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Equivariant completions of affine spaces

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Abstract. We survey recent results on open embeddings of the affine space $\mathbb{C}^n$ into a complete algebraic variety X such that the action of the vector group $\mathbb{G}_a^n$ on $\mathbb{C}^n$ by translations extends to an action of $\mathbb{G}_a^n$ on X. We begin with the Hassett–Tschinkel correspondence describing equivariant embeddings of $\mathbb{C}^n$ into projective spaces and present its generalization for embeddings into projective hypersurfaces. Further sections deal with embeddings into flag varieties and their degenerations, complete toric varieties, and Fano varieties of certain types.

Bibliography: 109 titles.

Keywords: affine space, algebraic variety, algebraic group, additive action, local algebra, projective space, quadric, flag variety, grading, locally nilpotent derivation, toric variety, Cox ring, Demazure root.

Contents

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Introduction

The survey is devoted to the study of completions of the affine space $\mathbb{C}^n$ by an algebraic variety X such that the action of the vector group $\mathbb{G}_a^n$ on $\mathbb{C}^n$ by translations can be extended to a regular action $\mathbb{G}_a^n \times X \to X$. To obtain such a completion means to construct an effective regular action with open orbit of the commutative unipotent group $\mathbb{G}_a^n$ on a complete algebraic variety X. We call an effective regular action $\mathbb{G}_a^n \times X \to X$ with open orbit an *additive action* on X. Another interpretation comes from the theory of group embeddings. Let G be a linear algebraic group. A *group embedding* is an embedding of G into an algebraic variety X as an open subset such that the actions of G on G by left and right translations can be extended to a regular action of the group $G \times G$ on X. In these terms we are going to study group embeddings of a commutative unipotent group.

The story began with a work by Hirzebruch. In [64], § 3.2, he considered complex analytic compactifications of the affine space $\mathbb{C}^n$. Problem 26 asks to determine all complex analytic compactifications of $\mathbb{C}^2$, and Problem 27 raises the same question for all spaces $\mathbb{C}^n$ under the restriction that the compactification must have second Betti number 1. These problems initiated the study of open embeddings of affine spaces, both in the analytic and algebraic categories. For more information on algebraic compactifications of affine spaces, see, for example, [56], [96], [33], and references therein.

Clearly, an algebraic variety X that contains an open subset U isomorphic to an affine space possesses some specific properties. In particular, X is rational, every invertible regular function on X is constant, and the divisor class group $\mathrm{Cl}(X)$ is a free finitely generated abelian group. More precisely, $\mathrm{Cl}(X)$ is freely generated by the classes of irreducible components of the complement $X \setminus U$. At the same time, the class of all compactifications of affine spaces is too wide, and it is natural to study compactifications satisfying some extra conditions.

The first variant is to consider algebraic manifolds X in the naive sense, that is, X can be covered by open subsets $U_1, \dots, U_m$ such that each U_i is isomorphic to an affine space. Manifolds of this type were considered by Gromov in [59], § 3.5.D. In [48], § 6.4, such manifolds are called *manifolds of class* $\mathcal{A}_0$. They appear in connection with the Oka principle and algebraic ellipticity. It is known that the class $\mathcal{A}_0$ includes the smooth projective rational surfaces, the smooth complete toric varieties, the flag varieties and, more generally, the smooth complete spherical varieties. Moreover, this class is closed under blowing up points. In [9], Theorem A.1, it was proved that any smooth complete rational variety with a torus action of complexity 1 belongs to the class $\mathcal{A}_0$. A wider class is the class of uniformly rational varieties. A variety X is *uniformly rational* if every point in X admits a Zariski open neighbourhood isomorphic to a Zariski open subset of the affine space. Some recent results on uniformly rational varieties can be found in [81].

The second variant is to involve algebraic group actions. Namely, if an algebraic group G acts on the affine space $\mathbb{C}^n$, we can study open embeddings of $\mathbb{C}^n$ into complete varieties X such that the action of G on $\mathbb{C}^n$ extends to an action of G on X. Taking $G = \mathbb{G}_a^n$ with the action $\mathbb{G}_a^n \times \mathbb{C}^n \to \mathbb{C}^n$ by parallel translations, we arrive at the theory of additive actions. This is the subject of the present survey.

A further motivation to investigate equivariant completions of affine spaces comes from arithmetic geometry. In their study of Manin's conjecture on the distribution of rational points on algebraic varieties, Chambert-Loir and Tschinkel [28] gave asymptotic formulae for the number of rational points of bounded height on smooth projective equivariant compactifications of the vector group. More generally, asymptotic formulae for the number of rational points of bounded height on quasi-projective equivariant embeddings of the vector group were obtained in [29]. The limited volume of this survey does not allow us to discuss these results. We recommend the articles [28], [29], [93], [102], [106], and references therein to the reader.

It is natural to compare the theory of additive actions with the theory of toric varieties. At the first glance the two theories should be similar since the formulations of problems are almost the same: in the toric case we study open equivariant embeddings of the group $\mathbb{G}_m^n$, and in the theory of additive actions we just replace the multiplicative group $\mathbb{G}_m$ of the ground field by the additive group $\mathbb{G}_a$. But it turns out that toric geometry and the theory of additive actions have almost nothing in common. Let us dwell a bit on this.

The theory of toric varieties plays an important role in modern algebra, combinatorics, geometry, and topology. This is caused by a beautiful description of toric varieties in terms of rational polyhedral cones and fans of such cones [37], [55]. There are several ways to generalize the theory of toric varieties. For example, one can consider arbitrary torus actions on algebraic varieties. A semi-combinatorial description of such actions in terms of so-called polyhedral divisors living on varieties of smaller dimensions was introduced recently [1], [2]. Another variant is to restrict the (complex) algebraic torus action on a toric variety to the maximal compact subtorus $(S^1)^n$, axiomatize this class of $(S^1)^n$-actions, and consider such actions on wider classes of topological spaces. This is an active research area, called toric topology [24]. One can also consider linear algebraic group actions with an open orbit by replacing the torus T by a non-abelian connected reductive group G.

In other words, one can study open equivariant embeddings of homogeneous spaces G/H, where H is an algebraic subgroup of G. This theory is well developed in the case when G/H is a spherical homogeneous space, that is, a Borel subgroup B of G acts on G/H with open orbit. Here a description of equivariant embeddings in terms of convex geometry is also available in the framework of the Luna–Vust theory, although it is more complicated than in the toric case [84], [107].

Returning to an 'additive analogue' of toric geometry, that is, to the case when we replace the acting torus T with the commutative unipotent group $\mathbb{G}_a^n$, we come across principal differences. Firstly, it is well known that every orbit of an action of a unipotent group on an affine variety is closed (see [95], § 1.3). In particular, if a unipotent group acts on an affine variety with open orbit, then this action is transitive. This means that, in contrast to the toric case, if an irreducible algebraic variety with a non-transitive action of a unipotent group U contains an open U-orbit, then it cannot be covered by U-invariant open affine charts. Secondly, any toric variety contains finitely many T-orbits, and if two toric varieties are isomorphic as abstract algebraic varieties, then they are isomorphic in the category of toric varieties (see [19], Theorem 4.1). In the additive case these two properties do not hold: consider two actions of $\mathbb{G}_a^2$ on the projective plane $\mathbb{P}^2$ given in homogeneous coordinates by

$$(a_1, a_2) \cdot [z_0 : z_1 : z_2] = [z_0 : z_1 + a_1 z_0 : z_2 + a_2 z_0]$$

and

$$(a_1, a_2) \cdot [z_0 : z_1 : z_2] = \left[z_0 : z_1 + a_1 z_0 : z_2 + a_1 z_1 + \left(\frac{a_1^2}{2} + a_2\right) z_0\right].$$

In the first case there is a line of fixed points, while there are exactly three $\mathbb{G}_a^2$-orbits for the second action.

At the same time, the absence of analogy with toric geometry is definitely not the end of the theory of additive actions. During the last decades, many general and classification results on varieties with an additive action were obtained and some original methods to deal with this class of actions were developed. Our survey aims to discuss these results and methods.

Let us describe the content of the paper. In § 1 we study additive actions on projective spaces. It is a certain surprise that the space $\mathbb{C}^n$ can be embedded equivariantly in $\mathbb{P}^n$ in many different ways. Hassett and Tschinkel [62] observed that such embeddings are in bijection with local commutative associative unital algebras of dimension $n+1$. This result also follows from a more general correspondence between finite-dimensional commutative associative unital algebras and open equivariant embeddings of commutative linear algebraic groups into projective spaces established by Knop and Lange [76]. We begin with the well-known structural theory and classification results on finite-dimensional commutative associative algebras and develop the Hassett–Tschinkel correspondence in complete generality. In particular, it includes a nice correspondence with certain subspaces of the polynomial algebra which are invariant under some differential operators with constant coefficients.

In § 2 we show how the technique proposed by Hassett and Tschinkel can be applied to the study of additive actions on projective varieties different from projective spaces. This started already in [62], where projective curves, smooth projective surfaces, and a special class of smooth projective 3-folds carrying additive actions were described. Using this technique, we give a proof of Sharoiko's theorem [103]. It claims that, in contrast to projective spaces, any non-degenerate projective quadric admits a unique additive action. We also explain how one can describe additive actions on degenerate projective quadrics [12], [10] and establish a generalization of the Hassett–Tschinkel correspondence to arbitrary projective hypersurfaces in terms of invariant multilinear forms [10], [18]. In Theorem 2.30 we find a correspondence between additive actions on non-degenerate projective hypersurfaces and Gorenstein local algebras. Finally, Theorem 2.32 generalizes Sharoiko's result and claims that a non-degenerate projective hypersurface admits at most one additive action. Theorems 2.30 and 2.32 and some other statements in § 2 are original results of this article.

Section 3 begins with some general background on varieties with additive actions. Then we show that if a flag variety G/P of a simple linear algebraic group G admits an additive action, then the parabolic subgroup P is maximal. We list all varieties G/P admitting an additive action following [3]. Then we discuss a uniqueness result which claims that if a flag variety is not isomorphic to the projective space, then it admits at most one additive action. This theorem was proved by Fu and Hwang [51] and independently by Devyatov [41]. The last part presents a construction due to Feigin [45] that degenerates an arbitrary flag variety to a variety with an additive action.

In § 4 we study additive actions on toric varieties following [11]. It is proved that if a complete toric variety admits an additive action, then it admits an additive action normalized by the acting torus. Moreover, we show that any two normalized additive actions are equivalent and give a combinatorial criterion of the existence of a normalized additive action on a toric variety. These results are based on the theory of Cox rings and Demazure roots of toric varieties. We also present two results of Dzhunusov. The first is a classification of additive actions on complete toric surfaces [43], and the second is a criterion of the uniqueness of an additive action on a complete toric variety [42].

In the last section, § 5, we discuss recent classification results due to Fu, Huang, Hwang, Montero, and Nagaoka on additive actions on generalized del Pezzo surfaces, Fano 3-folds, and varieties with high index (see [51], [53], [54], [65], and [89]). A special subsection is devoted to Euler-symmetric projective varieties introduced by Fu and Hwang. Every Euler-symmetric variety admits an additive action. Moreover, for wide classes of varieties including toric varieties and flag varieties the condition to be Euler symmetric is equivalent to the existence of an additive action.

We end the text with a list of open problems and possible directions for further research.

1. Equivariant embeddings into projective spaces

In this section we study additive actions on projective spaces. In 1999 Hassett and Tschinkel [62] established a remarkable correspondence between such actions and commutative associative local Artinian unital algebras. This correspondence led to classification results and allowed one to employ new methods, which were subsequently generalized to some other classes of projective varieties. The main goal of this section is to introduce all objects and concepts needed to establish the Hassett–Tschinkel correspondence, formulate this correspondence in complete generality and with detailed proofs, and discuss related results and corollaries. We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

In § 1.1 we begin with basic facts on finite-dimensional commutative associative algebras. Any finite-dimensional commutative associative algebra is a direct sum of local ones. So finite-dimensional local algebras are important building blocks in many problems in algebra and geometry, comparable with finite simple groups or finite fields. Although the classification of local algebras of small dimension has been known for many years, it is not easy to find it in explicit form in the literature. In Table 1 we list all local algebras up to dimension 6.[1] We also introduce the Hilbert–Samuel sequence of a local algebra and define Gorenstein local algebras.

