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Polyhedral models for K-theory of toric and flag varieties

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Abstract. In 1992, Pukhlikov and Khovanskii provided a description of the cohomology ring of toric variety as a quotient of the ring of differential operators on spaces of virtual polytopes. Later Kaveh generalized this construction to the case of cohomology rings for full flag varieties.

In this paper we extend Pukhlikov–Khovanskii type presentation to the case of K-theory of toric and flag varieties. First we study the Gorenstein duality quotients of the group algebra of free abelian group (possibly of infinite rank). Then we specialize to the K-ring of integer (virtual) polytopes with a fixed normal fan. Finally we show that the K-theory of toric and flag varieties can be realized as polytope K-rings and describe the classes of toric orbits or Schubert varieties in them.

Keywords: K-theory, toric varieties, flag varieties

1 Introduction

This paper is devoted to the study of K-theory of toric and flag varieties. In [12], A. Khovanskii and A. Pukhlikov described the cohomology ring of a toric variety X as the quotient of the ring of differential operators with constant coefficients modulo the annihilator of the volume polynomial of the moment polytope of X. This construction was generalized by K. Kaveh [5] who observed that the cohomology ring of a full flag variety can be obtained by applying the same construction to a Gelfand–Zetlin polytope. This description was later used by V. Kiritchenko, E. Smirnov, and V. Timorin [8] to provide a polyhedral model for Schubert calculus on full flag varieties.

In this paper we generalize the above results to the K-theory. First, in Section 2 we generalize the construction of algebras with Gorenstein duality pairing via Macaulay inverse systems (see [7]) to the case of shift operators with constant coefficients. Then we use this to define a polytope K-ring by considering the algebra of shift operators

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with constant coefficients modulo the annihilator of the Ehrhart polynomial and study its properties in the case when polytope is integrally simple.

In Section 3 we prove that the K-theory of a toric variety coincides with the K-ring of its moment polytope. Finally, in Section 4 we extend this to full flag varieties showing that K-theory of G/B for $G = \mathrm{GL}(n)$ is given by the K-ring of Gelfand–Zetlin polytope. Moreover, we identify the structure sheaves of Schubert varieties of G/B in the K-ring of Gelfand–Zetlin polytope. This provides a polyhedral model for K-theoretic Schubert calculus generalizing results of [8].

This is a report on the forthcoming paper [10]. In particular, we skip most of the proofs in this text and refer the reader to [10] for more details and complete proofs.

2 Polytope algebra

In this section we associate an algebra with duality to any function $f\colon \Lambda \to \mathbf{k}$ on a lattice. We then study a particular example of this construction given by the Ehrhart function on the lattice of integral (virtual) polytopes with a given normal fan.

2.1 Algebra associated to a function on a lattice

Let $\mathbf{k}$ be any field and let A be a commutative algebra with identity over $\mathbf{k}$. In further subsections of the paper we will stick with $\mathbf{k} = \mathbb{Q}$, however we decided to work with more general setting here. A linear function $\ell\colon A \to \mathbf{k}$ defines a symmetric, bilinear pairing on A via:

$$\langle a, b\rangle_\ell := \ell(a \cdot b) \text{ for any } a, b \in A.$$

Definition 2.1. A pairing $\langle \cdot, \cdot \rangle_\ell$ on algebra A is called *Gorenstein duality pairing* if it is non-degenerate.

The main objective of this subsection is to give a construction of algebras with Gorenstein duality pairing from a $\mathbf{k}$-valued function on a lattice. Everywhere in this text by a lattice we mean a (possibly infinitely generated) free abelian group.

Let Λ be a lattice and let $\mathbf{k}$ be any field. We will denote by $\mathbf{k}^\Lambda$ the set of maps $f : \Lambda \to \mathbf{k}$. Let us further denote by $\mathbf{k}[\Lambda]$ the group algebra of Λ. The group algebra $\mathbf{k}[\Lambda]$ is acting on the set of functions $\mathbf{k}^\Lambda$ via shift operators. That is, for $t = \sum_{i=1}^r a_i \lambda_i \in \mathbf{k}[\Lambda]$ and $f \in \mathbf{k}^\Lambda$ we have

$$t \cdot f(x) = \sum_{i=1}^{r} a_i f(x + \lambda_i), \text{ for any } x \in \Lambda.$$

In what follows we identify the group algebra $\mathbf{k}[\Lambda]$ with the algebra of shift operators with constant coefficients $\mathrm{Sh}(\Lambda)$ on Λ. We will further denote by t_x the shift operator by $-x$ and by $D_x = 1 - t_x$ the corresponding difference operator for any $x \in \Lambda$.

