathbb{Z}_{m^u})$, $\bar{M}^2 = M_s \sqcup (\mathbb{R}^2 \times \mathbb{Z}_{m^u})$ and denote by $p_u : \bar{M}^2 \to M^2$ the natural projection. Let $p_{u,1} = p_u|_{M_s}, p_{u,2} = p_u|_{\mathbb{R}^2\times\mathbb{Z}_{m^u}}$. Then the flow f^t on the surface of M^2 is determined by the formula

$$f^t = \begin{cases} p_{u,1}\Big(X^t_s\big(p^{-1}_{u,1}(x)\big)\Big) & x \in p_{u,1}(M_s) \\ p_{u,2}\Big(b^{-t}\big(p^{-1}_{u,2}(x)\big)\Big) & x \in p_{u,2}(\mathbb{R}^2 \times \{i\}), i \in \mathbb{Z}_{m^u}. \end{cases}$$

By construction, the nonwandering set of the flow f^t consists of $k^u + k^s$ saddle, $k^s + 1$ sink and m^u sources, all of them are fixed hyperbolic points. The surface M^2 is closed, its orientability (nonorientability) is determined by the presence (absence) of points with negative spin in the scheme S, and its genus g is calculated by the Poincaré–Hopf formula (see, for example, [10])

$$g = \frac{k^u - m^u + 1}{2} \ \ (g = k^u - m^u + 1).$$

4. AN EFFICIENT ALGORITHM FOR DISTINGUISHING ABSTRACT CIRCULAR SCHEMES

Recall that a *graph* Γ is an ordered pair (B, E), where B is a nonempty set of *vertices* and E is a set of pairs of vertices, called *edges*. Each vertex a, b of the edge $e = ab$ is called an *incident* to the edge e, and one says that a, b are joined by the edge e.

The valency of the vertex is the number of edges incident to it. If the edges are ordered pairs of vertices, then the graph is called *oriented*. A graph is called *connected* if any two of its vertices a, b can be joined *by a path* from edges, the number of edges included in the path is called *path length*. If the beginning and the end of the path coincide, then the path is called a *cycle*. If both vertices of an edge coincide, then the edge is called a *loop*. *A subgraph* of the graph Γ is a pair $(\tilde{B}, \tilde{E})$, where $\tilde{B} \subset B$, $\tilde{E} \subset E$.

Next, we call an operation in which the edge e is removed from the graph and new vertices $c_1, c_2, \ldots, c_k$ with edges $ac_1, c_1c_2, \ldots, c_{k-1}c_k, c_kb$ are added a *k-subdivision* of the edge $e = ab$.

Then we call an operation in which the edge e is removed from the graph and new vertices $c_1, c_2, \ldots, c_k, d$ with edges $ac_1, c_1c_2, \ldots, c_{k-1}c_k, c_kb, c_1d$ are added a *k^*-subdivision* of the edge $e = ab$.

A graph is called *simple* if it does not contain loops and multiple edges. A graph is called *planar* if there is an embedding of it in the plane. If there is an embedding of a graph in a surface, then the graph is called *embeddable in the surface*.

Two graphs Γ and Γ' are called *isomorphic* if there is a map that sends the vertices and the edges of the graph Γ into the vertices and the edges of the graph Γ', respectively.

Next, we will prove Theorem 3: let $S = (\Sigma, L^s, L^u)$ and $S' = (\Sigma, L'^s, L'^u)$ be abstract circular schemes such that $k^s = k'^s$, $k^u_- = k'^u_-$, $k^u_+ = k'^u_+$. Then there is an efficient (polynomial-dependent on $k = k^s + k^u_- + k^u_+$) algorithm for distinguishing their isomorphism.

Proof. Let $S = (\Sigma, L^s, L^u)$ be an abstract circular scheme. Next, we construct a simple graph Γ_S from it as follows. The intersection points will be the vertices of the graph and the arcs of the circle will be the edges. Connect the edges of the paired points of $L^s \cup L^u$. Let us apply a 1-subdivision to each edge joining the paired points from L^u with spin $+$, a 1*-subdivision to each edge joining the paired points from L^u with spin $-$ and a 2-subdivision to each edge joining the paired points from L^s (see Fig. 11). Notice that the graph Γ_S is uniquely constructed by the circular scheme S. Next, show that the converse is also true.

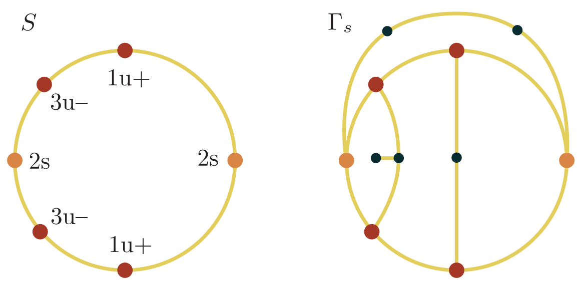

Fig. 11. Graph Γ^F_{ft} and simple graph $\hat{\Gamma}_{ft}$.

The initial points of the circular scheme S are the vertices of the graph Γ_S of valency 3 that do not have neighboring vertices of valency 1. The vertices of the graph Γ_S correspond to the paired points of the set L^s if they are joined by a path of length 3 that does not contain other similar vertices. The vertices of the graph correspond to the paired points of the set L^u if they are connected by a path of length 2 that does not contain other similar vertices. If the vertices in such a path have valency 2 (3), then the path corresponds to the spin $+$ $(-)$.

By construction, the graph Γ_S has $m = k + 2k^s + k^u_+ + k^u_- = 2k + k^s \leqslant 3k$ vertices and, by Theorem 2, it can be embedded in a surface of genus g. Then, the graphs Γ_S and $\Gamma_{S'}$ have the same number of vertices m and they can be embedded in a surface of genus $p = \max\{g, g'\}$. According to [9], the isomorphism of two simple m-vertex graphs embedded in a surface of genus p can be checked in time $O(m^{O(p)})$. Thus, there is a polynomial-dependent k algorithm for distinguishing the circular schemes S and S'.

Applications. Typical gradient flows are a special case of the systems considered in this work. A prototypical gradient flow is the diffusion equation that governs heat propagation in a physical medium. The formal derivation of diffusion equations dates back to the nineteenth-century treatise of Joseph Fourier on the Analytic Theory of Heat [3]. Such equations have a plethora of applications in physics and have also been used in image processing, computer vision [12], and graph neural networks [1].

5. FUNDING

This work was supported by the Russian Science Foundation (Project No. 23-71-30008), except for Section 4, which was supported by the Laboratory of Dynamic Systems and Applications of the HSE, grant of the Ministry of Science and Higher Education of the Russian Federation, Agreement No. 075-15-2019-1931.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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