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ON A STRUCTURE OF NON-WANDERING SET OF AN Ω-STABLE 3-DIFFEOMORPHISM POSSESSING A HYPERBOLIC ATTRACTOR

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Abstract. This paper belongs to a series of papers devoted to the study of the structure of the non-wandering set of an A-diffeomorphism. We study such set $NW(f)$ for an Ω-stable diffeomorphism f, given on a closed connected 3-manifold M^3. Namely, we prove that if all basic sets in $NW(f)$ are trivial except attractors, then every non-trivial attractor is either one-dimensional non-orientable or two-dimensional expanding.

1. **Introduction and formulation of results.** Let M^n be a smooth closed connected n-manifold with a Riemannian metric d and $f : M^n \to M^n$ be a diffeomorphism. A set $\Lambda \subset M^n$ is called an *invariant set* if $f(\Lambda) = \Lambda$. An invariant compact set $\Lambda \subset M^n$ is called *hyperbolic* if there is a continuous Df-invariant splitting of the tangent bundle $T_\Lambda M^n$ into *stable* and *unstable subbundles* $E^s_\Lambda \oplus E^u_\Lambda$, $\dim E^s_x + \dim E^u_x = n$ $(x \in \Lambda)$ such that for $i > 0$ and for some fixed $C_s > 0$, $C_u > 0$, $0 < \lambda < 1$

$$\|Df^i(v)\| \le C_s\lambda^i\|v\|, \qquad v \in E^s_\Lambda,$$
$$\|Df^{-i}(w)\| \le C_u\lambda^i\|w\|, \qquad w \in E^u_\Lambda.$$

The hyperbolic structure of Λ implies the existence of stable and unstable manifolds W^s_x, W^u_x respectively for any point $x \in \Lambda$:

$$W^s_x = \{y \in M^n : \lim_{j\to+\infty} d(f^j(x), f^j(y)) = 0\},$$
$$W^u_x = \{y \in M^n : \lim_{j\to+\infty} d(f^{-j}(x), f^{-j}(y)) = 0\},$$

which are smooth injective immersions of the E^s_x and E^u_x into M^n. Moreover, W^s_x, W^u_x are tangent to E^s_x and E^u_x at x respectively. For $r > 0$ we will denote by $W^s_{x,r}$, $W^u_{x,r}$ the immersions of discs on the subbundles E^s_x, E^u_x of the radius r.

Recall that a point $x \in M^n$ is *non-wandering* if for any neighborhood U of x the inequation $f^n(U) \cap U \neq \emptyset$ holds for infinitely many integers n. Then $NW(f)$, the *non-wandering set* of f, defined as the set of all non-wandering points, is an f-invariant closed set.

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If the non-wandering set $NW(f)$ of f is hyperbolic and periodic points are dense in $NW(f)$ then f is called *an A-diffeomorphism* [22]. In this case the non-wandering set is a finite union of pairwise disjoint sets, called *basic sets*

$$NW(f) = \Lambda_1 \sqcup \cdots \sqcup \Lambda_m,$$

each of which is compact, invariant and topologically transitive. A basic set Λ_i of an A-diffeomorphism $f : M^n \to M^n$ is called *trivial* if it coincides with a periodic orbit and *non-trivial* in the opposite case.

By [3], every non-trivial basic set Λ_i, similarly to a periodic orbit, is uniquely expressed as a finite union of compact subsets

$$\Lambda_i = \Lambda_{i_1} \sqcup \cdots \sqcup \Lambda_{i_{q_i}}, q_i \geqslant 1$$

such that $f^{q_i}(\Lambda_{i_j}) = \Lambda_{i_j}, f(\Lambda_{i_j}) = \Lambda_{i_{j+1}}, j \in \{1, \dots, q_i\}\,(\Lambda_{i_{q_i+1}} = \Lambda_{i_1})$. These subsets $\Lambda_{i_{q_i}}$, $q_i \geqslant 1$ are called *periodic components* of the set Λ_i[1]. For every point x of a periodic component Λ_{i_j} the set $W^s_x \cap \Lambda_{i_j}$ $(W^u_x \cap \Lambda_{i_j})$ is dense in Λ_{i_j}.

Without loss of generality, everywhere below we will assume that Λ_i consists of a unique periodic component and, in addition, $f|_{W^u_{\Lambda_i}}$ preserves orientation if Λ_i is trivial.

A sequence of basic sets $\Lambda_1, \dots, \Lambda_l$ of an A-diffeomorphism $f : M^n \to M^n$ is called *a cycle* if $W^s_{\Lambda_i} \cap W^u_{\Lambda_{i+1}} \neq \emptyset$ for $i = 1, \dots, l$, where $\Lambda_{l+1} = \Lambda_1$. A-diffeomorphisms without cycles form the set of Ω*-stable* diffeomorphisms; if, in addition, the stable and the unstable manifolds of every non-wandering point intersect transversaly then f is *structurally stable* (see, for example, [21]).

A non-trivial basic set Λ_i is called *orientable* if for any point $x \in \Lambda_i$ and any fixed numbers $\alpha > 0$, $\beta > 0$ the intersection index[2] (+1 or −1) [9]. Otherwise, the basic set is called *non-orientable*.

A basic set Λ_i is called an *attractor* if there exists a compact neighborhood U_{Λ_i} (*a trapping neighborhood*) of Λ_i such that $f(U_{\Lambda_i}) \subset \operatorname{int} U_{\Lambda_i}$ and $\Lambda_i = \bigcap\limits_{i=0}^{\infty} f^i(U_{\Lambda_i})$. Due to [23], a non-trivial attractor Λ_i of f is said to be *expanding* if $\dim\, \Lambda_i = \dim\, W^u_x$, $x \in \Lambda_i$.

The main result of this paper is following.

Theorem 1.1. *Let $f : M^3 \to M^3$ be an Ω-stable diffeomorphism whose basic sets are trivial except attractors. Then every non-trivial attractor is either one-dimensional non-orientable or two-dimensional expanding.*

Notice, that the attractors of both types described in the Theorem 1.1 are realized. In particular, the Figure 1 shows a phase portrait of a structurally stable

[1] R. Bowen [3] called these components C-dense.

[2] Let $J^k : \mathbb{R}^k \to M^3$ be immersions, D^k be open balls of finite radii in $\mathbb{R}^k$, $k = 1, 2$. Then the restrictions $J^k : D^k \to M$ are embeddings and their images $W^k = J^k(D^k)$ are smooth embedded submanifolds of the manifold M^3. Let U^k be a tubular neighborhood of W^k, which are images of embeddings in M^3 of spaces of $(3-k)$-dimensional vector bundles on W^k [13, Chapter 4, par. 5]. Since the balls D^k are contractible, then these bundles are trivial and, hence, $U^2 \setminus W^2$ consists of two connected components U^2_+ and U^2_-. It allows to define a function $\sigma : U^2_+ \cup U^2_- \to \mathbb{Z}$, such that $\sigma(x) = 1$ if $x \in U^2_+$ and $\sigma(x) = 0$ if $x \in U^2_-$. If submanifolds W^1 and W^2 intersect transversally at a point $x = J^1(t)$, $t \in D^1$ then there exists a number $\delta > 0$ such that $J^1(t - 2\delta, t + 2\delta) \subset U^2$. The number

$$\operatorname{Ind}_x(W^1, W^2) = \sigma(t + \delta) - \sigma(t - \delta)$$

is called an *intersection index* of submanifolds W^1 and W^2 in the point x. Notice, that this definition does not require orientability of the manifold M^3.

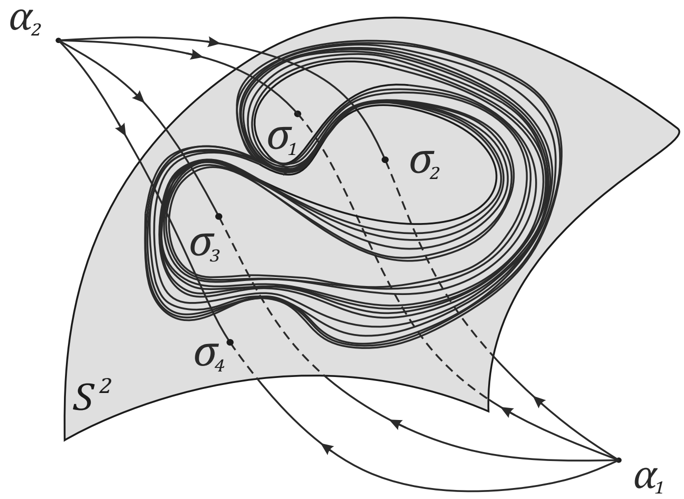

FIGURE 1. Ω-stable diffeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ with the unique non-trivial basic set which is Plykin attractor

diffeomorphism of a 3-sphere, whose non-wandering set consists of a one-dimensional non-orientable Plykin attractor, four saddle points with a two-dimensional unstable manifold and two sources. The DA-diffeomorphism of 3-torus on Figure 2 is an example of a combination of an orientable two-dimensional expanding attractor with a source in the non-wandering set of a structurally stable diffeomorphism. An example of a diffeomorphism with non-orientable 2-dimensional expanding attractor will be constructed in section 6.