Subsection 1.2 is devoted to results due to Suprunenko and Tyshkevich [105]. We explain how information on maximal commutative nilpotent subalgebras of a matrix algebra can be used to study abstract commutative algebras and groups. In particular, one can deduce the classification of local algebras in Table 1 from the classification results in [105]. That book contains many important facts and observations that are useful for our purposes, but it is not easy to extract them from the text. We hope that a subsection with unified formulations and, where it is possible, short proofs can help the reader to understand the results of Suprunenko and Tyshkevich better.

In § 1.3 we prove a result due to Knop and Lange [76]. It establishes a bijective correspondence between the effective actions of commutative linear algebraic groups on the projective space $\mathbb{P}^n$ with open orbit and the commutative associative unital algebras A of dimension $n+1$. We also characterize the actions with finitely many orbits.

Subsection 1.4 contains preparatory results on a duality between subspaces of the polynomial algebra $\mathbb{K}[x_1,\dots,x_n]$ and the algebra $\mathbb{K}\left[\dfrac{\partial}{\partial x_1},\dots,\dfrac{\partial}{\partial x_n}\right]$ of differential operators with constant coefficients. In general, this duality is not bijective, but it defines a bijection when restricted to finite-dimensional subspaces of $\mathbb{K}[x_1,\dots,x_n]$ and subspaces of finite codimension of $\mathbb{K}\left[\dfrac{\partial}{\partial x_1},\dots,\dfrac{\partial}{\partial x_n}\right]$. Moreover, let us define a generating subspace in $\mathbb{K}[x_1,\dots,x_n]$ as a translation invariant subspace that generates the algebra $\mathbb{K}[x_1,\dots,x_n]$. It turns out that this duality provides a bijection between the generating subspaces of dimension m and the non-degenerate ideals of codimension m in $\mathbb{K}\left[\dfrac{\partial}{\partial x_1},\dots,\dfrac{\partial}{\partial x_n}\right]$ supported at the origin.

[1] Starting from dimension 7 the number of isomorphy classes of such algebras becomes infinite.

Following Hassett and Tschinkel [62], in § 1.5 we establish a correspondence between

(a) the faithful cyclic representations $\rho\colon \mathbb{G}_a^n \to \mathrm{GL}_m(\mathbb{K})$;
(b) the pairs (A, U), where A is a local commutative associative unital algebra of dimension m with maximal ideal $\mathfrak{m}$, and $U \subseteq \mathfrak{m}$ is a subspace of dimension n generating the algebra A;
(c) the non-degenerate ideals $I \subseteq \mathbb{K}[S_1, \dots, S_n]$ of codimension m supported at the origin;
(d) the generating subspaces $V \subseteq \mathbb{K}[x_1, \dots, x_n]$ of dimension m.

We give complete proofs including arguments for 'up to isomorphism' statements, which are usually ignored in the literature. An effective algorithm that finds the generating subspace corresponding to a pair (A, U) is presented. We illustrate the theory by explicit computations in low-dimensional cases. It is also shown that the $\mathbb{G}_a^n$-modules A and V are dual to each other.

In § 1.6 we show that by limiting either the Knop–Lange theorem to the case of a unipotent group or the Hassett–Tschinkel correspondence to the case $m = n+1$ we obtain a bijection between the additive actions on $\mathbb{P}^n$ and the local commutative associative unital algebras A of dimension $n+1$. In this case we arrive at a remarkable class of generating subspaces, which we call basic subspaces. Such a subspace represents an automorphism of the open orbit of $\mathbb{G}_a^n$ in $\mathbb{P}^n$ that conjugates an additive action to the standard action by translations in the automorphism group of the affine space. We show that there is a unique additive action with finitely many orbits on $\mathbb{P}^n$ and describe additive actions of modality 1. Finally, we observe that an additive action has a unique fixed point if and only if the corresponding local algebra is Gorenstein.

1.1. Finite-dimensional algebras. In this subsection we recall basic structural and classification results on Artinian commutative algebras or, equivalently, finite-dimensional commutative associative unital algebras over the ground field $\mathbb{K}$; see [14], Chap. 8, for more information. In what follows an *algebra* means a finite-dimensional commutative associative unital algebra. The base field $\mathbb{K}$ is embedded into an algebra as the linear span of the unity.

Definition 1.1. An algebra A is called *local* if it contains a unique maximal ideal $\mathfrak{m}$.

Lemma 1.2. *An algebra A is local if and only if A is the direct sum of subspaces $\mathbb{K} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is an ideal consisting of nilpotent elements.*

Proof. Let $A = \mathbb{K} \oplus \mathfrak{m}$. The ideal $\mathfrak{m}$ is maximal since its codimension equals 1. Any element of $A \setminus \mathfrak{m}$ is the sum of an invertible scalar and a nilpotent element, hence it is invertible and cannot belong to any proper ideal. Thus the ideal $\mathfrak{m}$ is a unique maximal ideal.

Conversely, let A be a local algebra with maximal ideal $\mathfrak{m}$. Let us show that any $a \in \mathfrak{m}$ is nilpotent. Since A is finite-dimensional, for some $k \in \mathbb{Z}_{>0}$ we have the equality of ideals $(a^k) = (a^{k+1})$, that is, $a^k = a^{k+1}b$ and $a^k(ab-1) = 0$ for some $b \in A$. Note that $ab - 1 \notin \mathfrak{m}$. Therefore, $ab-1$ does not belong to any proper ideal and so is invertible. This implies that $a^k = 0$.

Denote by $L_a\colon A \to A$ the operator of multiplication by $a \in A$. Let λ be an eigenvalue of L_a. Then $L_{a-\lambda\cdot 1}$ is non-invertible, whence $a - \lambda \cdot 1$ is non-invertible and belongs to the maximal ideal $\mathfrak{m}$. Together with the relation $\mathbb{K} \cap \mathfrak{m} = 0$, this implies that $A = \mathbb{K} \oplus \mathfrak{m}$. □

The following lemma is a particular case of Theorem 8.7 in [14].

Lemma 1.3. *Every algebra is the direct sum of some local ideals of it.*

Proof. As above, denote by $L_a\colon A \to A$ the operator of multiplication by $a \in A$. Recall that the generalized eigenspace of an operator $L \in \operatorname{End}(V)$ with respect to an eigenvalue λ is the subspace $V^\lambda = \{v \in V \colon (L - \lambda\,\mathrm{id}_V)^k v = 0 \text{ for some } k \in \mathbb{Z}_{>0}\}$. Let us prove that A is a direct sum of some ideals V_i of A lying in generalized eigenspaces of L_a for any $a \in A$. Indeed, take some $a \in A$ and consider the generalized eigenspace decomposition $A = \bigoplus V_i'$ with respect to L_a. All generalized eigenspaces are ideals since A is commutative. Repeating the decomposition procedure for those V_i' that do not lie in a generalized eigenspace of L_b for some $b \in A$, we obtain the desired decomposition.

The components $\varepsilon_i \in V_i$ of the unity in A are the unities in V_i. By the construction of V_i, for any $a_i \in V_i$ there is $\lambda \in \mathbb{K}$ such that the action of $(L_{a_i} - \lambda\,\mathrm{id}_A)\big|_{V_i} = L_{a_i - \lambda\varepsilon_i}\big|_{V_i}$ on V_i is nilpotent. Applying this operator to $\varepsilon_i \in V_i$ we obtain that $a_i - \lambda\varepsilon_i$ is nilpotent in V_i. So the algebra V_i is local by Lemma 1.2. □

Let A be a local algebra and $\mathfrak{m}$ be its maximal ideal. Consider the following series of ideals in A:

$$A \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots \supset \mathfrak{m}^{l-1} \supset \mathfrak{m}^l = 0.$$

The number l is called the *length* of the algebra A. Set $r_i := \dim \mathfrak{m}^i - \dim \mathfrak{m}^{i+1}$. In particular, $r_0 = 1$. The sequence $r_0, r_1, r_2, \dots, r_{l-1}$ is called the *Hilbert–Samuel sequence* of the algebra A.

The *socle* of A is the ideal $\operatorname{Soc} A = \{a \in A \colon \mathfrak{m}a = 0\}$. The algebras with $\dim \operatorname{Soc} A = 1$ are called *Gorenstein*. Note that $\mathfrak{m}^{l-1} \subseteq \operatorname{Soc} A$, but the inclusion can be strict. So A is Gorenstein if and only if $\mathfrak{m}^{l-1} = \operatorname{Soc} A$ and $\dim \mathfrak{m}^{l-1} = r_{l-1} = 1$.

Theorem 1.4. *For $m \leqslant 6$ the number of isomorphism classes of local algebras of dimension m is finite. For $m \geqslant 7$ there are infinite series of non-isomorphic local algebras. The number of such classes is as follows*:

m	1	2	3	4	5	6	$\geqslant 7$
	1	1	2	4	9	25	∞

The local algebras of dimension at most 6 are listed in Table 1. Gorenstein algebras are marked with 'G'. It was observed in [62] that this result can be extracted from the 1968 book by Suprunenko and Tyshkevich [105]; see 2)–5) in the next subsection for details. The same classification was obtained independently and using other methods in the 1980 article by Mazolla (see [88], § 2), where schemes parametrizing commutative nilpotent associative multiplications on the affine space were studied. One more approach to such a classification can be found in [94].

There are many classification results on Gorenstein local algebras (see [26], [44], and [74], for instance). In general, local algebras and their Hilbert–Samuel sequences

Table 1. Local algebras of dimension at most 6

No	Local algebra A	$r_0, r_1, \ldots, r_{l-1}$	

$\dim A = 1$

1	$\mathbb{K}$	1	G

$\dim A = 2$

2	$\mathbb{K}[x_1]/(x_1^2)$	1, 1	G

$\dim A = 3$

3	$\mathbb{K}[x_1]/(x_1^3)$	1, 1, 1	G
4	$\mathbb{K}[x_1, x_2]/(x_1^2, x_1x_2, x_2^2)$	1, 2	

$\dim A = 4$

5	$\mathbb{K}[x_1]/(x_1^4)$	1, 1, 1, 1	G
6	$\mathbb{K}[x_1, x_2]/(x_1x_2, x_1^2 - x_2^2)$	1, 2, 1	G
7	$\mathbb{K}[x_1, x_2]/(x_1^3, x_1x_2, x_2^2)$	1, 2, 1	
8	$\mathbb{K}[x_1, x_2, x_3]/(x_i^2, x_ix_j)$	1, 3	

$\dim A = 5$

9	$\mathbb{K}[x_1]/(x_1^5)$	1, 1, 1, 1, 1	G
10	$\mathbb{K}[x_1, x_2]/(x_1x_2, x_1^3 - x_2^2)$	1, 2, 1, 1	G
11	$\mathbb{K}[x_1, x_2]/(x_1^3, x_2^3, x_1x_2)$	1, 2, 2	
12	$\mathbb{K}[x_1, x_2]/(x_1^4, x_2^2, x_1x_2)$	1, 2, 1, 1	
13	$\mathbb{K}[x_1, x_2]/(x_1^3, x_2^2, x_1^2x_2)$	1, 2, 2	
14	$\mathbb{K}[x_1, x_2, x_3]/(x_1x_2, x_1x_3, x_2x_3, x_1^2 - x_2^2, x_1^2 - x_3^2)$	1, 3, 1	G
15	$\mathbb{K}[x_1, x_2, x_3]/(x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2^2 - x_3^2)$	1, 3, 1	
16	$\mathbb{K}[x_1, x_2, x_3]/(x_1^3, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3)$	1, 3, 1	
17	$\mathbb{K}[x_1, x_2, x_3, x_4]/(x_i^2, x_ix_j)$	1, 4	