Theorem 2.2. *Let $f \in \mathbf{k}^{\Lambda}$ be any function, then $\operatorname{Ann}(f) = \{s \in \operatorname{Sh}(\Lambda) \,|\, s \cdot f \equiv 0\}$ is an ideal in $\operatorname{Sh}(\Lambda)$. Moreover, the quotient algebra $A_f := \operatorname{Sh}(\Lambda)/\operatorname{Ann}(f)$ has a Gorenstein duality pairing defined by a function*

$$\ell_f \colon \operatorname{Sh}(\Lambda) \to \mathbf{k}, \quad \ell_f \colon t \mapsto (t \cdot f)(0).$$

In particular, linear function ℓ_f descends to a well-defined linear function on A_f.

The construction from Theorem 2.2 is quite general as the following Theorem shows.

Theorem 2.3. *Let A be a commutative algebra generated by invertible elements $a_1, \ldots, a_s \in A^\times$ with Gorenstein duality pairing given by a function $\ell \colon A \to \mathbf{k}$. Then $A \simeq A_f$ with*

$$f \colon \mathbb{Z}^s \to \mathbf{k}, \quad f \colon (n_1, \ldots, n_s) \mapsto \ell(a_1^{n_1} \ldots a_s^{n_s}).$$

Remark 2.4. A version of Theorem 2.2 is also true for any commutative semigroup with cancellation Λ. In this more general form Theorem 2.2 is closely related to the explicit version of Macaulay duality recently studied in [7] (see also [4, 6] for related results).

Note that for a general function $f : \Lambda \to \mathbf{k}$ the ideal $\operatorname{Ann}(f)$ might be trivial (see Example 2.5). In fact, as illustrated by Lemma 2.6, for most functions f, $\operatorname{Ann}(f) = 0$. However, for some classes of functions f, the ideal A_f is non-trivial. For instance, if f is a polynomial on the lattice Λ, the quotient algebra A_f is Artinian, i.e. is a finite dimensional vector space over $\mathbf{k}$.

Example 2.5. Let $g(x) = e^{e^x}$ be a double exponential function on one-dimensional lattice $\mathbb{Z}$ and let $T = \sum_{i=1}^{s} \lambda_i t_{k_i}$ with $k_1 < \ldots < k_s$. Then $|t \cdot g(x)| > 0$ for $x \gg 0$, in particular $t \cdot g \not\equiv 0$ so $\operatorname{Ann}(g)$ is trivial.

Lemma 2.6. *Let $\Lambda = \mathbb{Z}$ be a one-dimensional lattice. Then for each nontrivial shift operator $T = \sum_{i=1}^{k} \lambda_i t_{r_i}$ the functions $f : \mathbb{Z} \to \mathbf{k}$ annihilated by T form a finite dimensional vector space.*

In particular, if $\mathbf{k}$ is countable, the set of functions f with nontrivial annihilator $Ann(f) \neq 0$ is countable, while the set of all functions $\mathbf{k}^{\mathbb{Z}}$ is uncountable.

Proposition 2.7. *Let $\Lambda \simeq \mathbb{Z}^r$ be of finite rank, and let f be a polynomial function on Λ. Then A_f is an Artinian algebra.*

Proof. Let $D_i = 1 - t_i$ be the standard difference operators for $i = 1, \ldots, r$. Then Laurent monomials $D_1^{k_1} \ldots D_r^{k_r}$ form a basis of $\mathbf{k}[\Lambda]$. The statement follows from the fact that for a polynomial f of degree d, one has $D_1^{k_1} \ldots D_r^{k_r} \cdot f = 0$ for $|k_1| + \ldots + |k_r| > d$. □

We finish this subsection with a description of the relation between two algebras A_f, A_g defined by pairs (Λ_g, g) and (Λ_f, f) such that there is a lattice homomorphism $\sigma \colon \Lambda_g \to \Lambda_f$ with $g = \sigma^* f$. Our description is parallel to [8, Proposition 2.4].

Proposition 2.8. *There exists an abelian group $M_{f,g}$ with an epimorphism $\pi \colon A_f \to M_{f,g}$ and a monomorphism $\iota \colon A_g \to M_{f,g}$ such that $\pi(\widetilde{\alpha}\widetilde{\beta}) = \iota(\alpha\beta)$ whenever $\pi(\widetilde{\alpha}) = \iota(\alpha)$ and $\pi(\widetilde{\beta}) = \iota(\beta)$.*

2.2 Ehrhart polynomial and the polytope algebra

In this subsection we briefly recall the definition of the Ehrhart polynomial on the space of virtual polytopes. For more detailed background on virtual polytopes we refer to [11].