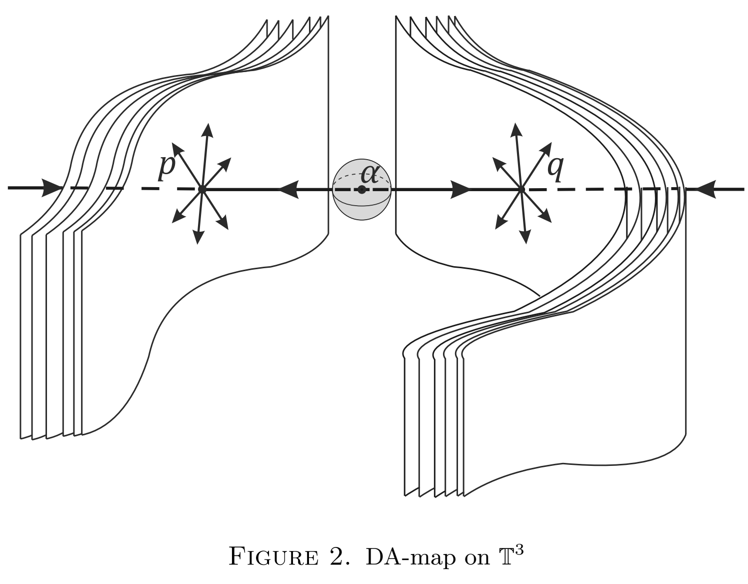

FIGURE 2. DA-map on $\mathbb{T}^3$

2. **Attractor, index of a hyperbolic point, filtration.**

2.1. **Attractors of an A-diffeomorphism.** Let $f : M^3 \to M^3$ be an A-diffeomorphism and Λ_i be its basic set. Then

$$\dim\, W^u_x + \dim\, W^s_x = 3,\; x \in \Lambda_i.$$

If Λ_i is a non-trivial then, moreover, $\dim\, W^u_x > 0$, $\dim\, W^s_x > 0$.

Now let Λ_i be a non-trivial attractor. It follows from [18] that

$$\Lambda_i = \bigcup_{x \in \Lambda_i} W^u_x$$

and, hence, $\dim\, \Lambda_i > 0$.

If $\dim\, \mathbf{\Lambda_i} = \mathbf{3}$ then $\Lambda_i = M^3 \cong \mathbb{T}^3$ [17].

If $\dim\, \mathbf{\Lambda_i} = \mathbf{2}$ then Λ_i is either expanding (as in the Figure 2) or an *Anosov torus* ($f|_{\Lambda_i}$ is conjugate to an Anosov algebraic automorphism of a torus $\mathbb{T}^2$) [4], [11]. Herewith, an expanding attractor Λ_i is locally homeomorphic to the product of $\mathbb{R}^2$ with a cantor set [19, 20]. There are both type of such attractor, orientable and non-orientable [24]. By [11] every Anosov torus Λ_i is a locally flat (possible non-smoothly [16]) embedded in M^3 and, hence, it is always orientable and has a trapping neighborhood U_{Λ_i} which is homeomorphic to $\mathbb{T}^2 \times [-1, 1]$.

If $\dim\, \mathbf{\Lambda_i} = \mathbf{1}$ then Λ_i is automatically expanding, derived from an expansions on a 1-dimensional branched manifold [23] and is the nested intersections of handlebodies [2]. Thus, any one-dimensional attractor Λ_i of an A-diffeomorphism $f : M^3 \to M^3$ has a trapping neighborhood U_{Λ_i} which is a handlebody. There are both type of such attractor, orientable and non-orientable, it is enough to consider $f = f_{DA} \times f_{NS}$ (see Figure 3) and $f = f_{Pl} \times f_{NS}$, where $f_{DA} : \mathbb{T}^2 \to \mathbb{T}^2$ is derived from Anosov diffeomorphism, $f_{NS} : \mathbb{S}^1 \to \mathbb{S}^1$ is a “source-sink” diffeomorphism, $f_{Pl} : \mathbb{S}^2 \to \mathbb{S}^2$ is a diffeomorphism with the Plykin attractor (as in the Figure 1) and four sources.

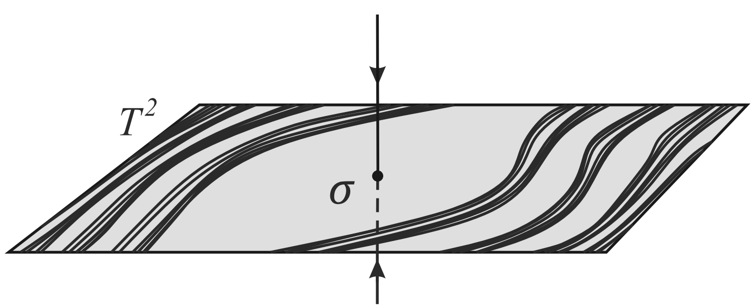

FIGURE 3. A one-dimensional attractor for a diffeomorphism $f_{DA} \times f_{NS}$

The most famous one-dimensional attractor is *Smale solenoid* (see Figure 4) which appears as intersection of the nested tori $f^k(\mathbb{D}^2 \times \mathbb{S}^1)$, $k \in \mathbb{N}$ for $f(d, z) = (d/10, 2z)$. An arbitrary one-dimensional attractor is sometimes called *Smale-Williams solenoid.*

It is well known that the presence of an attractor with certain properties in a non-wandering set of an A-diffeomorphism can determine both the character of the remaining basic sets and the topology of the ambient manifold.

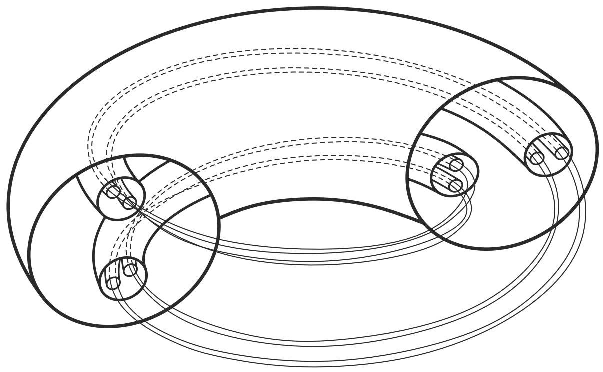

FIGURE 4. Smale's solenoid

- If $f : M^3 \to M^3$ is a structurally stable diffeomorphism whose non-wandering set $NW(f)$ contains a two-dimensional expanding attractor Λ_i, then it is orientable, $M^3 \cong \mathbb{T}^3$ and the set $NW(f) \setminus \Lambda_i$ consists of a finite number of isolated sources and saddles [12], [24].
- If $f : M^3 \to M^3$ is an A-diffeomorphism whose every basic set is two-dimensional then its attractors are either all Anosov tori or all expanding [1].
- If $f : M^3 \to M^3$ is a structural stable diffeomorphism whose every basic set is two-dimensional then its attractors are all Anosov tori and M^3 is a mapping torus [8].
- An orientable manifold M^3 admits an A-diffeomorphism $f : M^3 \to M^3$ with the non-wandering set which is a union of finitely many Smale solenoids if and only if M^3 is a Lens space $L_{p,q}$, $p \neq 0$. Every such a diffeomorphism is not structurally stable [15].

2.2. **Orientability of the basic set and index of the hyperbolic point.** In this section let M be a compact smooth n-manifold M (possibly with a non-empty boundary) and $f : M \to f(M)$ be a smooth embedding of a compact n-manifold M to itself and $Fix(f)$ be its set of the fixed points.

Let $p \in Fix(f)$ be an isolated hyperbolic point. By [22, Proposition 4.11] the *index* $I(p) = I(p, f)$ of p is defined by the formula

$$I(p) = (-1)^{\dim W^u_p} \Delta_p,$$

where $\Delta_p = +1$ if f preserves orientation on W^u_p and $\Delta_p = -1$ if f reverses it.

Lemma 2.1. *If Λ_i is an orientable hyperbolic attractor with* $\dim W^u_x = 1, x \in \Lambda_i$ *for f then $I(p) = I(q)$ for any $p, q \in (Fix(f) \cap \Lambda_i)$.*

Proof. Suppose the contrary: there are different points $p, q \in (Fix(f) \cap \Lambda_i)$ such that $I(p) = -I(q)$. As p, q belongs to the same basic set Λ_i then $\dim W^u_p = \dim W^u_q$ and, hence, $\Delta_p = -\Delta_q$. Let us assume for the definiteness that $\Delta_q = -1$ and $\Delta_p = +1$. As Λ_i is an attractor then $W^u_p, W^u_q \subset \Lambda_i$, moreover, $\mathrm{cl}\, W^u_p = \mathrm{cl}\, W^u_p = \Lambda_i$. Denote by ℓ^1_p, ℓ^2_p; ℓ^1_q, ℓ^2_q the connected components of the sets $W^u_p \setminus p$; $W^u_q \setminus q$. By [10] every such a component is dense in Λ_i. Due to hyperbolicity of Λ_i there is a point x_1 of the transversal intersection $\ell^1_q \cap W^s_p$ (see Figure 5).