$\dim A = 6$

18	$\mathbb{K}[x_1]/(x_1^6)$	1, 1, 1, 1, 1, 1	G
19	$\mathbb{K}[x_1, x_2]/(x_1x_2, x_1^4 - x_2^2)$	1, 2, 1, 1, 1	G
20	$\mathbb{K}[x_1, x_2]/(x_1x_2, x_1^3 - x_2^3)$	1, 2, 2, 1	G
21	$\mathbb{K}[x_1, x_2]/(x_1^3, x_2^2)$	1, 2, 2, 1	G
22	$\mathbb{K}[x_1, x_2]/(x_1^5, x_1x_2, x_2^2)$	1, 2, 1, 1, 1	
23	$\mathbb{K}[x_1, x_2]/(x_1^4, x_1x_2, x_2^3)$	1, 2, 2, 1	
24	$\mathbb{K}[x_1, x_2]/(x_1^3, x_1^2x_2, x_1x_2^2, x_2^3)$	1, 2, 3	
25	$\mathbb{K}[x_1, x_2]/(x_1^4, x_1^2x_2, x_1^3 - x_2^2)$	1, 2, 2, 1	
26	$\mathbb{K}[x_1, x_2]/(x_1^4, x_1^2x_2, x_2^2)$	1, 2, 2, 1	
27	$\mathbb{K}[x_1, x_2, x_3]/(x_1^2, x_2^2, x_3^2, x_1x_2 - x_1x_3)$	1, 3, 2	
28	$\mathbb{K}[x_1, x_2, x_3]/(x_2^2, x_3^2, x_1x_2, x_1^2 - x_2x_3)$	1, 3, 2	
29	$\mathbb{K}[x_1, x_2, x_3]/(x_1^2, x_2^2, x_3^2, x_2x_3)$	1, 3, 2	

30	$\mathbb{K}[x_1,x_2,x_3]/(x_1^2,x_2^2,x_1x_3,x_2x_3,x_1x_2-x_3^3)$	1, 3, 1, 1	G
31	$\mathbb{K}[x_1,x_2,x_3]/(x_1^2-x_3^3,x_2^2,x_1x_2,x_1x_3,x_2x_3)$	1, 3, 1, 1	
32	$\mathbb{K}[x_1,x_2,x_3]/(x_1^3,x_2^2,x_3^2,x_1x_2,x_1x_3)$	1, 3, 2	
33	$\mathbb{K}[x_1,x_2,x_3]/(x_1^2,x_2^2,x_3^2,x_1x_2-x_1x_3-x_2x_3)$	1, 3, 2	
34	$\mathbb{K}[x_1,x_2,x_3]/(x_1^3,x_2^2,x_1x_3,x_2x_3,x_1x_2-x_3^2)$	1, 3, 2	
35	$\mathbb{K}[x_1,x_2,x_3]/(x_1^4,x_2^2,x_3^2,x_1x_2,x_1x_3,x_2x_3)$	1, 3, 1, 1	
36	$\mathbb{K}[x_1,x_2,x_3]/(x_1^3,x_2^3,x_3^2,x_1x_2,x_1x_3,x_2x_3)$	1, 3, 2	
37	$\mathbb{K}[x_1,x_2,x_3]/(x_1^3,x_2^2,x_3^2,x_1^2x_2,x_1x_3,x_2x_3)$	1, 3, 2	
38	$\mathbb{K}[x_1,x_2,x_3,x_4]/(x_i^2-x_j^2,x_ix_j,i\neq j)$	1, 4, 1	G
39	$\mathbb{K}[x_1,x_2,x_3,x_4]\Big/\begin{pmatrix}x_1^2,x_2^2,x_4^2,x_1x_3,x_1x_4,x_2x_3,\\ x_2x_4,x_3x_4,x_1x_2-x_3^2\end{pmatrix}$	1, 4, 1	
40	$\mathbb{K}[x_1,x_2,x_3,x_4]/(x_i^2,x_1x_3,x_1x_4,x_2x_3,x_2x_4,x_3x_4)$	1, 4, 1	
41	$\mathbb{K}[x_1,x_2,x_3,x_4]/(x_1^3,x_2^2,x_3^2,x_4^2,x_ix_j,i\neq j)$	1, 4, 1	
42	$\mathbb{K}[x_1,x_2,x_3,x_4,x_5]/(x_i^2,x_ix_j)$	1, 5	

have been studied intensively in connection with punctual Hilbert schemes and collections of commuting nilpotent matrices (see, for example, [69]–[71], [90], [16], and references therein).

1.2. The Suprunenko–Tyshkevich classification. In this subsection we present and discuss some results in [105]. This monograph deals with collections of commuting matrices in the matrix algebra $\mathrm{Mat}_m(\mathbb{K})$. Our goal is to demonstrate applications of these results to the investigation of abstract commutative algebras and groups. In particular, a classification of maximal commutative nilpotent subalgebras of $\mathrm{Mat}_m(\mathbb{K})$ for $m \leqslant 6$ leads to a classification of local algebras of dimension at most 6 (see Theorem 1.4).

Let us start with a short historical overview. There is an immense number of results and publications on maximal commutative subalgebras and subgroups in various contexts and under various constraints. The earliest one was [50] by Frobenius. Circa 1920–35, Kravchuk studied a canonical form of maximal commutative subalgebras, called the Kravchuk normal form in [105], and obtained many results on criteria for conjugacy using this form (see [105], §§ 2.5 and 2.6).

As concerns the dimension function of commutative subalgebras of $\mathrm{Mat}_m(\mathbb{K})$, it dates back to Schur's work [98], where the upper bound $[m/4]^2+1$ for the field $\mathbb{K}=\mathbb{C}$ was established. Jacobson [73] extended this result to an arbitrary field. In [57], Gerstenhaber proved that the dimension of the algebra generated by two commuting matrices in $\mathrm{Mat}_m(\mathbb{K})$ is at most m (for other proofs of this fact and more discussion, see also [15], [109], and [78]). In [78] and [60] dimension bounds for algebras generated by a pair and a triple of elements were studied. The dual problem of the minimum dimension was discussed in [35] and [77]. It turns out that there are maximal commutative subalgebras of $\mathrm{Mat}_m(\mathbb{K})$ of dimension smaller than $m-1$. Various constructions of maximal commutative subalgebras of $\mathrm{Mat}_m(\mathbb{K})$ can be found in [22], [21], and [104].

As Handelman observed in [61], relations between maximal commutative subalgebras and maximal commutative subgroups had been established for the first time by Charles [30]–[32]. Such relations were studied systematically in [105]. Let us present the corresponding results.

As above, all algebras are supposed to be finite-dimensional, commutative, and associative. If an algebra is not said to be nilpotent, then we also assume that it has a unity element. All results are formulated over an algebraically closed field $\mathbb{K}$ of characteristic zero.

1) *Local algebras and indecomposable subalgebras.* Let us introduce some notation. A set A of elements in $\mathrm{Mat}_m(\mathbb{K})$ is called *decomposable* if $\mathbb{K}^m$ is the direct sum of proper subspaces that are invariant under the tautological action of A on $\mathbb{K}^m$; otherwise A is called *indecomposable*.

In [105], § 2.2 (see Theorem 2.2 in [105] and the text below), it was proved that any maximal commutative subalgebra of $\mathrm{Mat}_m(\mathbb{K})$ is a direct sum of indecomposable maximal commutative subalgebras of the $\mathrm{Mat}_{m_i}(\mathbb{K})$ for some $m_1+\dots+m_r=m$.

An algebra A is an indecomposable maximal commutative subalgebra of $\mathrm{Mat}_m(\mathbb{K})$ if and only if $A=\mathbb{K}\oplus\mathfrak{m}$, where $\mathbb{K}$ is the subalgebra of scalar matrices and $\mathfrak{m}$ is a maximal commutative nilpotent subalgebra of $\mathrm{Mat}_m(\mathbb{K})$ (see [105], Theorems 2.3 and 2.4). Together with Lemma 1.2, this implies that the set of indecomposable maximal commutative subalgebras of $\mathrm{Mat}_m(\mathbb{K})$ coincides with the set of local maximal commutative subalgebras of $\mathrm{Mat}_m(\mathbb{K})$.

2) *A classification of nilpotent subalgebras.* In § 3.3 of [105] a classification, up to conjugation, of maximal commutative nilpotent subalgebras of the algebra $\mathrm{Mat}_m(\mathbb{K})$ for $m\leqslant 6$ was presented. The number of conjugacy classes of such subalgebras is as follows:

m	1	2	3	4	5	6	$\geqslant 7$
	1	1	3	7	18	57	∞

For a nilpotent algebra $\mathfrak{m}$, denote by l its index of nilpotency, so that $\mathfrak{m}^l=0$ and $\mathfrak{m}^{l-1}\neq 0$. The classification is derived from the following cases: a classification, up to conjugation, of maximal commutative nilpotent subalgebras of $\mathrm{Mat}_m(\mathbb{K})$ for $l=2$ (§ 2.3, Theorem 2.7), $l=m$ (§ 2.4, Theorem 2.8), $l=m-1$ (§ 3.1, Theorem 3.1), $l=m-2$ (§ 3.2, Theorem 3.2) for an arbitrary m, and a classification of commutative nilpotent algebras of dimension 5 for $l=3$ (§ 2.9, Theorem 2.18 and § 3.3).

3) *Regular subgroups and subalgebras.* We call a commutative subgroup $G\subseteq \mathrm{GL}_{n+1}(\mathbb{K})$ *regular* if the tautological action of G on $\mathbb{K}^{n+1}$ has an open orbit, that is, there exists $v\in\mathbb{K}^{n+1}$ with open orbit $Gv\subseteq\mathbb{K}^{n+1}$. A commutative subalgebra $A\subseteq\mathrm{Mat}_{n+1}(\mathbb{K})$ is *regular* if there is a cyclic vector $v\in\mathbb{K}^{n+1}$, that is, $Av=\mathbb{K}^{n+1}$. A commutative nilpotent subalgebra $\mathfrak{m}\subseteq\mathrm{Mat}_{n+1}(\mathbb{K})$ is called *regular* if there is a vector $v\in\mathbb{K}^{n+1}$ with $\dim\mathfrak{m}v=n$; in this case we also call such a vector v cyclic.

Lemma 1.5. *Let G be a commutative algebraic group that acts effectively on an irreducible algebraic variety X with open orbit. Then G is connected and* $\dim G=\dim X$.

Proof. Let $Gx_0 \subseteq X$ be an open orbit. Since G is commutative, the stabilizers of all points in Gx_0 coincide. Any element of G that acts trivially on Gx_0 acts trivially on X as well. Therefore, by the effectivity of the action, the stabilizer of x_0 is trivial, and the mapping $G \hookrightarrow X$ defined by $g \mapsto gx_0$ is an equivariant open embedding. This implies the assertion. $\square$

Lemma 1.6. *Every regular subgroup* $G \subseteq \mathrm{GL}_{n+1}(\mathbb{K})$ (*regular subalgebra* $A \subseteq \mathrm{Mat}_{n+1}(\mathbb{K})$; *regular nilpotent subalgebra* $\mathfrak{m} \subseteq \mathrm{Mat}_{n+1}(\mathbb{K})$) *is maximal among the commutative subgroups of* $\mathrm{GL}_{n+1}(\mathbb{K})$ (*commutative subalgebras of* $\mathrm{Mat}_{n+1}(\mathbb{K})$; *commutative nilpotent subalgebras of* $\mathrm{Mat}_{n+1}(\mathbb{K})$, *respectively*). *Moreover*, G *is connected*, $\dim G = \dim A = n+1$, *and* $\dim \mathfrak{m} = n$.