We will call a polytope $\Delta \subset \mathbb{R}^d$ integral if all vertices of Δ belong to the integer lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. Denote by $\mathcal{P}^+$ the set of all integral polytopes in $\mathbb{R}^d$. The set $\mathcal{P}^+$ has a structure of an abelian semigroup with respect to Minkowski addition:

$$\Delta_1 + \Delta_2 = \{x + y \mid x \in \Delta_1, y \in \Delta_2\}.$$

It is easy to check that $\mathcal{P}^+$ has the cancellation property, i.e.

$$\Delta_1 + \Delta = \Delta_2 + \Delta \quad \text{if and only if} \quad \Delta_1 = \Delta_2.$$

Thus $\mathcal{P}^+$ is embedded into its Grothendieck group, which we denote by $\mathcal{P}$ i.e. the group of formal differences of elements in $\mathcal{P}^+$. The space of integral virtual polytopes $\mathcal{P}$ has a structure of a free abelian group.

A virtual polytope Δ is uniquely described by its support function $H_\Delta : (\mathbb{R}^d)^\vee \to \mathbb{R}$ given by

$$H_\Delta(\psi) := \min_{x \in \Delta} \psi(x),$$

for a convex polytope Δ and extended by linearity to virtual polytopes. For an integral virtual polytope Δ its support function H_Δ is a piecewise linear function which attains integer values on $(\mathbb{Z}^d)^\vee$. The cones of linearity of H_Δ form a fan in $(\mathbb{R}^d)^\vee$. For a given fan $\Sigma \subset (\mathbb{R}^d)^\vee$, let us denote by $\mathcal{P}_\Sigma^+$ the set of integer convex polytopes Δ with H_Δ linear on the cones of Σ. Similarly, by $\mathcal{P}_\Sigma$ we denote the set of integer virtual polytopes with support function linear on the cones of Σ. Clearly, $\mathcal{P}_\Sigma$ is a free abelian group and $\mathcal{P}_\Sigma^+$ is its subsemigroup.

Let $e_1, \dots, e_r$ be the primitive ray generators of Σ. Then a virtual polytope is uniquely determined by the evaluation of H_Δ on $e_1, \dots, e_r$. This defines an embedding $\mathcal{P}_\Sigma \hookrightarrow \mathbb{Z}^r$ as a sublattice. Notice however that $\mathcal{P}_\Sigma$ is in general a proper sublattice of $\mathbb{Z}^r$. In fact, it can both have smaller rank and be non-saturated.

An Ehrhart polynomial $\mathrm{Ehr}\colon \mathcal{P} \to \mathbb{Z}$ is the unique polynomial on the space of virtual polytopes such that

$$\mathrm{Ehr}(\Delta) = |\Delta \cap \mathbb{Z}^d|, \text{ for any } \Delta \in \mathcal{P}^+.$$

It is shown in [12] that the polynomial Ehr is well defined, i.e. that the number of lattice points is the restriction of a polynomial function on $\mathcal{P}^+$.

Definition 2.9. Let Σ be a complete rational fan and $\mathcal{P}_\Sigma$ be the corresponding space of integer virtual polytopes. We define the polytope K-ring to be the algebra K_Σ defined by the pair $(\mathcal{P}_\Sigma, \mathrm{Ehr})$

$$K_\Sigma = \mathrm{Sh}(\mathcal{P}_\Sigma) / \mathrm{Ann}(\mathrm{Ehr}).$$

Note that since $\mathcal{P}_\Sigma$ is a finite rank lattice and Ehr is a polynomial, K_Σ is an Artinian algebra by Proposition 2.7.

2.3 Structure of polytope K-ring

In this subsection we will study the algebra K_Σ in more detail. First let us assume that Σ is a smooth fan with $|\Sigma(1)| = r$. In this case, the evaluation of H_Δ on integer ray generators of Σ canonically identifies the lattice $\mathcal{P}_\Sigma$ with $\mathbb{Z}^r$. We will denote by $\Delta_1, \dots, \Delta_r$ the corresponding basis of $\mathcal{P}_\Sigma$. In other words Δ_i is a virtual polytope in $\mathcal{P}_\Sigma$ such that

$$H_{\Delta_i}(e_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise,} \end{cases}$$

where $e_1, \dots e_r$ are the primitive ray generators of Σ. We further denote by $t_1, \dots, t_r$ the shift operators with respect to $\Delta_1, \dots, \Delta_r$.

Theorem 2.10. *Let $\Sigma \subset (\mathbb{R}^n)^\vee$ be a smooth complete fan with $\Sigma(1) = r$ and primitive ray generators $e_1, \dots, e_r$. Then the polytope ring A_Σ is given by*

$$A_f \simeq \mathbb{Q}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]/(I + J),$$

where I is generated by products $(1 - t_{i_1}^{-1}) \cdots (1 - t_{i_s}^{-1})$ such that $\rho_{i_1}, \dots, \rho_{i_s} \in \Sigma(1)$ are distinct and do not form a cone in Σ and $J = \left\langle \prod_{i=1}^r t_i^{\langle u, e_i \rangle} - 1 \mid u \in \mathbb{Z}^n \right\rangle$.