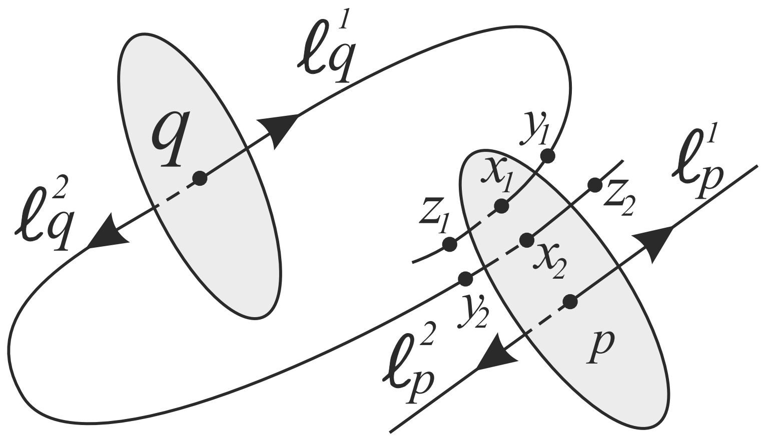

FIGURE 5. Illustration to the proof of Lemma 2.1

As $\Delta_q = -1$ then $x_2 = f(x_1)$ belongs to ℓ^2_q. Let $(y_1, z_1) \subset \ell^1_q$ be a neighbourhood of the point x_1 and $y_2 = f(y_1)$, $z_2 = f(z_1)$. Then the arc $(y_2, z_2) \subset \ell^2_q$ be a neighbourhood of the point x_2. By the orientability of Λ_i we get that y_1, y_2 are separated by W^s_p. By λ-lemma (see, for example, [21]) the iteration of (y_1, z_1), (y_2, z_2) with respect to f are C^1-closed to W^u_p. By continuity of f we conclude that $f(\ell^1_p) = \ell^2_p$. Thus, $\Delta_p = -1$, that contradicts to the assumption. □

Denote by $f_{*k} : H_k(M) \to H_k(M)$, $k \in \{0, \dots, n\}$ the induced automorphism of the k-th homology group $H_k(M)$ of M with real coefficients. The number

$$\Lambda(f) = \sum_{k=0}^{n} (-1)^k \mathrm{tr}(f_{*k})$$

is called a *Lefschetz number* of f [5].

Suppose f has only hyperbolic fixed points and their set $Fix(f)$ is finite. The following equality is named *Lefschetz-Hopf theorem*.

$$\sum_{p \in Fix(f)} I(p) = \Lambda(f). \tag{1}$$

Denote by N_m, $m \in \mathbb{N}$ the number of points in $Fix(f^m)$. Let $\lambda_{*k,j}$, $j \in \{1, \dots,$ $\dim H_k(M)\}$ be eigenvalues of f_{*k}. If $I(p, f^m) = I(q, f^m)$ for any $p, q \in Fix(f^m)$ then the Lefschetz-Hopf theorem has the following form

$$N_m = \left| \sum_{k=0}^{n} (-1)^k \left(\sum_{j=1}^{\dim H_k(M)} \lambda^m_{*k,j} \right) \right|. \tag{2}$$

Sometimes it is convenient to pass from homology groups to cohomology groups. Let us prove the following lemma for this aim.

Lemma 2.2. *Let M be an n-dimensional orientable smooth manifold with boundary ∂M, $f : M \to M$ be a diffeomorphism, $k \in \{0, 1, \dots, n\}$, $f_* : H_k(M) \to H_k(M)$, $\tilde{f}_* : H_{n-k}(M, \partial M) \to H_{n-k}(M, \partial M)$ and $f^* : H^k(M) \to H^k(M)$ be induced automorphisms for groups with real coefficients. Then:*

- *if λ is an eigenvalue for f_*, then $\tilde{\lambda} = \pm\lambda^{-1}$ is an eigenvalue for $\tilde{f}_*$;*

- *if* $\tilde{\lambda}$ *is an eigenvalue for* $\tilde{f}_*$, *then* $\lambda = \pm\tilde{\lambda}^{-1}$ *is an eigenvalue for* f^*.

In the both cases a sign $+$ *corresponds to an orientation-preserving diffeomorphism and a sign* $-$ *is used in the opposite situation.*

Proof. According to the strong part of the Poincare-Lefschetz duality groups $H_k(M)$ and $H_{n-k}(M, \partial M)$ have bases $e_1, \dots, e_m$ and $\varepsilon_1, \dots, \varepsilon_m$, dual with respect to the intersection form $\mathrm{Ind} : H_k(M) \times H_{n-k}(M, \partial M) \to \mathbb{R}$. The duality means that the following equalities take place

$$\mathrm{Ind}(e_i, \varepsilon_j) = \delta_{ij}, \quad i, j = 1, \dots, m.$$

Let A and B be matrices of automorphisms f_* and $\tilde{f}_*$ in the bases $e_1, \dots, e_m$ and $\varepsilon_1, \dots, \varepsilon_m$ correspondingly. Then

$$f_*(e_i) = \sum_{s=1}^{m} a_{is} e_s, \quad \tilde{f}_*(\varepsilon_j) = \sum_{t=1}^{m} b_{jt} \varepsilon_t$$

Herewith

$$\mathrm{Ind}(f_*(e_i), \tilde{f}_*(\varepsilon_j)) = \sum_{s,t=1}^{m} a_{is} b_{jt} \,\mathrm{Ind}(e_s, \varepsilon_t) = \sum_{s,t=1}^{m} a_{is} b_{jt} \delta_{st} = \sum_{s=1}^{m} a_{is} b_{js}. \tag{3}$$

On the other hand, since $\deg f = \pm 1$, then

$$\mathrm{Ind}(f_*(e_i), \tilde{f}_*(\varepsilon_j)) = \pm \mathrm{Ind}(e_i, \varepsilon_j) = \pm \delta_{ij}. \tag{4}$$

(3) and (4) imply $B^T = \pm A^{-1}$. Therefore, the roots of the characteristic equations $|A - \lambda E| = 0$ and $|B - \tilde{\lambda} E| = 0$ are related by the equation $\tilde{\lambda} = \pm\lambda^{-1}$. Thus, the first statement is proved.

For the Poincare-Lefschetz isomorphism $l : H^k(M) \to H_{n-k}(M, \partial M)$ the following diagram is commutative

$$\begin{array}{ccc} H^k(M) & \xleftarrow{\;f^*\;} & H^k(M) \\ {\scriptstyle \pm l}\downarrow & & \downarrow{\scriptstyle l} \\ H_{n-k}(M, \partial M) & \xrightarrow{\;\tilde{f}_*\;} & H_{n-k}(M, \partial M). \end{array} \tag{5}$$

Let $v \in H_{n-k}(M, \partial M)$, $v \neq 0$, $\tilde{\lambda} \in \mathbb{R}$ and $\tilde{f}_*(v) = \tilde{\lambda} v$. Then $\tilde{f}_*^{-1}(v) = \tilde{\lambda}^{-1} v$. Set $\alpha = l^{-1}(v)$. Since l is an isomorphism, then $\alpha \neq 0$. According to (5) we have

$$f^*(\alpha) = \pm l^{-1} \circ \tilde{f}_*^{-1} \circ l(\alpha) = \pm l^{-1} \circ \tilde{f}_*^{-1}(v) = \pm l^{-1}(\tilde{\lambda}^{-1} v) = \pm\tilde{\lambda}^{-1} l^{-1}(v) = \pm\tilde{\lambda}^{-1}\alpha.$$

Thus, $\lambda = \tilde{\lambda}^{-1}$ is an eigenvalue of the automorphism f^* corresponding to the eigenvector $\alpha \in H^k(M)$. □

According to the lemma proved above for the eigenvalues $\lambda^*_{k,j}$, $j \in \{1, \dots, \dim H^k(M)\}$ of f^*_k and f^m such that $I(p, f^m) = I(q, f^m)$ for any $p, q \in Fix(f^m)$ the following equality takes place

$$N_m = \left| \sum_{k=0}^{n} (-1)^{n-k} \left(\sum_{j=1}^{\dim H^k(M)} \lambda^{*m}_{k,j} \right) \right|. \tag{6}$$

2.3. **Filtration.** Let $f : M^n \to M^n$ be an Ω-stable diffeomorphism. As f has no cycles then $\prec$ is a partial order relation on the basic sets

$$\Lambda_i \prec \Lambda_j \iff W^s_{\Lambda_i} \cap W^u_{\Lambda_j} \neq \emptyset.$$

Intuitively the definition means that "everything trickles down" towards "smaller elements". The partial order $\prec$ extends to the order relation, i.e. the basic sets can be enumerated $\Lambda_1, \dots, \Lambda_m$ in accordance with the relation $\prec$:

$$\text{if}\, \Lambda_i \prec \Lambda_j, \text{ then}\, i \leq j.$$

We pick a sequence of nested subsets of the ambient manifold M^n in the following way. Let the first subset of M^n be a neighborhood M_1 of the basic set Λ_1, let the next subset M_2 be the union of M_1 and some neighborhood of the unstable manifold of the element Λ_2. If we continue this process we get the entire manifold M^n. This construction gives the idea to the following notion of filtration.