Proof. From Lemma 1.5 as applied to the tautological action of G on $\mathbb{K}^{n+1}$ we conclude that G is connected and has dimension $n+1$. Any commutative subgroup $\widetilde{G}$ of $\mathrm{Mat}_{n+1}(\mathbb{K})$ such that $\widetilde{G} \supseteq G$ is regular as well, so G and $\widetilde{G}$ are two connected algebraic groups of the same dimension $n+1$ and $\widetilde{G} = G$. This implies maximality.

If A is a regular subalgebra of $\mathrm{Mat}_{n+1}(\mathbb{K})$ with a cyclic vector v, then the map $A \to \mathbb{K}^{n+1}$, $a \mapsto av$, is a surjection. Any $a \in A$ in the kernel of this map equals zero since $a\mathbb{K}^{n+1} = aAv = Aav = 0$. Thus, A is isomorphic to $\mathbb{K}^{n+1}$ as a vector space. Maximality can be proved as above.

For a regular nilpotent subalgebra $\mathfrak{m} \subseteq \mathrm{Mat}_{n+1}(\mathbb{K})$ consider the direct sum $\mathbb{K} \oplus \mathfrak{m}$ with the subspace of scalar matrices. It is a regular unital subalgebra. Indeed, let $\dim \mathfrak{m}v = n$ for some $v \in \mathbb{K}^{n+1}$; then $\dim(\mathbb{K} + \mathfrak{m})v = n+1$ since $v \notin \mathfrak{m}v$ by the nilpotency of $\mathfrak{m}$. $\square$

4) *Regular representations.* Let us discuss a connection between abstract commutative algebras and commutative subalgebras of $\mathrm{Mat}_{n+1}(\mathbb{K})$. Any algebra A of dimension $n+1$ has the *regular representation* $R\colon A \to \mathrm{End}(A)$ defined by the operators of multiplication. Different identifications $\varphi\colon A \xrightarrow{\sim} \mathbb{K}^{n+1}$ give conjugate subalgebras $R'(A)$ of $\mathrm{Mat}_{n+1}(\mathbb{K})$: see the diagram below. We say that a subalgebra A comes from the regular representation if $A = R'(A)$ for some identification $A \cong \mathbb{K}^{n+1}$:

$$
\begin{tikzcd}
A \arrow[r, "R(a)"] \arrow[d, "\varphi"', "\wr"] & A \arrow[d, "\varphi"', "\wr"] \\
\mathbb{K}^{n+1} \arrow[r, "R'(a)"] & \mathbb{K}^{n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "R(A)"] \arrow[d, "\varphi"', "\wr"] & A \arrow[d, "\varphi"', "\wr"] \\
\varphi(1) \arrow[r, "R'(A)"] & \mathbb{K}^{n+1}
\end{tikzcd}
$$

The regular representation of an algebra A is faithful, provided that A has a unity. If $\mathfrak{m}$ is a nilpotent algebra of dimension n, we can add an element e and construct a unital algebra $A = \mathbb{K}e \oplus \mathfrak{m}$ of dimension $n+1$ defined by the relations $e^2 = e$ and $ae = ea = a$ for any $a \in \mathfrak{m}$. The regular representation of A induces a faithful representation of $\mathfrak{m}$ in $\mathrm{Mat}_{n+1}(\mathbb{K})$, which is also called *regular*.

Lemma 1.7. *A commutative subalgebra* (*commutative nilpotent subalgebra*) *of* $\mathrm{Mat}_{n+1}(\mathbb{K}))$ *comes from the regular representation if and only if it is a regular subalgebra* (*regular nilpotent subalgebra*, *respectively*). *In particular*, *there is a bijection between the isomorphism classes of commutative algebras of dimension* $n+1$ (*the commutative nilpotent algebras of dimension* n) *and the conjugacy classes of regular subalgebras* (*the regular nilpotent subalgebras*, *respectively*) *of* $\mathrm{Mat}_{n+1}(\mathbb{K})$.

Proof. First consider unital algebras. Any subalgebra $R'(A)$ of $\mathrm{Mat}_{n+1}(\mathbb{K})$ coming from the regular representation is regular with a cyclic vector $v=\varphi(1)$ since $R'(A)\varphi(1)=\varphi(A)$: see the diagrams above. Conversely, if A is a regular subalgebra with $Av=\mathbb{K}^{n+1}$, $v\in\mathbb{K}^{n+1}$, then A comes from its regular representation via the identification $\varphi(a)=av$.

Let the nilpotent subalgebra $R'(\mathfrak{m})$ come from the regular representation. Then $\mathbb{K}\oplus R'(\mathfrak{m})$ is a regular subalgebra of $\mathrm{Mat}_{n+1}(\mathbb{K})$, and $R'(\mathfrak{m})$ is regular with the same cyclic vector $v=\varphi(1)$ since $R'(\mathfrak{m})\varphi(1)=\varphi(\mathfrak{m})$. Conversely, if $\mathfrak{m}\subseteq\mathrm{Mat}_{n+1}(\mathbb{K})$ is a regular nilpotent subalgebra, then $A=\mathbb{K}\oplus\mathfrak{m}$ is a regular subalgebra, and by the above arguments it comes from its regular representation.

5) *Classification results on abstract algebras.* According to the above, a classification of local algebras of dimension $n+1$ is equivalent to a classification of the images of the regular representations of their maximal nilpotent ideals, that is, a classification of regular nilpotent subalgebras of $\mathrm{Mat}_{n+1}(\mathbb{K})$. Thus if we want to obtain a classification of local algebras of dimension at most 6 up to isomorphism, from the list of subalgebras in [105], § 3.3 (see item 2) above), then we have to choose those that are regular. Moreover, Theorem 2.15 in [105] says that a maximal commutative nilpotent subalgebra of $\mathrm{Mat}_{n+1}(\mathbb{K})$ is regular if and only if its so-called first Kravchuk number $\nu=n+1-\dim\mathfrak{m}\mathbb{K}^{n+1}$ equals 1, that is, $\dim\mathfrak{m}\mathbb{K}^{n+1}=n$. Thus, Table 1 can be obtained from results in [105], § 3.3.

Example 1.8. Consider $n+1=4$. By the classification in [105], § 3.3, there are seven maximal commutative nilpotent subalgebras of $\mathrm{Mat}_4(\mathbb{K})$:

$$l=2:\ (1)\left\{\begin{pmatrix}0&0&0&0\\a&0&0&0\\b&0&0&0\\c&0&0&0\end{pmatrix}\right\},\quad (2)\left\{\begin{pmatrix}0&0&0&0\\0&0&0&0\\a&b&0&0\\c&d&0&0\end{pmatrix}\right\},\quad (3)\left\{\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\c&b&a&0\end{pmatrix}\right\},$$

$$l=3:\ (4)\left\{\begin{pmatrix}0&0&0&0\\a&0&0&0\\b&a&0&0\\c&0&0&0\end{pmatrix}\right\},\quad (5)\left\{\begin{pmatrix}0&0&0&0\\a&0&0&0\\b&a&0&c\\0&0&0&0\end{pmatrix}\right\},\quad (6)\left\{\begin{pmatrix}0&0&0&0\\a&0&0&0\\b&a&0&c\\c&0&0&0\end{pmatrix}\right\},$$

$$l=4:\ (7)\left\{\begin{pmatrix}0&0&0&0\\a&0&0&0\\b&a&0&0\\c&b&a&0\end{pmatrix}\right\},\qquad a,b,c,d\in\mathbb{K}$$

(see item 2) above). The subalgebras (1), (4), (6), and (7) are regular. with a cyclic vector $v=(1,0,0,0)$. They correspond to four commutative algebras of dimension 4, namely, nos. 8, 7, 6, and 5 in Table 1. For the subalgebras (2), (3), and (5) the first Kravchuk numbers are 2, 3, and 2, respectively, so these subalgebras are not regular.

6) *Infinite series.* While there is a finite number of nilpotent algebras of dimension n with index of nilpotency 2, $n-2$, $n-1$, or n, there exist infinitely many non-isomorphic nilpotent algebras of dimension 6 with index of nilpotency 3. It follows that there is an infinite number of local algebras of dimension at least 7.

More precisely, consider algebras with Hilbert–Samuel sequence $(1, 4, 2)$. Since the index of nilpotency of the maximal ideal of such an algebra equals 3, multiplication is determined by a bilinear symmetric map $\mathfrak{m}/\mathfrak{m}^2 \times \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^2$. We have $\dim \mathfrak{m}/\mathfrak{m}^2 = 4$ and $\dim \mathfrak{m}^2 = 2$, so such maps form a space of dimension $20 = 4(4+1)/2 \cdot 2$. An isomorphism between such algebras corresponds to a change of coordinates in $\mathfrak{m}/\mathfrak{m}^2$ and $\mathfrak{m}^2$, that is, we consider maps up to the action of the group $\mathrm{GL}(4) \times \mathrm{GL}(2)$. It has dimension $20 = 4^2 + 2^2$ and acts on the space of such maps with a one-dimensional inefficiency kernel. Since $19 < 20$, it follows that there are infinitely many generic pairwise non-isomorphic algebras of this type. A discussion about algebras of similar type can be found in Example 3.6 in [62] and the text before and after it. For more information on Hilbert–Samuel sequences corresponding to infinitely many non-isomorphic local algebras, see [83].

Let us give an explicit example. For $n = 7$ consider the algebras A_α of the form

$$A_\alpha = \mathbb{K}[x_1, x_2, x_3, x_4]/(x_1^2 + x_3^2 - 2x_2^2, x_4^2 - x_2^2 - \alpha(x_3^2 - x_2^2), x_i x_j, \, i \neq j).$$

It was shown in [105], § 2.8, that any isomorphy class of algebras of the form A_α contains a finite number of algebras. For $n > 7$, to the algebra A_α we can add the variables $x_5, \dots, x_{n-3}$ such that $x_i x_k = 0$ for any $1 \leqslant i \leqslant n-3$ and $5 \leqslant k \leqslant n-3$. Then we obtain an infinite series of pairwise non-isomorphic algebras of dimension n.

1.3. The Knop–Lange theorem. In this section we study actions of arbitrary connected commutative linear algebraic groups on projective spaces with open orbit. It is well known that such a group G is isomorphic to $\mathbb{G}_m^r \times \mathbb{G}_a^s$ for some $r, s \in \mathbb{Z}_{\geqslant 0}$ (see [66], Theorem 15.5). The numbers r and s are called the *rank* and the *corank* of G, respectively.

Definition 1.9. Actions $\alpha_i \colon G_i \times X_i \to X_i$ of algebraic groups G_i on algebraic varieties X_i, $i = 1, 2$, are said to be *equivalent* if there is a group isomorphism $\psi \colon G_1 \to G_2$ and a variety isomorphism $\varphi \colon X_1 \to X_2$ such that $\varphi \circ \alpha_1 = \alpha_2 \circ (\psi \times \varphi)$.

The following theorem was proved in [76], Proposition 5.1.

Theorem 1.10. *There is a bijection between the following objects*:

(a) *the effective actions of connected commutative linear algebraic groups G on $\mathbb{P}^n$ with an open orbit*;

(b) *the commutative associative unital algebras A of dimension $n + 1$.*

This bijection is considered up to an equivalence of actions and algebra isomorphisms. Moreover, if G is of rank r, then A contains exactly $r + 1$ maximal ideals. The number of isomorphism classes is presented in Table 2.

Proof. (b) $\to$ (a) The group of invertible elements $A^\times$ of an algebra A is a connected commutative linear algebraic group, which is open in A. The factor group $G = \mathbb{P}(A^\times) := A^\times / \mathbb{K}^\times$ by the subgroup of invertible scalars $\mathbb{K}^\times \cdot 1$ is a connected commutative linear algebraic group. It acts on $\mathbb{P}(A) = \mathbb{P}^n$ in a canonical way with open orbit isomorphic to $\mathbb{P}(A^\times)$.