Remark 2.11. The statement of Theorem 2.10 is also true over $\mathbb{Z}$. We will treat this case in the full version [10].

The difference operators appearing in the definition of ideal I will play an important role in what follows. We will denote them by $D_i = 1 - t_i^{-1}$. More generally, for any cone $\sigma \in \Sigma$ we denote by D_σ the corresponding product of difference operators

$$D_\sigma = \prod_{\rho_i \in \Sigma} D_i.$$

The proof of Theorem 2.10 is done in two steps. The first part is to show that the relations I and J are satisfied in the algebra K_Σ. Indeed, the relations J are satisfied since

$$\prod_{i=1}^r t_i^{\langle u, e_i \rangle} \cdot \mathrm{Ehr}(\Delta) = \mathrm{Ehr}\left(\Delta + \sum_{i=1}^r \langle u, e_i \rangle \Delta_i\right) = \mathrm{Ehr}(\Delta + u) = \mathrm{Ehr}(\Delta) \text{ for any } u \in \mathbb{Z}^n.$$

The relations from I are subject of the following lemma.

Lemma 2.12. *Let Σ be a smooth fan and $\Delta \in \mathcal{P}_\Sigma$ an integer (possibly virtual) polytope. Then*

$$D_{i_1} \cdots D_{i_s} \cdot \mathrm{Ehr}(\Delta) = \mathrm{Ehr}(F_\sigma),$$

where σ is a cone spanned by $\rho_{i_1}, \dots, \rho_{i_s}$ and F_σ is a corresponding (virtual) face of Δ. In particular, if $\rho_{i_1}, \dots, \rho_{i_s}$ do not form a cone, $D_{i_1} \cdots D_{i_t} \cdot \mathrm{Ehr}(\Delta) = 0$.

Since both sets of relations I and J are satisfied in K_Σ, there exists a surjection

$$\psi\colon \mathbb{Z}[t_1^{\pm 1},\dots,t_r^{\pm 1}]/(I+J) \to A_\Sigma, \quad \psi\colon t_i \mapsto \Delta_i.$$

To show that ψ is an isomorphism, we will then use a Morse theoretic argument which is parallel to results in [13].

In what follows we will also have to work with non-smooth fans Σ and their K-rings K_Σ. It is convenient in this case to reduce to the smooth case by considering any smooth subdivison Σ' of Σ and using Proposition 2.8. Indeed, $\mathcal{P}_\Sigma \subset \mathcal{P}_{\Sigma'}$ and the Ehrhart polynomial on $\mathcal{P}_\Sigma$ is the restriction of Ehrhart polynomial on $\mathcal{P}_{\Sigma'}$. Thus there is an abelian group $M_{\Sigma',\Sigma}$ with an epimorphism $\pi\colon K_{\Sigma'} \to M_{\Sigma',\Sigma}$ and a monomorphism $\iota\colon K_\Sigma \to M_{\Sigma',\Sigma}$ such that $\pi(\widetilde{\alpha}\widetilde{\beta}) = \iota(\alpha\beta)$ whenever $\pi(\widetilde{\alpha}) = \iota(\alpha)$ and $\pi(\widetilde{\beta}) = \iota(\beta)$. This allows to perform the computations in $K_{\Sigma'}$ instead of K_Σ which we understand better.

3 Toric varieties

In this section we will apply the results of Section 2 to computation of the K-theory of toric varieties. In what follows we assume the basic knowledge of toric geometry and refer to [2] for further details and references.

For a smooth algebraic variety X we denote by $K_0(X)$ the free abelian group generated by isomorphism classes of coherent sheaves on X up to the relation $[\mathcal{V}]+[\mathcal{U}]=[\mathcal{W}]$ whenever there is a short exact sequence $0 \to \mathcal{V} \to \mathcal{W} \to \mathcal{U} \to 0$. The subgroup generated by classes of vector bundles is denoted by $K^0(X)$. For a smooth variety X the inclusion $K^0(X) \hookrightarrow K_0(X)$ is an isomorphism. In this case, we define the ring structure on $K_0(X)$ via $[\mathcal{V}][\mathcal{U}] = [\mathcal{V}\otimes\mathcal{U}]$. In what follows we will work with rational K-theory $K_0(X)\otimes_{\mathbb{Z}}\mathbb{Q}$. To simplify the notation we will denote the rational K-theory of X by $K_0(X)$. Finally, K-theory admits a proper push-forward. In particular, for a trivial map $f\colon X \to \mathrm{pt}$, the pushforward $f_*\colon K_0(X) \to K_0(\mathrm{pt}) \simeq \mathbb{Q}$ is a linear function on $K_0(X)$ which is equal to the holomorphic Euler characteristic on the classes of sheaves:

$$f_*([\mathcal{F}]) = \chi(X,\mathcal{F}).$$

For a more detailed introduction to K-theory we refer to [9].