A sequence $M_1, \dots, M_{m-1}$ of compact n-submanifolds of M^n, each having a smooth boundary, and such that $M^n = M_m \supset M_{m-1} \supset \cdots \supset M_1 \supset M_0 = \emptyset$ is called a *filtration* for a diffeomorphism f with its ordered basic sets $\Lambda_1 \prec \cdots \prec \Lambda_m$ if for each $i = 1, \dots, m$ the following holds:

1. $f(M_i) \subset \text{int}\, M_i$;
2. $\Lambda_i \subset \text{int}\,(M_i \setminus M_{i-1})$;
3. $\Lambda_i = \bigcap\limits_{l \in \mathbb{Z}} f^l(M_i \setminus M_{i-1})$;
4. $\bigcap\limits_{l \geq 0} f^l(M_i) = \bigcup\limits_{j \leq i} W^u_{\Lambda_j} = \bigcup\limits_{j \leq i} cl(W^u_{\Lambda_j})$.

Below we describe following from [7] interrelations between actions f on cohomology groups $H^k(M^n)$, $H^k(M_i, M_{i-1})$ and homology group $H_k(M^n)$ with real coefficients. If an action in these group is *nilpotent* then all eigenvalues equal zero and if it is *unipotent* then it has only roots of unity as eigenvalues.

Proposition 2.3. *Let $f : M^n \to M^n$ be an Ω-stable diffeomorphism and $M^n = M_m \supset M_{m-1} \supset \cdots \supset M_1 \supset M_0 = \emptyset$ be a filtration for its ordered basic sets $\Lambda_1 \prec \cdots \prec \Lambda_m$. Then*

1. *If λ is an eigenvalue of $f^*_k : H^k(M^n) \to H^k(M^n)$, then there is an $i \in \{1, \dots, m\}$ such that $f^*_k : H^k(M_i, M_{i-1}) \to H^k(M_i, M_{i-1})$ has λ as an eigenvalue.*
2. *If Λ_i is a trivial basic set then $f^*_k : H^k(M_i, M_{i-1}) \to H^k(M_i, M_{i-1})$ is nilpotent unless $k = \dim W^u_x$, $x \in \Lambda_i$ and $f^*_k : H^k(M_i, M_{i-1}) \to H^k(M_i, M_{i-1})$ is unipotent for $k = \dim W^u_x$, $x \in \Lambda_i$.*

3. **Proof of theorem 1.1.** In this section we prove that if $f : M^3 \to M^3$ is an Ω-stable diffeomorphism whose basic sets are trivial except attractors, then every non-trivial attractor is either one-dimensional non-orientable or two-dimensional expanding. We will use in this proof some results, which will be proven in the next section. As above, the symbols $H_k(X, A)$ and $H^k(X, A)$ will denote homology and cohomology groups with real coefficients. For homology groups with integer coefficients, the notation $H_k(X, A; \mathbb{Z})$ will be used.

Proof. Suppose the contrary: $NW(f)$ contains a non-trivial attractor A such that A is either one-dimensional orientable or two-dimensional Anosov torus. Without loss of generality we can assume that in the order $\prec$, first positions occupied by attractors and A is the last of them. Let $M^n = M_k \supset M_{k-1} \supset \cdots \supset M_1 \supset M_0 = \emptyset$

be a filtration for the ordered basic sets $\Lambda_1 \prec \cdots \prec \Lambda_k$. Then $\tilde{M}_i = M^n \setminus \operatorname{int} M_{k-i}$ is the filtration for the basic sets $\tilde{\Lambda}_i = \Lambda_{k-i}$ of the diffeomorphism $g = f^{-1}$. Let $A = \tilde{\Lambda}_{i_0}$. Without loss of generality we can assume that the manifold $\tilde{M}_{i_0}$ is connected (in the opposite case let us consider its connected component containing A). Then $g(\tilde{M}_{i_0}) \subset \operatorname{int} \tilde{M}_{i_0}$. Notice, that $i_0 > 1$ since any Ω-stable diffeomorphism has non-empty sets of attractors and repellers.

Let N_m be a number of points in $Fix(g^m)$. As the non-trivial basic set A belongs to $\tilde{M}_{i_0}$ then $\lim\limits_{m\to\infty} N_m = \infty$. Since A is orientable then the Lemma 2.1 and the formula (6) gives the existence of an eigenvalue λ with absolute value greater than 1 for $g_k^* : H^k(\tilde{M}_{i_0}) \to H^k(\tilde{M}_{i_0})$ for some $k \in \{0, \dots, 3\}$.

First of all, let us show that it is impossible for orientable M^3. We will prove it separately for each dimension $k = 0, 1, 2, 3$.

a) $k = 0$. Eigenvalues of the automorphism $g^* : H^0(\tilde{M}_{i_0}) \to H^0(\tilde{M}_{i_0})$ are roots of unity by the lemma 4.7.

b) $k = 3$. The group $H_3(\tilde{M}_{i_0}; \mathbb{Z})$ is trivial when $\partial\tilde{M}_{i_0} \neq \emptyset$ and is isomorphic to $\mathbb{Z}$ when $\partial\tilde{M}_{i_0} = \emptyset$. In the first case we have $H^3(\tilde{M}_{i_0}) = 0$ and so $g^* : H^3(\tilde{M}_{i_0}) \to H^3(\tilde{M}_{i_0})$ does not have eigenvalues. In the second case, $g^* = \pm\,\mathrm{id}$ by the lemma 4.6.

c) $k = 1$. Suppose, that the automorphism $g^* : H^1(\tilde{M}_{i_0}) \to H^1(\tilde{M}_{i_0})$ has an eigenvalue λ, for which $\lambda^2 \neq 1$. Then it follows from the item 1 of the proposition 2.3, that there exists a number i, $1 \leqslant i \leqslant i_0$ such that the automorphism $g^* : H^1(M_i, M_{i-1}) \to H^1(M_i, M_{i-1})$ also has the eigenvalue λ.

As all basic sets of g before A in the Smale order $\prec$ are trivial then by the item 2 of proposition 2.3 for $i < i_0$ we get that the automorphisms g^* on $H^1(M_i, M_{i-1})$ are either nilpotent or uniponent. Hence, it is precisely the automorphism $g^* : H^1(\tilde{M}_{i_0}, \tilde{M}_{i_0-1}) \to H^1(\tilde{M}_{i_0}, \tilde{M}_{i_0-1})$ must have the eigenvalue λ.

Let $\dim A = 1$. In this case $\tilde{M}_{i_0} = Q_g \cup \tilde{M}_{i_0-1}$, where Q_g is a handlebody of a genus $g \geqslant 0$ such that $Q_g \cap \tilde{M}_{i_0-1} = \partial Q_g$. By lemma 4.2 $H_1(\tilde{M}_{i_0}, \tilde{M}_{i_0-1}; \mathbb{Z}) = 0$. Then $H^1(\tilde{M}_{i_0}, \tilde{M}_{i_0-1}) = 0$ and therefore λ cannot be an eigenvalue of the automorphism g^*.

If $\dim A = 2$, then $\tilde{M}_{i_0} = Q \cup \tilde{M}_{i_0-1}$, where $Q \cong \mathbb{T}^2 \times [0,1]$ and $Q \cap \tilde{M}_{i_0-1} = \partial Q$. In this situation by the lemma 4.3 $H_1(\tilde{M}_{i_0}, \tilde{M}_{i_0-1}; \mathbb{Z}) \cong \mathbb{Z}$. From here and from the lemma 4.6 it follows, that $g^* = \pm\,\mathrm{id}$. Thus, we obtain a contradiction for $k = 1$ as well.

d) $k = 2$. Let us finally assume that $g^* : H^2(\tilde{M}_{i_0}) \to H^2(\tilde{M}_{i_0})$ has as eigenvalue λ, for which $\lambda^2 \neq 1$. Due to lemma 2.2, in such a situation the automorphism $g^* : H^1(\tilde{M}_{i_0}, \partial\tilde{M}_{i_0}) \to H^1(\tilde{M}_{i_0}, \partial\tilde{M}_{i_0})$ has an eigenvalue $\tilde{\lambda} = \pm\lambda^{-1}$.