Equivalence. An algebra isomorphism $\varphi \colon A_1 \to A_2$ induces a group isomorphism $\mathbb{P}(A_1^\times) \to \mathbb{P}(A_2^\times)$ and a variety isomorphism $\mathbb{P}(A_1) \to \mathbb{P}(A_2)$. They define an equivalence between the actions of $\mathbb{P}(A_i^\times)$ on the $\mathbb{P}(A_i)$ for $i = 1, 2$.

Table 2. The number of algebras of small dimension

$\dim A$	1	2	3	4	5	6	7	8	9	10	11	12
$r=0$	1	1	2	4	9	25	∞	∞	∞	∞	∞	$\dots$
$r=1$		1	1	3	6	16	42	∞	∞	∞	∞	$\dots$
$r=2$			1	1	3	7	18	49	∞	∞	∞	$\dots$
$r=3$				1	1	3	7	19	51	∞	∞	$\dots$
$r=4$					1	1	3	7	19	52	∞	$\dots$
$r=5$						1	1	3	7	19	52	$\dots$
$\dots$							$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$
total	1	2	4	9	20	53	∞	∞	∞	∞	∞	$\dots$

(a) $\to$ (b) Lemma 1.5 implies that $\dim G = n$. Since G acts on $\mathbb{P}^n$ effectively, we can regard G as a subgroup of $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbb{K})$.

Denote by $\pi\colon \mathrm{GL}_{n+1}(\mathbb{K}) \to \mathrm{PGL}_{n+1}(\mathbb{K})$ the canonical projection and let $H := \pi^{-1}(G)$. Let us prove that H is a connected commutative linear algebraic group of dimension $n+1$. First note that H contains the group $\mathbb{K}^\times$ of invertible scalar matrices since $G \ni 1$. Then

$$\dim H = \dim G + \dim \operatorname{Ker} \pi\big|_H = n+1.$$

Further, H is connected as $\pi(H) = G$ and $\operatorname{Ker}\pi\big|_H$ are connected. Finally, we prove that H is commutative. Consider the commutant $[H,H]$ of H. Since G is commutative, we have $[H,H] \subseteq \operatorname{Ker}\pi\big|_H = \mathbb{K}^\times$. On the other hand $[H,H]$ is connected as the commutant of a connected group, so $[H,H] = \{1\}$ or $[H,H] = \mathbb{K}^\times$. The latter is impossible since the commutant consists of matrices with determinant 1. It follows that $[H,H]$ is trivial and H is commutative.

Consider $\mathrm{GL}_{n+1}(\mathbb{K})$ as an open subset of $\mathrm{Mat}_{n+1}(\mathbb{K})$ and denote by A the associative subalgebra of $\mathrm{Mat}_{n+1}(\mathbb{K})$ generated by H. Clearly, A is a commutative unital algebra. Let us prove that $\dim A = n+1$.

Note that the tautological action of $H \subseteq \mathrm{GL}_{n+1}(\mathbb{K})$ on $\mathbb{K}^{n+1}$ has an open orbit. The group of invertible elements $A^\times \subseteq \mathrm{GL}_{n+1}(\mathbb{K})$ is open in A. It is commutative, acts on $\mathbb{K}^{n+1}$ effectively, and this action has an open orbit since the action of $H \subseteq A^\times$ has. By Lemma 1.5 we obtain $\dim A^\times = \dim \mathbb{K}^{n+1} = n+1$, so $\dim A = n+1$. Moreover, $H = A^\times$ since H is an algebraic subgroup of $A^\times$ of the same dimension.

Equivalence. Let $\psi\colon G_1 \to G_2$ and $\varphi\colon \mathbb{P}^n \to \mathbb{P}^n$ define an equivalence of two actions. Since $\varphi \in \mathrm{PGL}_n(\mathbb{K})$, there is $\Phi \in \mathrm{GL}_{n+1}(\mathbb{K})$ that induces φ on $\mathbb{P}(\mathbb{K}^{n+1})$. The isomorphism of vector spaces Φ induces an isomorphism of operator algebras $\Psi\colon \mathrm{Mat}_{n+1}(\mathbb{K}) \to \mathrm{Mat}_{n+1}(\mathbb{K})$, $\Psi(X) = \Phi X \Phi^{-1}$. Regarding G_i as a subgroup of $\mathrm{PGL}_n(\mathbb{K})$ and setting $H_i = \pi^{-1}(G_i)$, $i = 1, 2$, we obtain

$$\Psi(H_1) = \Phi\pi^{-1}(G_1)\Phi^{-1} = \pi^{-1}(\varphi G_1 \varphi^{-1}) = \pi^{-1}(G_2) = H_2.$$

Hence $\Psi(A_1) = A_2$ is the desired algebra isomorphism.

Let us check that the two maps constructed are inverse to each other. Let A be an algebra as in part (b). Then we have an action of the group $G = A^\times/\mathbb{K}^\times$ on $\mathbb{P}(A)$ as in part (a). We can regard G as a subgroup of $\mathrm{PGL}(A)$. According to the implication (a) $\to$ (b), this action corresponds to the associative subalgebra of $\mathrm{Mat}_{n+1}(\mathbb{K})$ generated by $\pi^{-1}(A^\times/\mathbb{K}^\times)=A^\times$, which coincides with A.

Conversely, let G act on $\mathbb{P}^n$ with open orbit. We have an algebra A as in (a) $\to$ (b), in particular, $A^\times=H=\pi^{-1}(G)$. Then $A^\times/\mathbb{K}^\times$ coincides with G in $\mathrm{PGL}_{n+1}(\mathbb{K})$.

For the second assertion note that if $A=\mathbb{K}\oplus\mathfrak{m}$ is local, then its group of invertible elements is $A^\times=\mathbb{K}^\times\oplus\mathfrak{m}=\mathbb{K}^\times\times(1+\mathfrak{m})$, where $(1+\mathfrak{m},\times)\cong(\mathfrak{m},+)\cong\mathbb{G}_a^n$ via the exponential map and $\mathbb{K}^\times\cong\mathbb{G}_m$. Since any commutative algebra A is a sum of local algebras by Lemma 1.3, the rank of the group $A^\times$ is the number of its local summands, which is equal to the number of maximal ideals. By construction, the rank of $A^\times=H$ is one greater than the rank of G.

The number of isomorphism classes of algebras of dimension $n+1$ can be found by direct computations using the number of local algebras of fixed dimension, which is given in Table 1. More precisely, any algebra of dimension $n+1$ decomposes into a sum of local algebras, and this decomposition is defined by an unordered tuple of local algebras of dimensions $m_1,\dots,m_r$, where $n+1=m_1+\dots+m_r$.

Remark 1.11. In [76], Proposition 5.1, the first assertion of Theorem 1.10 was proved for an arbitrary ground field $\mathbb{K}$.

Remark 1.12. Theorem 2.1 of Suprunenko and Tyshkevich [105] states that there is a one-to-one correspondence between the maximal commutative subalgebras of $\mathrm{Mat}_{n+1}(\mathbb{K})$ and the maximal commutative subgroups of $\mathrm{GL}_{n+1}(\mathbb{K})$. More precisely, for a subalgebra $A\subseteq\mathrm{Mat}_{n+1}(\mathbb{K})$ and a subgroup $H\subseteq\mathrm{GL}_{n+1}(\mathbb{K})$ this bijection is defined by $A\mapsto A^\times$ and $\operatorname{Span}H \leftarrow H$. Let us reformulate the proof of the Knop–Lange theorem in these terms.

It is easy to see that the correspondence in Theorem 2.1 restricts to a bijection between the regular subalgebras of $\mathrm{Mat}_{n+1}(\mathbb{K})$ and the regular subgroups of $\mathrm{GL}_{n+1}(\mathbb{K})$. On the one hand regular subalgebras of $\mathrm{Mat}_{n+1}(\mathbb{K})$ correspond to abstract algebras of dimension $n+1$ by Lemma 1.7. On the other hand the arguments in the proof of the Knop–Lange theorem show that the regular subgroups $H\subseteq\mathrm{GL}_{n+1}(\mathbb{K})$ are in bijection with the commutative subgroups $G\subseteq\mathrm{PGL}_{n+1}(\mathbb{K})=\mathrm{Aut}(\mathbb{P}^n)$ such that the corresponding action of the group G on $\mathbb{P}^n$ has an open orbit: the correspondence is given by $G=\pi(H)$ and $H=\pi^{-1}(G)$, where π is the canonical projection $\pi\colon \mathrm{GL}_{n+1}(\mathbb{K})\to\mathrm{PGL}_{n+1}(\mathbb{K})$. Thus we obtain a bijection between the G-actions on $\mathbb{P}^n$ with open orbit and the algebras of dimension $n+1$.

Now we arrive at a description of orbits of a commutative group acting on $\mathbb{P}^n$ in terms of the corresponding algebra.

Corollary 1.13. *The correspondence in Theorem 1.10 defines a bijection between the G-orbits on $\mathbb{P}^n$ and the non-zero principal ideals in the algebra A.*

Proof. First we establish a bijection between the G-orbits on $\mathbb{P}^n$ and the association classes of non-zero elements in the algebra A. If for $a,b\in A$ there exists $c\in A^\times$ such that $a=cb$, then $[b]\in\mathbb{P}(A)$ is obtained from $[a]\in\mathbb{P}(A)$ by the action of $[c]\in A^\times/\mathbb{K}^\times$. Conversely, if $[a]=[c]\cdot[b]$ for $a,b\in A$ and $c\in A^\times$, then $a=\lambda cb$, $\lambda\in\mathbb{K}^\times$. Hence a and b are associated. $\square$

It remains to notice that a generator of a principal ideal is defined up to association.

For the following statement, see [62], Proposition 3.5.

Corollary 1.14. *There is a unique action of $\mathbb{G}_a^n$ on $\mathbb{P}^n$ with finitely many orbits. It corresponds to the truncated polynomial algebra $A = \mathbb{K}[S]/(S^{n+1})$.*

Proof. By Corollary 1.13 we have to investigate local $(n+1)$-dimensional algebras A with finite number of principal ideals. First note that the algebra $\mathbb{K}[S]/(S^{n+1})$ is local and has a finite number of principal ideals (S^k), $0 \leqslant k \leqslant n+1$. Let us prove the converse statement using induction on n. Let A be a local algebra of dimension $n+1$ with finitely many principal ideals. The set of fixed points in $\mathbb{P}^n = \mathbb{P}(A)$ coincides with $\mathbb{P}(\operatorname{Soc} A)$, so that $\dim \operatorname{Soc} A = 1$. Notice that $\operatorname{Soc} A$ is an ideal in A, so we can consider the factor algebra $A/\operatorname{Soc} A$. It is n-dimensional and has a finite number of principal ideals as well, so by the inductive hypothesis it is isomorphic to $\mathbb{K}[s]/(s^n)$. Let $S + \operatorname{Soc} A \in A/\operatorname{Soc} A$ correspond to s. Then A is the direct sum of the vector spaces $\operatorname{Soc} A$ and $\langle S^k, 0 \leqslant k \leqslant n-1\rangle$. Moreover, it follows that $S^n \in \operatorname{Soc} A$, hence $S^{n+1} = 0$. If $S^n = 0$, then $S^{n-1} \cdot S = 0$ and $S^{n-1} \operatorname{Soc} A = 0$ imply that $S^{n-1}\mathfrak{m} = 0$, which is in contradiction with $S^{n-1} \notin \operatorname{Soc} A$. Thus $A = \langle S^k, 0 \leqslant k \leqslant n\rangle$.