Let $T \simeq (\mathbb{C}^*)^n$ be an algebraic torus, $M \simeq \mathbb{Z}^n$ its character lattice and $N = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ its dual lattice. We denote by $M_{\mathbb{R}} = M\otimes_{\mathbb{Z}}\mathbb{R}$, $N_{\mathbb{R}} = N\otimes_{\mathbb{Z}}\mathbb{R}$ vector spaces spanned by M and N respectively. Let further Σ be a smooth complete fan and Y_Σ the corresponding toric variety. We denote by $\Sigma(1) = \{\rho_1,\dots,\rho_r\}$ the set of rays of Σ and by $D_1,\dots,D_r$ the corresponding T-invariant divisors. Finally, for a cone $\sigma\in\Sigma$ of dimension ≥ 1 we will denote by X_σ the closure of the T-orbit corresponding to σ.

The main input from toric geometry for us comes from the computation of holomorphic Euler characteristic of line bundles on Y_Σ in terms of combinatorics of polytopes.

More concretely, every line bundle $\mathcal{L}$ on Y_Σ can be linearized, and this is equivalent (as a sheaf) to $\mathcal{O}(\sum_{i=1}^r h_i D_i)$ for some $h_1, \ldots, h_r \in \mathbb{Z}$. Therefore, there is a surjection

$$\mathcal{P}_\Sigma \to \mathrm{Pic}(Y_\Sigma), \quad \Delta_h \mapsto \mathcal{O}\left(\sum_{i=1}^r h_i D_i\right).$$

We will denote the line bundle corresponding to a polytope $\Delta \in \mathcal{P}_\Sigma$ by $\mathcal{L}_\Delta$.

Proposition 3.1. *Let $\Delta \in \mathcal{P}_\Sigma$ be an integer virtual polytope and $\mathcal{L}_\Delta$ the corresponding line bundle on Y_Σ. Then $\chi(Y_\Sigma, \mathcal{L}_\Delta) = \mathrm{Ehr}(\Delta)$.*

Theorem 3.2. *Let Σ be a smooth, complete fan and let Y_Σ be the corresponding toric variety. Then we have an isomorphism*

$$K_0(Y_\Sigma) \simeq K_\Sigma = \mathrm{Sh}(\mathcal{P}_\Sigma)/\operatorname{Ann}(\mathrm{Ehr}).$$

Proof. First, since Y_Σ has an algebraic cell decomposition, the Euler characteristic provides a Gorenstein duality pairing on $K_0(Y_\Sigma)$, i.e. the Euler pairing

$$\langle \mathcal{F}, \mathcal{G} \rangle_{Eu} := \chi(Y_\Sigma, \mathcal{F} \otimes \mathcal{G}), \quad \mathcal{F}, \mathcal{G} \in K_0(Y_\Sigma)$$

is non-degenerate. Moreover, $K_0(Y_\Sigma)$ is generated by the Picard lattice $\mathrm{Pic}(Y_\Sigma)$. Therefore, by Theorem 2.3 we get $K_0(Y_\Sigma) \simeq \mathrm{Sh}(\mathrm{Pic}(Y_\Sigma))/\operatorname{Ann}(\chi)$. Finally the theorem follows from the correspondence of $\mathcal{P}_\Sigma$ with the Picard lattice $\mathrm{Pic}(Y_\Sigma)$ and Proposition 3.1. □

As a corollary of Theorems 2.10 and 3.2 we obtain the following statement.

Corollary 3.3. *Let Σ be a smooth fan and Y_Σ the corresponding toric variety. Then*

$$K_0(Y_\Sigma) \simeq \mathbb{Q}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]/(I + J),$$

where I is generated by monomials $D_{i_1} \cdots D_{i_s}$ such that $\rho_{i_1}, \ldots, \rho_{i_s} \in \Sigma(1)$ are distinct and do not form a cone in Σ and $J = \left\langle \prod_{i=1}^r t_i^{\langle u, e_i \rangle} - 1 \mid u \in \mathbb{Z}^n \right\rangle$.

The above presentation for $K_0(Y_\Sigma)$ can be also obtained by applying the Chern character isomorphism to the usual presentation of the Chow ring of Y_Σ.

We will finish this section with a description of the classes of structure sheaves $\mathcal{O}_{X_\sigma}$ for orbit closures in the polytope K-ring K_Σ.