Consider the following diagram

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H^0(\partial\tilde{M}_{i_0}) & \xrightarrow{\delta^*} & H^1(\tilde{M}_{i_0}, \partial\tilde{M}_{i_0}) & \xrightarrow{j^*} & H^1(\tilde{M}_{i_0}) & \longrightarrow & \dots \\
 & & \Big\downarrow g^* & & \Big\downarrow g^* & & \Big\downarrow g^* & & \\
\dots & \longrightarrow & H^0(\partial\tilde{M}_{i_0}) & \xrightarrow{\delta^*} & H^1(\tilde{M}_{i_0}, \partial\tilde{M}_{i_0}) & \xrightarrow{j^*} & H^1(\tilde{M}_{i_0}) & \longrightarrow & \dots,
\end{array} \tag{7}$$

where the rows are taken from the cohomological sequence of the pair $(\tilde{M}_{i_0}, \partial\tilde{M}_{i_0})$ and the vertical arrows denote the mappings induced by the diffeomorphism g. All squares of the diagram are commutative, and the middle automorphism g^* from (7) has an eigenvalue $\tilde{\lambda}$. From this, by [7, Lemma 3] it follows that for one of the extreme vertical automorphisms of the diagram (7) $\tilde{\lambda}$ is also an eigenvalue. Since $\tilde{\lambda}^2 \neq 1$, then for the automorphism $g^* : H^1(\tilde{M}_{i_0}) \to H^1(\tilde{M}_{i_0})$ this is impossible according to proven in c). Since the manifold $\tilde{M}_{i_0}$ is compact, its boundary $\partial\tilde{M}_{i_0}$ consists of a finite set of connected components. Then by Lemma 4.7 all eigenvalues of the automorphism $g^* : H^0(\partial\tilde{M}_{i_0}) \to H^0(\partial\tilde{M}_{i_0})$ are roots of unity. Thus, in this case we also obtain a contradiction.

If M^n is non-orientable then, by lemma 5.2, there is an oriented two-fold covering $p : \bar{M}^n \to M^n$ and a lift $\bar{g} : \bar{M}^n \to \bar{M}^n$ of the diffeomorphism[3] g. Herewith, by lemma 5.3, $\bar{A} = p^{-1}(A)$ is orientable, like A. So we can apply all arguments from an orientable case to $\bar{g}$ and get a contradiction. □

4. **Homology and induced automorphisms.** In this section, we calculate the homology groups of some topological pairs and study the properties of automorphisms of cohomology groups induced by homeomorphisms.

4.1. **Calculations.** In this section we calculate relative homology for the following situation. Let M and N be smooth 3-manifolds with boundaries such that $P = M \cup N$ is connected, $M \cap N = \partial M$ and connected components of ∂M are some of connected components of ∂N. Let us calculate the relative homology groups of the pair (P, N).

Firstly notice that $H_0(P, N; \mathbb{Z}) = 0$ as the manifold P is connected and N is not empty. For the calculation of other relative homology groups we need the following fact.

Lemma 4.1. *For every natural k the following isomorphism takes place*

$$H_k(P, N; \mathbb{Z}) \cong H_k(M, \partial M; \mathbb{Z}).$$

Proof. By [13, Theorem 6.1, Chapter 4] the boundary ∂N possesses a collar in N. As connected components of ∂M are connected components of ∂N then there is an embedding $\phi : \partial M \times [0, 1) \to N$ such that $\phi(a, 0) = a$ for every $a \in \partial M$. Let $V = \phi(\partial M \times [0, 1))$, $B = N \setminus V$ and $\operatorname{cl} B$ be the closure of $B \subset P$ in P, $\operatorname{int} N$ be the interior of $N \subset P$ in P. By the construction $\operatorname{cl} B = B \subset N \setminus \partial M = \operatorname{int} N$. The excision theorem [6, Corollary 7.4, Chapter III] claims in such case that

$$H_k(P, N; \mathbb{Z}) \cong H_k(P \setminus B, N \setminus B; \mathbb{Z}) = H_k(M \cup V, V; \mathbb{Z}).$$

But the pair $(M \cup V, V)$ is homotopically equivalent to the pair $(M, \partial M)$. Hence, $H_k(M \cup V, V; \mathbb{Z}) \cong H_k(M, \partial M; \mathbb{Z})$ for a natural k. □

Below we calculate $H_k(M, \partial M; \mathbb{Z})$ in two cases: 1) M is a handlebody of a genus $g \geqslant 0$, 2) $M \cong \mathbb{T}^2 \times [0, 1]$.

Lemma 4.2. *If M is a handlebody of a genus $g \geqslant 0$ then*

$$H_3(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}, \quad H_2(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}^g, \quad H_1(M, \partial M; \mathbb{Z}) = 0. \tag{8}$$

Proof. As $H_3(M; \mathbb{Z}) = 0$ and $H_0(M, \partial M; \mathbb{Z}) = 0$ then the homological sequence of the pair $(M, \partial M)$ has the following form [6, Proposition 4.4, Chapter III]:

$$0 \longrightarrow H_3(M, \partial M; \mathbb{Z}) \xrightarrow{\partial^3_*} H_2(\partial M; \mathbb{Z}) \xrightarrow{\imath^2_*} H_2(M; \mathbb{Z}) \xrightarrow{\jmath^2_*} H_2(M, \partial M; \mathbb{Z}) \xrightarrow{\partial^2_*}$$
$$\longrightarrow H_1(\partial M; \mathbb{Z}) \xrightarrow{\imath^1_*} H_1(M; \mathbb{Z}) \xrightarrow{\jmath^1_*} H_1(M, \partial M; \mathbb{Z}) \xrightarrow{\partial^1_*}$$

[3]We have not found a reference for this fact, so we prove it in the section 5 below.

$$\longrightarrow H_0(\partial M;\mathbb{Z}) \xrightarrow{\imath^0_*} H_0(M;\mathbb{Z}) \xrightarrow{\jmath^0_*} 0. \quad (9)$$

Handlebody M of a genus g is the 3-ball with glued g 3-handles of the index 1. That is M is homotopically equivalent to the bouquet of g circles. Therefore,

$$H_2(M;\mathbb{Z}) = 0, \quad H_1(M;\mathbb{Z}) \cong \mathbb{Z}^g, \quad H_0(M;\mathbb{Z}) \cong \mathbb{Z}.$$

On the other side the boundary ∂M is homeomorphic to the surface S_g of the genus g. Hence,

$$H_2(\partial M;\mathbb{Z}) \cong \mathbb{Z}, \quad H_1(\partial M;\mathbb{Z}) \cong \mathbb{Z}^{2g}, \quad H_0(\partial M;\mathbb{Z}) \cong \mathbb{Z}.$$

Substituting the latter in (9), we get the exact sequence

$$0 \longrightarrow H_3(M,\partial M;\mathbb{Z}) \xrightarrow{\partial^3_*} \mathbb{Z} \xrightarrow{\imath^2_*} 0 \xrightarrow{\jmath^2_*} H_2(M,\partial M;\mathbb{Z}) \xrightarrow{\partial^2_*}$$
$$\longrightarrow \mathbb{Z}^{2g} \xrightarrow{\imath^1_*} \mathbb{Z}^g \xrightarrow{\jmath^1_*} H_1(M,\partial M;\mathbb{Z}) \xrightarrow{\partial^1_*} \mathbb{Z} \xrightarrow{\imath^0_*} \mathbb{Z} \xrightarrow{\jmath^0_*} 0. \quad (10)$$

As $\imath^1_*$ is an epimorphism then (10) decomposes into short exact sequences

$$0 \longrightarrow H_3(M,\partial M;\mathbb{Z}) \xrightarrow{\partial^3_*} \mathbb{Z} \xrightarrow{\imath^2_*} 0,$$
$$0 \longrightarrow H_2(M,\partial M;\mathbb{Z}) \xrightarrow{\partial^2_*} \mathbb{Z}^{2g} \xrightarrow{\imath^1_*} \mathbb{Z}^g \longrightarrow 0,$$
$$0 \longrightarrow H_1(M,\partial M;\mathbb{Z}) \xrightarrow{\partial^1_*} \mathbb{Z} \xrightarrow{\imath^0_*} \mathbb{Z} \longrightarrow 0,$$

from which follows the statement of the lemma. □

Lemma 4.3. *If* $M = \mathbb{T}^2 \times [0,1]$ *then*

$$H_3(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}, \quad H_2(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}^2, \quad H_1(M,\partial M;\mathbb{Z}) = \mathbb{Z}. \quad (11)$$

Proof. As M is homotopically equivalent to $\mathbb{T}^2$ and ∂M is homeomorphic to $\mathbb{T}^2 \times \mathbb{S}^0$ then $H_k(M;\mathbb{Z}) \cong H_k(\mathbb{T}^2;\mathbb{Z})$ and $H_k(\partial M;\mathbb{Z}) \cong H_k(\mathbb{T}^2;\mathbb{Z}) \times H_k(\mathbb{T}^2;\mathbb{Z})$. In such situation the homological sequence (9) of the pair $(M,\partial M)$ has the following form:

$$0 \longrightarrow H_3(M,\partial M;\mathbb{Z}) \xrightarrow{\partial^3_*} \mathbb{Z}^2 \xrightarrow{\imath^2_*} \mathbb{Z} \xrightarrow{\jmath^2_*} H_2(M,\partial M;\mathbb{Z}) \xrightarrow{\partial^2_*}$$
$$\longrightarrow \mathbb{Z}^4 \xrightarrow{\imath^1_*} \mathbb{Z}^2 \xrightarrow{\jmath^1_*} H_1(M,\partial M;\mathbb{Z}) \xrightarrow{\partial^1_*} \mathbb{Z}^2 \xrightarrow{\imath^0_*} \mathbb{Z} \xrightarrow{\jmath^0_*} 0. \quad (12)$$