For positive integers n and r we denote by $p_r(n)$ the number of partitions $n = n_1 + \dots + n_r$ such that $n_1 \geqslant \dots \geqslant n_r \geqslant 1$. □

Corollary 1.15. *Let G be a connected commutative linear algebraic group of dimension n and rank r. Then there exist precisely $p_r(n)$ effective actions of G on $\mathbb{P}^n$ with finite number of orbits. The corresponding algebras A are precisely the algebras of the form $\mathbb{K}[S]/(f(S))$, where $f(S)$ is a polynomial of degree n with precisely r distinct roots.*

Proof. By Corollary 1.13 the number of G-orbits in $\mathbb{P}^n$ is equal to the number of principal ideals in the corresponding algebra A. Let

$$A = A_1 \oplus \dots \oplus A_n$$

be the decomposition into a sum of local ideals (see Lemma 1.3). Principal ideals of A are precisely sums of principal ideals in A_i, so the number of principal ideals in A is finite if and only if it is finite for every local summand. By Corollary 1.14 this holds if and only if every A_i is isomorphic to $\mathbb{K}[S]/(S^{n_i})$, where $n_i = \dim A_i$. Hence the algebra A is of the required form and is uniquely determined by the dimensions $n_1, \dots, n_r$. □

Example 1.16. Consider the algebra $A = \mathbb{K}^{n+1}$ with coordinatewise multiplication. Then $A^\times = (\mathbb{K}^\times)^{n+1}$, and the group $A^\times/\mathbb{K}^\times$ is isomorphic to $\mathbb{G}_m^n$: an element $(t_1, \dots, t_n) \in \mathbb{G}_m^n$ corresponds to the class of $(1, t_1, \dots, t_n) \in A^\times$ and acts by multiplication on the classes of elements $(z_0, \dots, z_n) \in A$:

$$(t_1, \dots, t_n) \cdot [z_0 : z_1 : \dots : z_n] = [z_0 : t_1 z_1 : \dots : t_n z_n].$$

It is an action of $\mathbb{G}_m^n$ on $\mathbb{P}^n$ with open orbit $\{z_i \neq 0, 0 \leqslant i \leqslant n\}$. The other orbits are parametrized by the set of indices $0 \leqslant i \leqslant n$ such that $z_i = 0$, so there are $2^{n+1} - 1$ orbits for this action.

Example 1.17. Consider the local algebra $A = \mathbb{K}[S_1, S_2]/(S_1^2, S_1S_2, S_2^2)$ with $\mathfrak{m} = \langle S_1, S_2\rangle$. Let us find the corresponding action of $A^\times/\,\mathbb{K}^\times$ on $\mathbb{P}(A)$.

Since $A^\times/\,\mathbb{K}^\times = (1+\mathfrak{m}, \times) \cong (\mathfrak{m}, +) \cong \mathbb{G}_a^2$ via the exponential map, the action of an element $(x_1, x_2) \in \mathbb{G}_a^2$ is given by multiplication by the class of $\exp(x_1S_1 + x_2S_2) \in A^\times$. Applying this to $[z_0 : z_1 : z_2] \in \mathbb{P}^2$ identified with the class of $z_0 + z_1S_1 + z_2S_2 \in A$ we obtain

$$\begin{aligned}(x_1, x_2)\cdot[z_0 : z_1 : z_2] &= \exp(x_1S_1 + x_2S_2)(z_0 + z_1S_1 + z_2S_2)\\ &= (1 + x_1S_1 + x_2S_2)(z_0 + z_1S_1 + z_2S_2)\\ &= z_0 + (z_1 + x_1z_0)S_1 + (z_2 + x_2z_0)S_2\\ &= [z_0 : z_1 + x_1z_0 : z_2 + x_2z_0].\end{aligned}$$

It is an action of $\mathbb{G}_a^2$ on $\mathbb{P}^2$ with open orbit $\{z_0 \neq 0\}$. The other orbits are fixed points, which form the line $\{z_0 = 0\}$, so there are infinitely many orbits in this case.

Example 1.18. Consider the remaining local algebra of dimension 3: $A = \mathbb{K}[S]/(S^3)$ with $\mathfrak{m} = \langle S, S^2\rangle$. As above, the action of $(x_1, x_2) \in \mathbb{G}_a^2$ on $[z_0 : z_1 : z_2] \in \mathbb{P}^2$ is given by

$$\begin{aligned}(x_1, x_2)\cdot[z_0 : z_1 : z_2] &= \exp(x_1S + x_2S^2)(z_0 + z_1S + z_2S^2)\\ &= \left(1 + x_1S + \left(x_2 + \frac{x_1^2}{2}\right)S^2\right)(z_0 + z_1S + z_2S^2)\\ &= z_0 + (z_1 + x_1z_0)S + \left(z_2 + x_1z_1 + \left(x_2 + \frac{x_1^2}{2}\right)z_0\right)S^2\\ &= \left[z_0 : z_1 + x_1z_0 : z_2 + x_1z_1 + \left(x_2 + \frac{x_1^2}{2}\right)z_0\right].\end{aligned}$$

It is an action of $\mathbb{G}_a^2$ on $\mathbb{P}^2$ with open orbit $\{z_0 \neq 0\}$. The other orbits are $\{z_0 = 0, z_1 \neq 0\}$ and $\{z_0 = z_1 = 0\}$, so there are three orbits for this action.

1.4. Polynomials and differential operators. We begin with some auxiliary definitions and bijections required for the Hassett–Tschinkel correspondence. Similar results were explained in [72] with a reference to [86]. Let $\mathbb{K}$ be a field of characteristic zero. Fix $n \in \mathbb{Z}_{>0}$ and consider two polynomial algebras $\mathbb{K}[x_1, \dots, x_n]$ and $\mathbb{K}[S_1, \dots, S_n]$. If we identify S_i with $\dfrac{\partial}{\partial x_i}$, $1 \leqslant i \leqslant n$, then $\mathbb{K}[S_1, \dots, S_n]$ can be considered as the polynomial algebra $\mathbb{K}\left[\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}\right]$ of differential operators with constant coefficients.

Construction 1.19. Consider the pairing between $\mathbb{K}[x_1, \dots, x_n]$ and $\mathbb{K}[S_1, \dots, S_n]$:

$$\mathbb{K}[S_1, \dots, S_n] \times \mathbb{K}[x_1, \dots, x_n] \to \mathbb{K}, \quad (g, f) \mapsto g[f]\big|_{(0,\dots,0)} =: \langle g \mid f\rangle. \tag{1.1}$$

In particular, $\langle S_1^{i_1} \dots S_n^{i_n} \mid x_1^{j_1} \dots x_n^{j_n}\rangle$ equals $i_1! \cdots i_n!$ if $i_k = j_k$, $1 \leqslant k \leqslant n$, and 0 otherwise. This pairing is non-degenerate:

- $f \in \mathbb{K}[x_1, \ldots, x_n]$ with $\langle g \mid f\rangle = 0$ for all $g \in \mathbb{K}[S_1, \ldots, S_n]$ implies that $f = 0$;
- $g \in \mathbb{K}[S_1, \ldots, S_n]$ with $\langle g \mid f\rangle = 0$ for all $f \in \mathbb{K}[x_1, \ldots, x_n]$ implies that $g = 0$.

Moreover, it induces the perfect pairing $\mathbb{K}[S_1, \ldots, S_n]_{\leqslant d} \times \mathbb{K}[x_1, \ldots, x_n]_{\leqslant d} \to \mathbb{K}$ between the polynomials and differential operators of total degree at most d, since these vector spaces are of finite dimension and the restriction of the pairing is non-degenerate as well.

For a subspace $V \subseteq \mathbb{K}[x_1, \ldots, x_n]$ one can define the subspace

$$I_V = \{g \in \mathbb{K}[S_1, \ldots, S_n]\colon \langle g \mid f\rangle = 0 \ \forall\, f \in V\},$$

and for a subspace $I \subseteq \mathbb{K}[S_1, \ldots, S_n]$ one can consider

$$V_I = \{f \in \mathbb{K}[x_1, \ldots, x_n]\colon \langle g \mid f\rangle = 0 \ \forall\, g \in I\}.$$

Example 1.20. Let $V = \langle x_1^2\rangle \subseteq \mathbb{K}[x_1]$. Then I_V consists of the elements $g = \sum_{i\geqslant 0} \alpha_i S_1^i$ such that $\langle g \mid x_1^2\rangle = 2!\,\alpha_2 = 0$, that is, $I_V = \langle S_1^i, i \neq 2\rangle$. Conversely, for $I = \langle S_1^i, i \neq 2\rangle \subseteq \mathbb{K}[S_1]$ we obtain $V_I = \langle x_1^2\rangle$ since any $f = \sum_{i\geqslant 0} \alpha_i x_1^i \in V_I$ satisfies $\langle S_1^i \mid f\rangle = i!\,\alpha_i = 0$ for all $i \neq 2$.

Example 1.21. Consider the ideal $I = (S_1^2 - 1) \subseteq \mathbb{K}[S_1]$, that is, $I = \langle S_1^{i+2} - S_1^i, i \geqslant 0\rangle$. Any $f = \sum_{i\geqslant 0} \alpha_i x_1^i \in V_I$ satisfies

$$\langle S_1^{i+2} - S_1^i \mid f\rangle = (i+2)!\,\alpha_{i+2} - i!\,\alpha_i = 0$$

for all $i \geqslant 0$. Then

$$0!\,\alpha_0 = 2!\,\alpha_2 = 4!\,\alpha_4 = \cdots$$

and

$$1!\,\alpha_1 = 3!\,\alpha_3 = 5!\,\alpha_5 = \cdots ;$$

hence $f = 0$ since it cannot contain infinitely many non-zero coefficients. Thus $V_I = \{0\}$. It follows that the correspondences between subspaces of $\mathbb{K}[x_1, \ldots, x_n]$ and $\mathbb{K}[S_1, \ldots, S_n]$ in Construction 1.19 are not bijective.

Lemma 1.22. *For fixed $d, m \in \mathbb{Z}_{\geqslant 0}$ Construction 1.19 defines a bijection between*

(a) *the subspaces $V \subseteq \mathbb{K}[x_1, \ldots, x_n]_{\leqslant d}$ with $\dim V = m$*

and

(b) *the subspaces $I \subseteq \mathbb{K}[S_1, \ldots, S_n]$ with $I \supseteq \mathbb{K}[S_1, \ldots, S_n]_{>d}$ and $\operatorname{codim}_{\mathbb{K}[S_1,\ldots,S_n]} I = m$.*

Proof. It is easy to see that $I_V \supseteq \mathbb{K}[S_1, \ldots, S_n]_{>d}$. Note that $\dim V = \operatorname{codim}_{\mathbb{K}[S_1,\ldots,S_n]} I_V$ because the pairing between $\mathbb{K}[x_1, \ldots, x_n]_{\leqslant d}$ and $\mathbb{K}[S_1, \ldots, S_n]_{\leqslant d}$ is perfect. Since $V \subseteq V_{(I_V)}$ and $\dim V = \operatorname{codim} I_V = \dim V_{(I_V)}$, we obtain $V = V_{(I_V)}$. Analogously, $I = I_{(V_I)}$. $\square$

Now we are going to precise the correspondence constructed in a series of lemmas. The main result of this subsection is formulated in Proposition 1.33.

Notice that there is a canonical action of the group $\mathbb{G}_a^n$ by translations on the linear span $\langle x_1, \dots, x_n\rangle$. It can be extended to an action of $\mathbb{G}_a^n$ on $\mathbb{K}[x_1, \dots, x_n]$: a group element $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{G}_a^n$ maps a polynomial $f(x) = f(x_1, \dots, x_n)$ to

$$f(x + \beta) = f(x_1 + \beta_1, \dots, x_n + \beta_n).$$

We recall Taylor's theorem:

$$f(x+\beta) = \sum_{i_1,\dots,i_n} \frac{\beta_1^{i_1}\cdots\beta_n^{i_n}}{i_1!\cdots i_n!}\,\frac{\partial^{i_1+\dots+i_n} f(x)}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}.$$

It follows that

$$f(x+\beta) = \exp(\beta_1 S_1 + \dots + \beta_n S_n)[f(x)].$$

Definition 1.23. A subspace $V \subseteq \mathbb{K}[x_1, \dots, x_n]$ is called *translation invariant* if the following equivalent conditions holds:

1) V is invariant under $S_i = \partial/\partial x_i$ for every $1 \leqslant i \leqslant n$;
2) V is invariant under the $\mathbb{G}_a^n$-action by translations.