Proposition 3.4. *Let Σ be a smooth fan and Y_Σ the corresponding toric variety. Let further $\sigma \in \Sigma$ be a cone and $X_\sigma \subset Y_\Sigma$ the corresponding orbit closure. Then the class of $\mathcal{O}_{X_\sigma}$ is represented in K_Σ by the operator $D_\sigma = \prod_{\rho_i \in \sigma} D_i$.*

Proof. Indeed, since $K_0(Y_\Sigma)$ is generated by $\mathrm{Pic}(Y_\Sigma)$, it is enough to check that

$$\chi(Y_\Sigma, \mathcal{O}_{X_\sigma} \otimes \mathcal{L}_\Delta) = D_\sigma \cdot \mathrm{Ehr}(\Delta)$$

for any $\Delta \in \mathcal{P}_\Sigma$. Hence the proposition follows from Lemma 2.12 and the fact that $\chi(Y_\Sigma, \mathcal{O}_{X_\sigma} \otimes \mathcal{L}_\Delta) = \chi(X_\sigma, \mathcal{L}_\Delta|_{X_\sigma}) = \mathrm{Ehr}(F_\sigma)$, where F_σ is a face of Δ corresponding to σ. □

4 Full flag varieties

Let G be a reductive algebraic group of rank n. Fix a Borel subgroup $B \subset G$; let T be the maximal torus corresponding to B. Denote by $\mathfrak{X} = \mathfrak{X}(T)$ the weight lattice of G (i.e., the character lattice of T). For a dominant highest weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{X}$, the dimension of the corresponding irreducible G-module $V(\lambda)$ is computed using the *Weyl dimension formula*:

$$\dim V_\lambda = \prod_{\alpha \in \Delta^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle},$$

where the product is taken over the system of positive roots Δ_+, $\langle \cdot, \cdot \rangle$ is the Cartan pairing on $\mathfrak{X}$, and $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Thus, $\dim V_\lambda$ is a polynomial in $\lambda_1, \ldots, \lambda_n$. It is called the *Weyl polynomial* and denoted by $F_G(\lambda)$. It is an inhomogeneous polynomial of degree $|\Delta^+|$.

Example 4.1. If $G = \mathrm{GL}(n)$, the Weyl polynomial equals $F_{\mathrm{GL}(n)}(\lambda) = \prod_{i<j} \frac{\lambda_i - \lambda_j - i + j}{j - i}$.

The previous discussion immediately implies the following theorem about the K-group of the full flag variety G/B.

Theorem 4.2. *We have an isomorphism* $K_0(G/B) \cong \mathrm{Sh}(\mathfrak{X}) / \operatorname{Ann} F_G$.

Proof. The proof is analogous to the proof of Theorem 3.2. Indeed, G/B has an algebraic cell decomposition and $K_0(G/B)$ is generated by $\mathrm{Pic}(G/B)$ which can be identified with $\mathfrak{X}$. Therefore, the statement follows from Theorem 2.3 and Borel–Weil–Bott theorem which states that $\chi(G/B, \mathcal{L}(\lambda)) = \dim V(\lambda) = F_G(\lambda)$. □

4.1 Gelfand–Zetlin polytopes

In this subsection we study $K_0(G/B)$ for $G = \mathrm{GL}(n)$ in more detail. For this we shall need the definition of Gelfand–Zetlin polytopes.

Take a strictly decreasing sequence of integers $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_n)$. Consider a triangular tableau of the following form (it is called a *Gelfand–Zetlin tableau*):

$$\begin{array}{ccccccccc}
\lambda_n & & \lambda_{n-1} & & \lambda_{n-2} & & \ldots & & \lambda_1 \\
& x_{1,n-1} & & x_{1,n-2} & & \ldots & & x_{11} & \\
& & x_{2,n-2} & & \ldots & & x_{21} & & \\
& & & \ddots & \vdots & & & & \\
& & & & x_{n-1,1} & & & &
\end{array} \qquad (4.1)$$

We will interpret x_{ij}, where $i + j \leq n$, as coordinates in $\mathbb{R}^N$, where $N = \frac{n(n-1)}{2}$. This tableau can be viewed as a set of inequalities on the coordinates in the following way:

for each triangle $\begin{smallmatrix} a & & b \\ & c & \end{smallmatrix}$ in this tableau, impose the inequalities $a \le c \le b$. This system of inequalities defines a bounded nondegenerate polytope in $\mathbb{R}^N$. This polytope is called a *Gelfand–Zetlin polytope*; we will denote it by $GZ(\lambda)$.