As the inclusion of every connected component of ∂M to M is a homotopical equivalence then $\imath^2_*$ and $\imath^1_*$ are epimorphisms. Herewith (12) decomposes into short exact sequences

$$0 \longrightarrow H_3(M,\partial M;\mathbb{Z}) \xrightarrow{\partial^3_*} \mathbb{Z}^2 \xrightarrow{\imath^2_*} \mathbb{Z} \longrightarrow 0,$$
$$0 \longrightarrow H_2(M,\partial M;\mathbb{Z}) \xrightarrow{\partial^2_*} \mathbb{Z}^4 \xrightarrow{\imath^1_*} \mathbb{Z}^2 \longrightarrow 0,$$
$$0 \longrightarrow H_1(M,\partial M;\mathbb{Z}) \xrightarrow{\partial^1_*} \mathbb{Z}^2 \xrightarrow{\imath^0_*} \mathbb{Z} \longrightarrow 0,$$

from which follows the statement of the lemma. □

4.2. **Eigenvalues of induced automorphisms.** In this section we again consider all homology groups $H_k(X, A; \mathbb{Z})$ with integer coefficients and cohomology groups $H^k(X, A)$ with real coefficients. Firstly, by the Universal Coefficient Formula [6, Section 7, Chapter VI], the previous subsection results give the following calculations.

Lemma 4.4. *If M is a handlebody of a genus $g \geqslant 0$ then*

$$H^3(M, \partial M) \cong \mathbb{R},\ H^2(M, \partial M) \cong \mathbb{R}^g,\ H^1(M, \partial M) = 0,\ H^0(M, \partial M) = 0.$$

Lemma 4.5. *If $M = \mathbb{T}^2 \times [0, 1]$ then*

$$H^3(M, \partial M) \cong \mathbb{R},\ H^2(M, \partial M) \cong \mathbb{R}^2,\ H^1(M, \partial M) = \mathbb{R},\ H^0(M, \partial M) = 0.$$

The groups $H^k(X, A) \cong \mathbb{R}^m$ admit many automorphisms even for $m = 1$. But in some cases only a small part of them can be induced by homeomorphisms of the topological space X.

Lemma 4.6. *Let X be a topological space, $A \subset X$ be its subspace, $f : X \to X$ be homeomorphism, and $f(A) \subset A$. Denote by $H'_k(X, A; \mathbb{Z})$ a free part of the group of k-dimensional singular homology of the pair (X, A), and by $H^k(X, A)$ its k-dimensional cohomology group with real coefficients. It $H'_k(X, A; \mathbb{Z}) \cong \mathbb{Z}$ for some k, then for the induced automorphism $f^* : H^k(X, A) \to H^k(X, A)$ the equality $f^* = \pm\,\mathrm{id}$ holds.*

Proof. Let the automorphism $f_* : H'_k(X, A; \mathbb{Z}) \to H'_k(X, A; \mathbb{Z})$ also induced by the homeomorphism f. The formula $f^*_h(q) = q \circ f_*$ defines the automorphism $f^*_h : \mathrm{Hom}(H'_k(X, A; \mathbb{Z}); \mathbb{R}) \to \mathrm{Hom}(H'_k(X, A; \mathbb{Z}); \mathbb{R})$. If $H'_k(X, A; \mathbb{Z}) \cong \mathbb{Z}$, then $f_* = \pm\,\mathrm{id}$. Moreover, $f^*_h(q) = q \circ (\pm\,\mathrm{id}) = \pm q$ for all $q \in \mathrm{Hom}(H'_k(X, A; \mathbb{Z}); \mathbb{R})$. Hence $f^*_h = \pm\,\mathrm{id}$.

It follows for the Universal Coefficient Formula for cohomology [6, Chapter VI, Section 7] that there exists the natural isomorphism $\kappa : H^k(X, A) \to \mathrm{Hom}(H'_k(X, A; \mathbb{Z}); \mathbb{R})$. The naturalness means commutativity of the diagram

$$\begin{array}{ccc} H^k(X, A) & \xrightarrow{\kappa} & \mathrm{Hom}(H'_k(X, A; \mathbb{Z}); \mathbb{R}) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*_h} \\ H^k(X, A) & \xrightarrow{\kappa} & \mathrm{Hom}(H'_k(X, A; \mathbb{Z}); \mathbb{R}). \end{array} \tag{13}$$

It follows from (13) and the equation $f^*_h = \pm\,\mathrm{id}$ that $f^* = \kappa^{-1} \circ f^*_h \circ \kappa = \kappa^{-1} \circ (\pm\,\mathrm{id}) \circ \kappa = \pm\,\mathrm{id}$. □

Lemma 4.7. *Let X be a topological space with a finite number of path-connected components, $f : X \to X$ be a homeomorphism, and $f^* : H^0(X) \to H^0(X)$ be an induced automorphism. Then any eigenvalue λ for f^* satisfies the equality $\lambda^2 = 1$.*

Proof. Firstly consider a case when X_1 and X_2 are path-connected topological spaces and $f_2 : X_1 \to X_2$ is a homeomorphism. All elements of groups $H^0(X_j)$ are constant functions $c_j : X_j \to \mathbb{R}$. Therefore, the formula $\nu_j(c_j) = \mathrm{im}\, c_j$ defines isomorphisms $\nu_j : H^0(X_j) \to \mathbb{R}$, $j = 1, 2$. The induced isomorphism $f^*_2 : H^0(X_2) \to H^0(X_1)$ is defined by the formula $f^*_2(c_2) = c_2 \circ f_2$. Since values of the functions c_2 and $c_2 \circ f_2$ are equal, then

$$\nu_1 \circ f^*_2 = \nu_2. \tag{14}$$

Now suppose that X consists of path-connected components $X_1, \ldots, X_m$. Then there exists a permutation $\sigma \in S_m$ such that f maps the component X_j onto the component $X_{\sigma(j)}$ homeomorphically. Thus, setting $f_{\sigma(j)}(x) = f(x)$ for all $x \in X_j$, we obtain homeomorphisms $f_{\sigma(j)} : X_j \to X_{\sigma(j)}$, $j = 1, \ldots, m$. Moreover, the induced homomorphisms $f_j^* : H^0(X_j) \to H^0(X_{\tau(j)})$ are defined, where $\tau = \sigma^{-1}$. By virtue of (14)

$$\nu_{\tau(j)} \circ f_j^* = \nu_j, \quad j = 1, \ldots, m. \tag{15}$$

For each element $c \in H^0(X)$ we set $c_j = c|_{X_j}$. Then $c_j \in H^0(X_j)$. Define isomorphisms $\mu : H^0(X) \to H^0(X_1) \times \cdots \times H^0(X_m)$ and $\nu : H^0(X_1) \times \cdots \times H^0(X_m) \to \mathbb{R}^m$ by the formulas $\mu(c) = (c_1, \ldots, c_m)$ and $\nu((c_1, \ldots, c_m)) = (\nu_1(c_1), \ldots, \nu_m(c_m))$. We construct the automorphism $p : \mathbb{R}^m \to \mathbb{R}^m$ such that the diagram is commutative

$$\begin{array}{ccccc} H^0(X) & \xrightarrow{\mu} & H^0(X_1) \times \cdots \times H^0(X_m) & \xrightarrow{\nu} & \mathbb{R}^m \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle (f_1^*, \ldots, f_m^*)} & & \downarrow{\scriptstyle p} \\ H^0(X) & \xrightarrow{\mu} & H^0(X_1) \times \cdots \times H^0(X_m) & \xrightarrow{\nu} & \mathbb{R}^m. \end{array} \tag{16}$$

For all $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ we set $\|y\| = \sqrt{y_1^2 + \cdots + y_m^2}$. Since f_j^* maps $H^0(X_j)$ onto $H^0(X_{\tau(j)})$, then it follows from the equality (15) and the diagram (16) that $p(y) = (y_{\tau(1)}, \ldots, y_{\tau(1)})$. Moreover, $\|p(y)\| = \|y\|$.