That conditions 1) and 2) are equivalent follows from the fact that the subspace V is $\mathbb{G}_a^n$-invariant if and only if it is $(\operatorname{Lie}\mathbb{G}_a^n)$-invariant.

Example 1.24. Consider the vector subspace $V = \langle 1, x_1, x_2\rangle \subseteq \mathbb{K}[x_1, x_2]$. It is invariant under $\partial/\partial x_1$ and $\partial/\partial x_2$. On the other hand it is invariant under translations: the corresponding representation of $(\beta_1, \beta_2) \in \mathbb{G}_a^2$ in V is given by

$$\begin{pmatrix} 1 & \beta_1 & \beta_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis 1, x_1, x_2.

Example 1.25. Let $V = \langle 1, x_1, x_2 + x_1^2/2\rangle \subseteq \mathbb{K}[x_1, x_2]$. It is translation invariant according to both definitions. Since applying $(\beta_1, \beta_2) \in \mathbb{G}_a^2$ to the basis vectors 1, x_1, and $x_2 + x_1^2/2$ gives

$$1, \quad x_1 + \beta_1, \quad \text{and} \quad x_2 + \beta_2 + \frac{(x_1+\beta_1)^2}{2} = x_2 + \frac{x_1^2}{2} + \beta_1 x_1 + \beta_2 + \frac{\beta_1^2}{2},$$

respectively, the corresponding representation of $\mathbb{G}_a^2$ in V is given by

$$\begin{pmatrix} 1 & \beta_1 & \beta_2 + \beta_1^2/2 \\ 0 & 1 & \beta_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Lemma 1.26. *Lemma 1.22 defines a bijection between the translation invariant subspaces of* $\mathbb{K}[x_1, \dots, x_n]$ *and the ideals in* $\mathbb{K}[S_1, \dots, S_n]$. *Moreover, in this case*

$$\begin{aligned} V_I &= \{f \in V \colon g[f] = 0 \ \forall\, g \in I\}, \\ I_V &= \{g \in I \colon g[f] = 0 \ \forall\, f \in V\}. \end{aligned} \tag{1.2}$$

Proof. Let I be an ideal and let $f \in V_I$, that is $\langle g \mid f\rangle = 0$ for any $g \in I$. Since $\widetilde{g}g \in I$ for any $\widetilde{g} \in \mathbb{K}[S_1,\dots,S_n]$, it follows that $0 = \langle \widetilde{g}g \mid f\rangle = \langle \widetilde{g} \mid g[f]\rangle$. Hence by the non-degeneracy of $\langle\cdot\mid\cdot\rangle$ we have $g[f] = 0$. This implies the first formula in (1.2), and thus V_I is $\partial/\partial x_i$-invariant for any $1 \leqslant i \leqslant n$.

Conversely, let V be a translation invariant subspace and let $g \in I_V$. Since $\widetilde{g}[f] \in V$ for any $\widetilde{g} \in \mathbb{K}[S_1,\dots,S_n]$ and $f \in V$, it follows that $0 = \langle g \mid \widetilde{g}[f]\rangle = \langle \widetilde{g} \mid g[f]\rangle$. Hence $g[f] = 0$. Then we obtain the second formula in (1.2), which implies that I_V is an ideal. □

Example 1.27. The translation invariant vector subspace $V = \langle 1, x_1, x_1^2\rangle \subseteq \mathbb{K}[x_1]$ corresponds to the ideal $I = (S_1^3) \subseteq \mathbb{K}[S_1]$.

Definition 1.28. We call a subspace $V \subseteq \mathbb{K}[x_1,\dots,x_n]$ *non-degenerate* if no non-zero operator in $\langle S_1,\dots,S_n\rangle$ annihilates V. A subspace $I \subseteq \mathbb{K}[S_1,\dots,S_n]$ is called *non-degenerate* if $I \cap \langle S_1,\dots,S_n\rangle = 0$.

The following lemma is straightforward.

Lemma 1.29. *The bijection in Lemma 1.22 restricts to a bijection between the non-degenerate subspaces of* $\mathbb{K}[x_1,\dots,x_n]$ *and* $\mathbb{K}[S_1,\dots,S_n]$.

Definition 1.30. We call a subspace $V \subseteq \mathbb{K}[x_1,\dots,x_n]$ *generating* if one of the following equivalent conditions hold:

1) V is translation invariant and non-degenerate;
2) V is translation invariant and generates $\mathbb{K}[x_1,\dots,x_n]$ as an algebra.

We prove that conditions 1) and 2) in Definition 1.30 are equivalent. Let V be translation invariant and generate $\mathbb{K}[x_1,\dots,x_n]$. Then there is no non-zero operator in $\langle S_1,\dots,S_n\rangle$ annihilating V since otherwise it would annihilate $\mathbb{K}[x_1,\dots,x_n]$.

Conversely, let a translation invariant and non-degenerate subspace V generate a subalgebra $A \subsetneq \mathbb{K}[x_1,\dots,x_n]$. Set $W = A \cap \langle x_1,\dots,x_n\rangle$. Choosing appropriate variables in $\mathbb{K}[x_1,\dots,x_n]$ we can assume that $W = \langle x_1,\dots,x_k\rangle$ for some $k < n$. Note that $\mathbb{K}[x_1,\dots,x_k] \subseteq A$ since it is generated by $W \subseteq A$. Let us prove that $A = \mathbb{K}[x_1,\dots x_k]$. Assume the converse and let f be a polynomial of smallest degree in $A \setminus \mathbb{K}[x_1,\dots,x_k]$. Since V is invariant under translations, A is translation invariant as well. Then the polynomials $\dfrac{\partial f}{\partial x_i}$ belong to A and are of degree less than that of f, hence $\dfrac{\partial f}{\partial x_i} \in \mathbb{K}[x_1,\dots,x_k]$ for every $1 \leqslant i \leqslant n$.

Let
$$f = \sum_j b_j x_n^j, \qquad b_j \in \mathbb{K}[x_1,\dots,x_{n-1}].$$

Since $\dfrac{\partial f}{\partial x_n} = \sum\limits_j j b_j x_n^{j-1}$ is an element of $\mathbb{K}[x_1,\dots,x_k]$, we have $f = b_1x_n + b_0$. For every i, $1 \leqslant i < n$, the polynomial $\dfrac{\partial f}{\partial x_i} = \dfrac{\partial b_1}{\partial x_i}x_n + \dfrac{\partial b_0}{\partial x_i}$ does not contain x_n either. Hence $\dfrac{\partial b_1}{\partial x_i} = 0$ for any i, that is $b_1 \in \mathbb{K}$. Thus x_n occurs only in a linear term in f. The same holds for $x_{k+1},\dots,x_{n-1}$, that is, f is a sum of a linear polynomial in $x_{k+1},\dots,x_n$ and an element $f_0 \in \mathbb{K}[x_1,\dots,x_k]$. Since $f, f_0 \in A$, this linear polynomial belongs to W. But $W = \langle x_1, \dots, x_k\rangle$. Hence this linear polynomial is zero, that is, $f = f_0 \in \mathbb{K}[x_1, \dots, x_k]$, which is a contradiction. Thus, $A = \mathbb{K}[x_1, \dots x_k]$. Then $\dfrac{\partial}{\partial x_n}$ annihilates A and therefore V, which contradicts the non-degeneracy of V.

Consider the canonical action of the group $\mathrm{GL}_n(\mathbb{K})$ on the vector space $\langle x_1, \dots, x_n\rangle$: $x \mapsto \varphi x$, where $x \in \langle x_1, \dots, x_n\rangle$ and $\varphi \in \mathrm{GL}_n(\mathbb{K})$. It induces an action of $\mathrm{GL}_n(\mathbb{K})$ on the algebra $\mathbb{K}[x_1, \dots, x_n]$: $(\varphi f)(x_1, \dots, x_n) := f(\varphi x_1, \dots, \varphi x_n)$. We define an action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathbb{K}[S_1, \dots, S_n]$ as follows: for $g \in \mathbb{K}[S_1, \dots, S_n]$ and $\varphi \in \mathrm{GL}_n(\mathbb{K})$ set $(\varphi g)[f] = g[\varphi^{-1} f]$ for any $f \in \mathbb{K}[x_1, \dots, x_n]$.

Definition 1.31. We say that two subspaces $V_1, V_2 \subseteq \mathbb{K}[x_1, \dots, x_n]$ ($I_1, I_2 \subseteq \mathbb{K}[S_1, \dots, S_n]$) are GL-*equivalent* if there exists $\varphi \in \mathrm{GL}_n(\mathbb{K})$ such that $\varphi V_1 = V_2$ ($\varphi I_1 = I_2$, respectively).

Lemma 1.32. *The bijection in Lemma 1.22 is well defined on classes of* GL-*equivalence.*

Proof. Let $\varphi V_1 = V_2$. Then

$$\begin{aligned} I_{V_2} &= \{h \in \mathbb{K}[S_1, \dots, S_n] \colon \langle h \mid \varphi f\rangle = 0 \ \forall\, f \in V_1\} \\ &= \{h \in \mathbb{K}[S_1, \dots, S_n] \colon \langle \varphi^{-1} h \mid f\rangle = 0 \ \forall\, f \in V_1\} \\ &= \{\varphi g \in \mathbb{K}[S_1, \dots, S_n] \colon \langle g \mid f\rangle = 0 \ \forall\, f \in V_1\} = \varphi I_{V_1}. \end{aligned}$$

In the same way $\varphi I_1 = I_2$ implies that $\varphi V_1 = V_2$. □

We say that an ideal $I \subseteq \mathbb{K}[S_1, \dots, S_n]$ is *supported at the origin* if I contains some powers of S_i for every $1 \leqslant i \leqslant n$. It can easily be checked that an ideal I is supported at the origin if and only if I contains $\mathbb{K}[S_1, \dots, S_n]_{>d}$ for some d.

From Lemmas 1.22, 1.26, 1.29, and 1.32 we obtain the following result.

Proposition 1.33. *Let $m \in \mathbb{Z}_{\geqslant 0}$. Formulae* (1.2) *give a bijection between the classes of* GL-*equivalence of*

(a) *generating subspaces $V \subseteq \mathbb{K}[x_1, \dots, x_n]$ of dimension m; and*

(b) *non-degenerate ideals $I \subseteq \mathbb{K}[S_1, \dots, S_n]$ of codimension m supported at the origin.*

Example 1.34. The generating subspace

$$V = \langle 1, x_1, x_2\rangle \subseteq \mathbb{K}[x_1, x_2]$$

corresponds to the ideal

$$I = (S_1^2, S_1S_2, S_2^2) \subseteq \mathbb{K}[S_1, S_2]$$

as the latter consists of the elements $g = \sum_{i,j\geqslant 0} \alpha_{ij} S_1^i S_2^j$ with $\alpha_{00} = \alpha_{01} = \alpha_{11} = 0$.

Example 1.35. The generating subspace

$$V = \left\langle 1, x_1, x_2 + \frac{x_1^2}{2} \right\rangle \subseteq \mathbb{K}[x_1, x_2]$$

corresponds to the ideal

$$I = (S_1^2 - S_2, S_1S_2) \subseteq \mathbb{K}[S_1, S_2],$$

since $g = \sum_{i,j\geqslant 0} \alpha_{ij} S_1^i S_2^j$ belongs to I if and only if $\alpha_{00} = \alpha_{10} = \alpha_{01} + 2!\,\alpha_{20}/2 = 0$.

1.5. The Hassett–Tschinkel correspondence. In this subsection we describe and study the correspondence presented in [62], § 2.4.