Gelfand–Zetlin polytopes were introduced by I. M. Gelfand and M. L. Zetlin[1] in 1950 (cf. [3]). The integer points in $GZ(\lambda)$ index a special basis, called the Gelfand–Zetlin basis, in the irreducible representation V_λ with the highest weight λ of the group $\mathrm{GL}(n)$. In particular, the number of integer points in $GZ(\lambda)$ equals $\dim V(\lambda)$. This can be viewed as follows: the map taking each row of the Gelfand–Zetlin tableau into its sum

$$\mathrm{pr}\colon \mathbb{R}^N \to \mathbb{R}^{n-1} \cong \mathfrak{X}, \qquad (x_{11}, x_{12}, \dots, x_{n-1,1}) \mapsto \left(\sum_{i=1}^{n-1} x_{1i}, \sum_{i=1}^{n-2} x_{2i}, \dots, x_{n-1,1} \right)$$

projects $GZ(\lambda)$ onto the weight polytope of $V(\lambda)$.

The following proposition is immediate.

Proposition 4.3. *For a given n, all Gelfand–Zetlin polytopes have the same normal fan. The Ehrhart polynomial of $GZ(\lambda)$ is equal to the Weyl polytope of type A_{n-1}:*

$$\mathrm{Ehr}(GZ(\lambda)) = F_{\mathrm{GL}(n)} = \prod_{i<j} \frac{\lambda_i - \lambda_j - i + j}{j - i}.$$

We denote the lattice of (possibly virtual) integer Gelfand–Zetlin polytopes by $\mathcal{P}_{GZ}$. Theorem 4.2 together with Proposition 4.3 implies the following corollary.

Corollary 4.4. *Let $\mathrm{Fl}(n) = \mathrm{GL}(n)/B$ be the variety of complete flags in $\mathbb{C}^n$. Its K-group is isomorphic to $K_0(\mathrm{Fl}(n)) \cong K_{GZ} = \mathrm{Sh}(\mathcal{P}_{GZ}) / \operatorname{Ann} \mathrm{Ehr}(GZ(\lambda))$.*

4.2 Faces of Gelfand-Zetlin polytopes

Let us describe the set of faces of the Gelfand–Zetlin polytope. The polytope is defined by a set of inequalities, represented by (4.1). Each face is obtained by turning some of these inequalities into equalities. In particular, each facet is defined by a unique equation: $x_{ij} = x_{i-1,j+1}$ or $x_{ij} = x_{i-1,j}$ for some pair (i, j), where $i + j \le n$. (We suppose that $x_{0,k} = \lambda_k$). Denote the facets of the first type by F_{ij}.

Definition 4.5. A face F of $GZ(\lambda)$ is called a *Kogan face* if it is obtained as the intersection of facets F_{ij} for some i, j. Equivalently, F is a Kogan face if it contains the vertex defined by the equations

$$\lambda_n = x_{1,n-1} = \dots = x_{n-1,1}, \quad \lambda_{n-1} = x_{1,n-2} = \dots = x_{n-2,1}, \quad \dots, \quad \lambda_2 = x_{1,1}.$$

[1] Sometimes also spelled Cetlin or Tsetlin.

We will represent Kogan faces of the Gelfand–Zetlin polytope symbolically by diagrams obtained from Gelfand–Zetlin tableaux by replacing all λ_i's and x_{ij}'s by dots, where each equality of type $x_{ij} = x_{i+1,j-1}$ is represented by an edge joining these dots. We shall assign to each Kogan face F a word $\underline{w}(F)$ in the alphabet $s_1, \dots, s_{n-1}$ of Coxeter generators of the symmetric group S_n, as follows. We mark the edge going from $x_{i-1,j+1}$ to x_{ij} by a simple transposition $s_{i+j-1} \in S_n$ (recall that $1 \le i, j$ and $i + j \le n$), as shown on Figure 1, and take the word in $s_1, \dots, s_{n-1}$ obtained by reading the letters on the edges *from bottom to top from left to right*.

Figure 1: Diagrams of Kogan faces

Definition 4.6. Let $\underline{w} = (s_{i_1}, \dots, s_{i_k})$ be a word. The *Demazure product* $\delta(\underline{w})$ of $\underline{w}$ is the permutation defined inductively as follows: $\delta(s_i) = s_i$, and $\delta(\underline{w}, s_i)$ equals $\delta(\underline{w})s_i$ if $\ell(\delta(\underline{w})s_i) > \ell(\delta(\underline{w}))$, and $\delta(\underline{w})$ otherwise. Note if $\underline{w}$ is a reduced word, then $\delta(\underline{w}) = s_{i_1} \dots s_{i_k}$.

Definition 4.7. Let F be a Kogan face of codimension k, and let $\underline{w}(F) = (s_{i_1}, \dots, s_{i_k})$ be the corresponding word. We shall say that F *corresponds* to the permutation $\delta(\underline{w}(F))$. A Kogan face is said to be *reduced* if the word $\underline{w}(F)$ is reduced, and *non-reduced* otherwise.