Finally, let $\lambda \in \mathbb{R}$, $c \in H^0(X)$, $c \neq 0$ and $f^*(c) = \lambda c$. We set $y = \nu \circ \mu(c)$. Then by virtue of (16) $p(y) = \mu \circ \nu(f^*(c)) = \mu \circ \nu(\lambda c) = \lambda y$. Hence, according to what was proved above, we obtain $\|y\|^2 = \|p(y)\|^2 = \lambda^2 \|y\|^2$. Hence, $\lambda^2 = 1$. □

5. **On oriented two-fold covering.** Let M be a non-orientable connected smooth n-manifold, $a \in M$ and $x : I \to M$ be a loop based at a point a. Let us consider continuous vector fields $X_1, \ldots, X_n$ along x such that $X_1(t), \ldots, X_n(t)$ linearly independent for each $t \in I$. Then there is a matrix $A = (a_i^j) \in \mathrm{GL}_n(\mathbb{R})$ such that

$$X_i(1) = a_i^j X_j(0), \quad i, j = 1, \ldots, n. \tag{17}$$

Let $\omega_a(x) = \operatorname{sign} \det A$. If y is a loop which based at the same starting point and $x \sim y$ then $\omega_a(x) = \omega_a(y)$. Therefore the formula $\omega_a([x]) = \omega_a(x)$ defines a homeomorphism $\omega_a : \pi_1(M, a) \to G$, where $G = \{1, -1\}$. The manifold M is orientable if and only if $\ker \omega_a = \pi_1(M, a)$.

Let $a, b \in M$, $z : I \to M$ be a path which starts in $z(0) = a$ and ends in $z(1) = b$ and $T_z : \pi_1(M, a) \to \pi_1(M, b)$ be the isomorphism defined by the formula $T_z([x]) = [z^{-1} x z]$. Then $z z^{-1} \sim 1_a$ and $z^{-1} z \sim 1_b$ implies commutativity of the diagram

$$\begin{array}{ccc} \pi_1(M, a) & \xrightarrow{\omega_a} & \mathbb{R} \\ {\scriptstyle T_z}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \pi_1(M, b) & \xrightarrow{\omega_b} & \mathbb{R}. \end{array} \tag{18}$$

Lemma 5.1. *Let M, N be connected smooth manifolds, $f : M \to N$ be a local diffeomorphism, $a \in M$, $b = f(a)$ and $f_* : \pi_1(M, a) \to \pi_1(N, b)$ be an induced homeomorphism. Then the following diagram is commutative*

$$\begin{array}{ccc} \pi_1(M, a) & \xrightarrow{\omega_a} & \mathbb{R} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \pi_1(N, b) & \xrightarrow{\omega_b} & \mathbb{R}. \end{array} \tag{19}$$

Proof. Let $[x] \in \pi_1(M,a)$, $X_1,\dots,X_n$ be continuous vector fields along x, linearly independent at each point $x(t)$, and the equality (17) is satisfied. Let $y = f \circ x$ and $Y_i(t) = df_{x(t)}(X_i(t))$ for every $i = 1,\dots,n$ and $t \in I$. Then $[y] \in \pi_1(N,b)$, $[y] = f_*([x])$ and $Y_1,\dots,Y_n$ are continuous vector fields along the loop y. According to the condition, $df_{x(t)} : T_{x(t)}M \to T_{y(t)}N$ are isomorphisms. Therefore $Y_1(t),\dots,Y_n(t)$ are linearly dependent for all $t \in I$. But

$$Y_i(1) = df_a(X_i(1)) = df_a(a_i^j X_j(0)) = a_i^j df_a(X_j(0)) = a_i^j Y_j(0).$$

from (17) and the linearity of the differential $df_a : T_aM \to T_bN$. Thus, $\omega_b([y]) = \operatorname{sign}\det A = \omega_a([x])$. □

Lemma 5.2. *Let M be a non-orientable connected smooth manifold and $f : M \to M$ be a diffeomorphism. Then there exists a connected smooth orientable manifold $\bar M$, a smooth two-fold cover $p : \bar M \to M$ and a diffeomorphism $\bar f : \bar M \to \bar M$ for which the diagram is commutative*

$$\begin{array}{ccc} \bar M & \xrightarrow{\ \bar f\ } & \bar M \\ p\big\downarrow & & \big\downarrow p \\ M & \xrightarrow{\ f\ } & M. \end{array} \tag{20}$$

Proof. Let $a \in M$. Then $\ker\omega_a$ is the normal divisor of the group $\pi_1(M,a)$. By the theorem of the existence of covers, there will be a connected smooth manifold $\bar M$, a regular smooth cover $p : \bar M \to M$ and a point $u \in \bar M$ such that $p(u) = a$ and the induced homomorphism $p_*^u : \pi_1(\bar M,u) \to \pi_1(M,a)$ has the image $\operatorname{im} p_*^u = \ker\omega_a$. As the manifold M is non-orientable then $\pi_1(M,a)/\ker\omega_a \cong G$. Therefore p is a two-fold covering. As $p_*^u : \pi_1(\bar M,u) \to \ker\omega_a$ is an isomorphism then by Lemma 5.1 we get $\ker\omega_u = \pi_1(\bar M,u)$. That means $\bar M$ is an orientable manifold.

Let $b = f(a)$ and $v \in p^{-1}(b)$. As the manifold $\bar M$ is connected then there is a path $\bar z : I \to \bar M$ with the starting in $\bar z(0) = u$ and the end in $\bar z(1) = v$. Let $z = p \circ \bar z$. Then $z(0) = a$, $z(1) = b$ and

$$\operatorname{im} p_*^v = T_z(\operatorname{im} p_*^u). \tag{21}$$

As $T_z : \pi_1(M,a) \to \pi_1(M,b)$ is an isomorphism then (18) implies

$$\ker\omega_b = T_z(\ker\omega_a). \tag{22}$$

Finitely, as $f_* : \pi_1(M,a) \to \pi_1(M,b)$ is an isomorphism, (19) implies the equality

$$\ker\omega_b = f_*(\ker\omega_a). \tag{23}$$

If follows from (21), (22), (23) and the equality $\operatorname{im} p_*^u = \ker\omega_a$ that

$$\operatorname{im}(f \circ p)_*^u = f_*(\operatorname{im} p_*^u) = f_*(\ker\omega_a) = \ker\omega_b = T_z(\ker\omega_a) = T_z(\operatorname{im} p_*^u) = \operatorname{im} p_*^v.$$

According to a theorem from the theory of covering, in such a situation there is a map $\bar f : \bar M \to \bar M$ such that $\bar f(u) = v$ and the diagram (20) is commutative. This mapping is uniquely defined and is smooth. Similarly, it is proved that for the inverse diffeomorphism $f^{-1} : M \to M$ there is a smooth map $\overline{f^{-1}} : \bar M \to \bar M$ such that $\overline{f^{-1}}(v) = u$ and the following diagram is commutative

$$\begin{array}{ccc} \bar M & \xrightarrow{\ \overline{f^{-1}}\ } & \bar M \\ p\big\downarrow & & \big\downarrow p \\ M & \xrightarrow{\ f^{-1}\ } & M. \end{array} \tag{24}$$

Adding (24) to (20) on the right and on the left, we get the equality $\overline{f^{-1}} \circ \bar{f} = \mathrm{id}$ and $\bar{f} \circ \overline{f^{-1}} = \mathrm{id}$. Therefore $\overline{f^{-1}} = \bar{f}^{-1}$ and $\bar{f}$ is a diffeomorphism. □

Lemma 5.3. *Let M be a smooth closed non-orientable connected 3-manifold and W^1, $W^2 \subset M$ be immersions of open balls D^1, D^2 accordingly, such that $\mathrm{Ind}_x(W^1, W^2) = \mathrm{Ind}_y(W^1, W^2)$ for every points x, $y \in (W^1 \cap W^2)$. If $p : \bar{M} \to M$ is an oriented double covering then $\bar{W}^1 = p^{-1}(W^1)$, $\bar{W}^2 = p^{-1}(W^2)$ be immersions of two copies of open balls D^1, D^2 accordingly, $\bar{W}^1 = \bar{W}^1_1 \sqcup \bar{W}^1_2$, $\bar{W}^2 = \bar{W}^2_1 \sqcup \bar{W}^2_2$, and $\mathrm{Ind}_{\bar{x}}(\bar{W}^1_i, \bar{W}^2_j) = \mathrm{Ind}_{\bar{y}}(\bar{W}^1_i, \bar{W}^2_j)$ for every points $\bar{x}$, $\bar{y} \in (\bar{W}^1_i \cap \bar{W}^2_j)$, i, $j = 1, 2$.*

Proof. Consider a tubular neighborhood U^k of the submanifolds W^k. Since the open subsets $U^k \subset M$, $k = 1, 2$, are contractible, they are regular covered neighborhoods. That is $p^{-1}(U^k) = \bar{U}^k_1 \cup \bar{U}^k_2$, where $\bar{U}^k_1 \cap \bar{U}^k_2 = \emptyset$ and $p|_{\bar{U}^k_i} : \bar{U}^k_i \to U^k$ are diffeomorphisms, $i = 1, 2$. Then the sets $\bar{U}^k_i$ are tubular neighborhoods of smooth submanifolds $\bar{W}^k_i \subset \bar{M}$, and the differences $\bar{U}^2_i \setminus \bar{W}^2_i$ consist of the connected components $\bar{U}^2_{i+}$ and $\bar{U}^2_{i-}$.

Let $\bar{\sigma}_i : \bar{U}^2_{i+} \cup \bar{U}^2_{i-} \to \mathbb{Z}$ be a function such that $\bar{\sigma}(\bar{x}) = 1$ for $\bar{x} \in \bar{U}^2_{i+}$ and $\bar{\sigma}(\bar{x}) = 0$ for $\bar{x} \in \bar{U}^2_{i-}$. As $\bar{W}^1_i = (p|_{\bar{W}^1_i})^{-1}(J^1(D^1))$ then the intersection index in $\bar{x} \in (\bar{W}^1_i \cap \bar{W}^2_j)$ is equal to $\mathrm{Ind}_{\bar{x}}(\bar{W}^1_i, \bar{W}^2_j) = \bar{\sigma}(t+\delta) - \bar{\sigma}(t-\delta)$, where δ is a small enough positive number. Then $\mathrm{Ind}_x(W^1, W^2) = \mathrm{Ind}_{\bar{x}}(\bar{W}^1_i, \bar{W}^2_j)$ and $\mathrm{Ind}_y(W^1, W^2) = \mathrm{Ind}_{\bar{y}}(\bar{W}^1_i, \bar{W}^2_j)$. So if $\mathrm{Ind}_x(W^1, W^2) = \mathrm{Ind}_y(W^1, W^2)$ for every points x, $y \in (W^1 \cap W^2)$ then $\mathrm{Ind}_{\bar{x}}(\bar{W}^1_i, \bar{W}^2_j) = \mathrm{Ind}_{\bar{y}}(\bar{W}^1_i, \bar{W}^2_j)$ for every points $\bar{x}$, $\bar{y} \in (\bar{W}^1_i \cap \bar{W}^2_j)$, i, $j = 1, 2$. □

6. **Example of a diffeomorphism with a non-orientable expanding 2-dimensional attractor.** Let us construct an example of an Ω-stable diffeomorphism of a closed connected 3-manifold M^3 the non-wandering set of which consists of trivial sources, saddles, and a non-orientable expanding 2-dimensional attractor Λ.

We will start with hyperbolic toral automorphism $L_A : \mathbb{T}^3 \to \mathbb{T}^3$ induced by linear map of $\mathbb{R}^3$ with a hyperbolic matrix $A \in GL(3, \mathbb{Z})$, eigenvalues λ_1, λ_2, λ_3 of which such that $0 < \lambda_1 < 1 < \lambda_2 \leqslant \lambda_3$. The involution $J : \mathbb{T}^3 \to \mathbb{T}^3$ defined by the formula $J(x) = -x \pmod 1$ has 8 fixed points in the 3-torus of the form (a, b, c), where $a, b, c \in \{0, \frac{1}{2}\}$. Notice that these points are also fixed for L^k_A for some $k \in \mathbb{N}$. Let us "blow up" these points like to the classical Smale surgery and such that the surgery commutes with the involution. We will obtain generalized DA-diffeomorphism $f_{GDA} : \mathbb{T}^3 \to \mathbb{T}^3$ with 8 fixed sources α_i, $i \in \{1, 2, \ldots, 8\}$ and one 2-dimensional expanding attractor obtained from the diffeomorphism L^k_A.

After that we will remove all sources and factorize the basin of the attractor to obtain a new manifold $\tilde{M}$, i.e. $\tilde{M} = (\mathbb{T}^3 \setminus \bigcup\limits_{i=1}^{8} \alpha_i)/_{x \sim -x}$. The natural projection $p : \mathbb{T}^3 \setminus \bigcup\limits_{i=1}^{8} \alpha_i \to \tilde{M}$ is a 2-fold cover. As $f_{GDA} J = J f_{GDA}$ then f_{GDA} is projected to $\tilde{M}$ by the diffeomorphism $\tilde{f} = p f_{GDA} p^{-1} : \tilde{M} \to \tilde{M}$ with one 2-dimensional expanding attractor Λ and $\tilde{M}$ is its basin. The set $\tilde{M} \setminus \Lambda$ consists of 8 connected components $\tilde{N}_i$ each of which is diffeomorphic to $\mathbb{R}P^2 \times \mathbb{R}$, where $\mathbb{R}P^2$ is the real projective plane.

To obtain a fundamental domain $\tilde{D}_i$ of $\tilde{f}|_{\tilde{N}_i}$ we can consider a local coordinates $(x, y, z) : U_i \to \mathbb{R}^3$ in a neighborhood U_i of α_i in which the diffeomorphism f_{GDA} has a form $f_{GDA}(x, y, z) = (2x, 2y, 2z)$. A fundamental domain of $f_{GDA}|_{W^u_{\alpha_i} \setminus \{\alpha_i\}}$

is $D_i = \{(x,y,z)\in\mathbb{R}^3\,|\,1\leqslant x^2+y^2+z^2\leqslant 4\}$ and then the desired fundamental domain $\tilde{D}_i = p(D_i)$. By the construction it is homeomorphic to $RP^2\times[0,1]$. The orbit space of $f_{GDA}|_{W^u_{\alpha_i}\setminus\{\alpha_i\}}$ is homeomorphic to $S^2\times S^1$ since each orientation preserving diffeomorphism of S^2 is homotopic to identity. Then the orbit space $\tilde{N}_i/\tilde{f}$ can be obtained as $S^2\times S^1|_{\tilde{J}}$, where $\tilde{J}$ is involution of $S^2\times S^1$ induced by J. Since $\tilde{N}_i/\tilde{f}$ is non-orientable, it follows from [14] that $\tilde{N}_i/\tilde{f}$ is either $S^2\tilde{\times}S^1$, $RP^2\times S^1$, or $RP^3\#RP^3$. The orbit space $\tilde{N}_i/\tilde{f}$ can also be obtained from the fundamental domain $\tilde{D}_i$ as a mapping torus $RP^2\times[0,1]|_{(x,0)\sim(\tilde{f}(x),1)}$. Hence a fundamental group of the orbit space $\pi_1(\tilde{N}_i/\tilde{f}) = \mathbb{Z}_2\rtimes_{\tilde{f}}\mathbb{Z}$ and then it can be only $RP^2\times S^1$.

Consider a gradient-like diffeomorphism $g_1:\mathbb{R}P^2\to\mathbb{R}P^2$ with exactly 3 fixed points: a source α, a sink ω and a saddle σ (see Fig. 6). Let $g_2:\mathbb{R}\to\mathbb{R}$ be

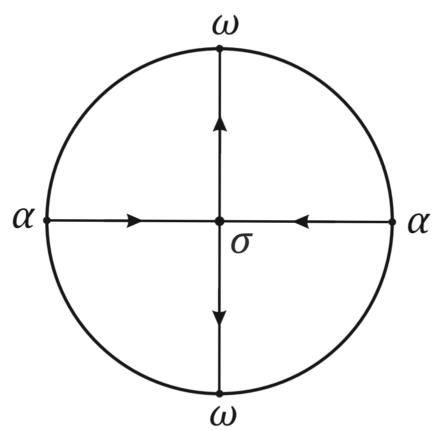

FIGURE 6. Diffeomorphism g on the projective plane

a diffeomorphism given by the formula $g_2(x) = 2x$ and $g(w,x) = (g_1(w), g_2(x)):\mathbb{R}P^2\times\mathbb{R}\to\mathbb{R}P^2\times\mathbb{R}$. Let us denote N_1, N_2 the connected components of $\mathbb{R}P^2\times(\mathbb{R}\setminus\{0\})$. Analogically with cases with $\tilde{N}_i$ the orbit spaces N_j/g are diffeomorphic to $\mathbb{R}P^2\times\mathbb{S}^1$.

As $\tilde{N}_i/\tilde{f}$ are diffeomorphic to N_j/g then there is a diffeomorphism $h:\tilde{N}_i\to N_j$ conjugating $\tilde{f}$ with g. Let $h_i:\tilde{N}_i\to N_1$, $i=1,3,5,7$ and $h_i:\tilde{N}_i\to N_2$, $i=2,4,6,8$ be such diffeomorphisms. For $\tilde{N} = \bigcup\limits_{i=1}^{8}\tilde{N}_i$ denote by $h:\tilde{N}\to(N_1\sqcup N_2)\times\mathbb{Z}_4$ a diffeomorphism composed by h_i, $i\in\{1,\dots,8\}$. Let $\tilde{P} = \mathbb{R}P^2\times\mathbb{R}\times\mathbb{Z}_4$ and $G:\tilde{P}\to\tilde{P}$ be a diffeomorphism composed by g on every copy of $\mathbb{R}P^2\times\mathbb{R}$. Finitely, let $M^3 = \tilde{M}\cup_h\tilde{P}$. Denote by $q:\tilde{M}\sqcup\tilde{P}\to M^3$ the natural projection. Then the desired diffeomorphism $f:M^3\to M^3$ coincides with the diffeomorphism $q\tilde{f}q^{-1}|_{q(\tilde{M})}$ on $q(\tilde{M})$ and with the diffeomorphism $qGq^{-1}|_{q(\tilde{P})}$ on $q(\tilde{P})$.

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