Example 4.8. Diagrams on Figure 1 produce the words (s_3, s_2, s_1, s_3) and $(s_3, s_2, s_3, s_1, s_3)$ respectively. Both of these faces correspond to the permutation $s_3 s_2 s_1 s_3 = (4231)$. The left of them is reduced, while the right one is not.

Finally, we can assign a collection of Kogan faces to each permutation $w \in W$. Denote by $\mathcal{F}(w)$ the set of all Kogan faces F corresponding to w, and by $\Gamma(w)$ the (set-theoretic) *union* of all these faces.

4.3 Characters of Demazure modules

In this subsection we give a definition of Demazure modules and recall a theorem by Kiritchenko, Smirnov, and Timorin relating their characters to faces of Gelfand–Zetlin polytopes.

Let $G = \mathrm{GL}(n)$. We denote the Weyl group of G by $W \cong S_n$. For $w \in W$, let $X_w = \overline{BwB/B} \subset G/B$ be the corresponding Schubert variety; in particular, for the longest element w_0 we have $X_{w_0} = G/B$. Let $\mathcal{L}(\lambda)$ be the G-equivariant line bundle on G/B defined as $G \times^B \mathbb{C}_{-\lambda} \to G/B$, and let $\mathcal{L}_w(\lambda)$ stand for the restriction of $\mathcal{L}(\lambda)$ to X_w.

Definition 4.9. Let $V(\lambda) = H^0(G/B, \mathcal{L}_\lambda)^*$ be the irreducible representation with the highest weight λ. For $w \in W$, define a *Demazure module* $V_w(\lambda)$ as follows. Take a vector of weight $w\lambda$ in $V(\lambda)$ (such a vector is unique up to a scalar) and consider the B-module of $V(\lambda)$ generated by this vector. We denote this B-module by $V_w(\lambda)$.

The following theorem is well-known (cf., for instance, [1, Sec. 3.3]).

Theorem 4.10. *For $\lambda \in \mathfrak{X}$, $w \in W$, we have a B-module isomorphism $H^0(X_w, \mathcal{L}_w(\lambda)) \cong V_w(\lambda)$.*

The character of a Demazure module $\operatorname{char} V_w(\lambda)$ is an element of the group algebra $A(T) := \mathbb{Z}[\mathfrak{X}]$ of the character group. The following theorem provides a relation between characters of Demazure modules and faces of Gelfand–Zetlin polytopes.

Theorem 4.11 ([8, Theorem 5.1])**.** *The character of a Demazure module $V_w(\lambda)$ is obtained as the sum over all integer points of the set of Kogan faces of $GZ(\lambda)$ corresponding to w:*

$$\operatorname{char} V_w(\lambda) = \sum_{x \in \Gamma(w) \cap \mathbb{Z}^N} e^{\operatorname{pr}(x)}.$$

Specializing the character at 0, we obtain the dimension formula for $V_w(\lambda)$:

Corollary 4.12. *The Euler characteristic of $\mathcal{L}_w(\lambda)$ is equal to*

$$\chi(\mathcal{L}_w(\lambda)) = \dim V_w(\lambda) = \#(\Gamma(w) \cap \mathbb{Z}^N).$$

We finish this section by identifying classes of structure sheaves of Schubert varieties in K_{GZ}. Since the Gelfand–Zetlin fan is not smooth it is more convenient for us to work with the module $M_{\widetilde{GZ},GZ}$ which is associated to its smooth subdivision (See Proposition 2.8 and the end of Section 2.3). Recall that there is an epimorphism $\pi\colon K_{\widetilde{GZ}} \to M_{\widetilde{GZ},GZ}$. Finally, define $\mathcal{D}_w \in K_{\widetilde{GZ}}$ via

$$\mathcal{D}_w = \sum_{\Gamma \in \mathcal{F}(w)} (-1)^{\ell(w) - \dim(\Gamma)} D_\Gamma.$$

Theorem 4.13. *Let $w \in W$. Then the class $[\mathcal{O}_w]$ of the structure sheaf of Schubert variety X_w is represented in K_{GZ} by $\pi(\mathcal{D}_w)$.*

Proof. Since $K_0(G/B)$ is generated by $\operatorname{Pic}(G/B)$, it is enough to check that $\chi(\mathcal{L}_w(\lambda)) = \mathcal{D}_w \cdot \operatorname{Ehr}(GZ(\lambda))$. Therefore the theorem follows from Corollary 4.12 and the fact that $\mathcal{D}_w \cdot \operatorname{Ehr}(GZ(\lambda)) = \#(\Gamma(w) \cap \mathbb{Z}^N)$ by the inclusion-exclusion formula. □

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