Definition 1.36. Let G be an algebraic group. Representations $\rho_1\colon G \to \mathrm{GL}(V_1)$ and $\rho_2\colon G \to \mathrm{GL}(V_2)$ are said to be *equivalent* if there exist an automorphism $\psi\colon G \to G$ and an isomorphism of vector spaces $\varphi\colon V_1 \to V_2$ such that $\varphi(\rho_1(g)v) = \rho_2(\psi(g))\varphi(v)$ for any $g \in G$, $v \in V_1$.

Definition 1.37. Consider pairs (A, U), where A is an algebra and $U \subseteq A$ is a subspace. Two such pairs (A_1, U_1) and (A_2, U_2) are *equivalent* if there is an algebra isomorphism $\varphi\colon A_1 \to A_2$ with $\varphi(U_1) = U_2$.

We arrive at the main result in this subsection.

Theorem 1.38. *Let $n, m \in \mathbb{Z}_{\geqslant 0}$. There exist one-to-one correspondences between*

(a) *the faithful cyclic representations $\rho\colon \mathbb{G}_a^n \to \mathrm{GL}_m(\mathbb{K})$;*

(b) *the pairs (A, U), where A is a local commutative associative unital algebra of dimension m with maximal ideal $\mathfrak{m}$ and $U \subseteq \mathfrak{m}$ is a subspace of dimension n generating the algebra A;*

(c) *the non-degenerate ideals $I \subseteq \mathbb{K}[S_1, \dots, S_n]$ of codimension m supported at the origin;*

(d) *the generating subspaces $V \subseteq \mathbb{K}[x_1, \dots, x_n]$ of dimension m.*

These correspondences are given up to equivalences as in Definitions 1.31, 1.36, and 1.37.

Proof. (a) $\to$ (b) Here we follow [12], § 1. Let $\rho\colon \mathbb{G}_a^n \to \mathrm{GL}_m(\mathbb{K})$ be a faithful representation. Its differential gives a representation $d\rho\colon \mathfrak{g} \to \mathfrak{gl}_m(\mathbb{K})$ of the tangent algebra $\mathfrak{g} = \mathrm{Lie}(\mathbb{G}_a^n)$. This defines a representation $\tau\colon \mathbb{U}(\mathfrak{g}) \to \mathrm{Mat}_m(\mathbb{K})$ of the universal enveloping algebra $\mathbb{U}(\mathfrak{g}) = \mathbb{K}[S_1, \dots, S_n]$.

Let $A := \tau(\mathbb{U}(\mathfrak{g}))$ and $U := \tau(\mathfrak{g})$. The subspace U generates the algebra A since $\mathfrak{g}$ generates $\mathbb{U}(\mathfrak{g})$. The group $\mathbb{G}_a^n$ is commutative, so $\mathfrak{g}$ is a commutative Lie algebra. Thus $\mathbb{U}(\mathfrak{g})$ is isomorphic to a polynomial algebra in n variables with maximal ideal $(\mathfrak{g})$ consisting of polynomials without constant term. The algebra A is a commutative associative unital algebra. Since $\mathbb{G}_a^n$ is a unipotent group, the image $d\rho(\mathfrak{g}) \subseteq \mathfrak{gl}_m(\mathbb{K})$ consists of commuting nilpotent matrices. By definition, $\tau\big|_{\mathfrak{g}} = d\rho$, so $(U) = \tau((\mathfrak{g}))$ is a nilpotent ideal in A of codimension 1 and the algebra A is local. Since ρ is faithful, it follows that $\tau\big|_{\mathfrak{g}}\colon \mathfrak{g} \to U$ is an isomorphism of vector spaces and $\dim U = n$.

Let v be a cyclic vector, that is, $\langle \rho(\mathbb{G}_a^n)v \rangle = \mathbb{K}^m$. Note that the subspace $Av = \tau(\mathbb{U}(\mathfrak{g}))v$ is $\mathfrak{g}$- and $\mathbb{G}_a^n$-invariant and contains v, hence $Av = \mathbb{K}^m$. Consider $\pi\colon A \to \mathbb{K}^m$, $a \mapsto av$. Note that $\operatorname{Ker} \pi = 0$. Indeed, if $av = 0$ for some $a \in A$, then $a\mathbb{K}^m = aAv = Aav = 0$, hence $a = 0$. Thus, π is an isomorphism of vector spaces and $\dim A = m$.

Equivalence. Let $\rho_1\colon \mathbb{G}_a^n \to \mathrm{GL}_m(\mathbb{K})$ and $\rho_2\colon \mathbb{G}_a^n \to \mathrm{GL}_m(\mathbb{K})$ be two equivalent representations, that is, there exist isomorphisms $\varphi\colon \mathbb{K}^m \to \mathbb{K}^m$ and $\psi\colon \mathbb{G}_a^n \to \mathbb{G}_a^n$

such that the first of the diagrams

$$
\begin{tikzcd}
\mathbb{K}^m \arrow[r, "\varphi", "\sim"'] \arrow[d, "\rho_1(g)"] & \mathbb{K}^m \arrow[d, "\rho_2(\psi(g))"] \\
\mathbb{K}^m \arrow[r, "\varphi", "\sim"'] & \mathbb{K}^m
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "\pi_1", "\sim"'] \arrow[d, "\text{mult. by } \tau_1(y)"] & \mathbb{K}^m \arrow[r, "\varphi", "\sim"'] \arrow[d, "\tau_1(y)"] & \mathbb{K}^m \arrow[r, "\pi_2^{-1}", "\sim"'] \arrow[d, "\tau_2(\Psi(y))"] & A_2 \arrow[d, "\text{mult. by } \tau_2(\Psi(y))"] \\
A_1 \arrow[r, "\pi_1", "\sim"'] & \mathbb{K}^m \arrow[r, "\varphi", "\sim"'] & \mathbb{K}^m \arrow[r, "\pi_2^{-1}", "\sim"'] & A_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{U}(\mathfrak{g}) \arrow[r, "\Psi"] \arrow[d, "\tau_1"] & \mathbb{U}(\mathfrak{g}) \arrow[d, "\tau_2"] \\
A_1 \arrow[r, "\pi_2^{-1}\varphi\pi_1", "\sim"'] & A_2
\end{tikzcd}
$$

is commutative for any $g \in \mathbb{G}_a^n$. If we differentiate it and extend $d\psi\colon \mathfrak{g} \to \mathfrak{g}$ to $\Psi\colon \mathbb{U}(\mathfrak{g}) \to \mathbb{U}(\mathfrak{g})$, we obtain the central part of the second diagram for every $y \in \mathbb{U}(\mathfrak{g})$. Denote a cyclic vector of ρ_1 by v_1 and set $v_2 = \varphi(v_1)$. Then v_2 is a cyclic vector for ρ_2. Identifying A_i with $\mathbb{K}^m$ by means of the correspondences π_i, $i = 1, 2$, and applying the diagram to $1 \in A_1$, we obtain that $\pi_2^{-1}\varphi\pi_1$ maps $\tau_1(y)$ to $\tau_2(\Psi(y))$ for any $y \in \mathbb{U}(\mathfrak{g})$, which implies that $\pi_2^{-1}\varphi\pi_1$ is an algebra isomorphism. The third diagram implies that $\pi_2^{-1}\varphi\pi_1(U_1) = U_2$, since $d\psi = \Psi\big|_{\mathfrak{g}}$ maps $\mathfrak{g}$ to $\mathfrak{g}$.

(b) $\to$ (a) Let A be a local algebra with maximal ideal $\mathfrak{m}$, let $U \subseteq \mathfrak{m}$ generate A, and let $\dim A = m$ and $\dim U = n$. Since U consists of nilpotent elements, one can consider the subgroup $\exp U \cong \mathbb{G}_a^n$ in $A^\times$ and its representation $\rho\colon \exp U \to \mathrm{GL}(A)$ that maps $a \in \exp U \subseteq A$ to the operator of multiplication by a in A.

Clearly, ρ is faithful. Let us prove that ρ is cyclic with a cyclic vector $1 \in A$. Let $W := \langle \exp U \rangle$. Note that W is $(\exp U)$-invariant. Therefore, W is $\mathrm{Lie}(\exp U)$-invariant, that is, W is invariant under multiplication by elements in U. Since U generates the algebra A, we obtain $W = A$.

Equivalence. Let $\varphi\colon A_1 \to A_2$ be an algebra isomorphism such that $\varphi(U_1) = U_2$. Then $\varphi(\exp U_1) = \exp U_2$, and for any $u \in U_1$ we have $\rho_1(\exp u) \circ \varphi = \varphi \circ \rho_2(\varphi(\exp u))$.

Let us show that the two maps constructed are inverse to each other. Given a representation ρ, we have $A = \tau(\mathbb{U}(\mathfrak{g})) \subseteq \mathrm{Mat}_m(\mathbb{K})$ and $U = \tau(\mathfrak{g}) = d\rho(\mathfrak{g})$. The corresponding representation maps $\exp U$ to the operators of multiplication by elements of $\exp U$ in A. It is equivalent to the original representation since $\exp U \subseteq \mathrm{Mat}_m(\mathbb{K})$ coincides with $\exp d\rho(\mathfrak{g}) = \rho(\mathbb{G}_a^n)$.

Conversely, given (A, U), let $\rho\colon \exp U \to \mathrm{GL}(A)$ be the corresponding representation. Then $d\rho\colon U \to \mathfrak{gl}(A)$ maps u to the operator of multiplication by u. Since the image of τ coincides with the associative algebra generated by $d\rho(U)$ and U generates A, we obtain the algebra of operators of multiplication by elements of A, which is isomorphic to A.

(b) $\to$ (c) Denote a basis of the vector space U by $s_1, \ldots, s_n$. Since U generates A, the algebra A is the image of a polynomial algebra under the projection $\pi\colon \mathbb{K}[S_1, \ldots, S_n] \to A$, $S_i \mapsto s_i$. Then $A \cong \mathbb{K}[S_1, \ldots, S_n]/I$ for some ideal $I \subseteq \mathbb{K}[S_1, \ldots, S_n]$. The s_i are nilpotent in A, so the ideal I contains some powers of all variables S_i. Since the s_i form a basis of U, it follows that $I \cap \langle S_1, \ldots, S_n \rangle = 0$ and I is non-degenerate. Since $\dim A = m$, we have $\operatorname{codim} I = m$.

Equivalence. First we check that the above construction does not depend on the choice of a basis in U. Let $(s_1, \ldots, s_n)$ and $(\widetilde{s}_1, \ldots, \widetilde{s}_n)$ be two bases of U, which correspond to ideals I and $\widetilde{I}$, respectively, and let $(s_1, \ldots, s_n) = (\varphi\widetilde{s}_1, \ldots, \varphi\widetilde{s}_n)$ for some $\varphi \in \mathrm{GL}_n(\mathbb{K})$. Then $g(S_1, \ldots, S_n) \in I$ if and only if $(\varphi g)(S_1, \ldots, S_n) = g(\varphi S_1, \ldots, \varphi S_n) \in \widetilde{I}$, hence I is equivalent to $\widetilde{I}$.

Now let (A_1, U_1) be equivalent to (A_2, U_2), that is, suppose there exists an isomorphism $\varphi\colon A_1 \to A_2$ such that $\varphi(U_1) = U_2$. According to the above, we can choose a basis in U_2 to be the φ-image of a basis in U_1 and obtain $I_1 = I_2 \subseteq \mathbb{K}[S_1, \dots S_n]$.

(c) $\to$ (b) Given an ideal $I \subseteq \mathbb{K}[S_1, \dots, S_n]$, let $A := \mathbb{K}[S_1, \dots, S_n]/I$, $s_i := S_i + I$, and $U := \langle s_1, \dots, s_n\rangle$.

The elements s_i are nilpotent since some powers of S_i belong to I. It follows that the ideal $(s_1, \dots, s_n)$ is nilpotent of codimension 1, hence the algebra A is local. As above, $\dim U = n$ since I is non-degenerate, and $\dim A = \operatorname{codim} I = m$.

Equivalence. If I_1 and I_2 are equivalent ideals, then we have an automorphism of $\mathbb{K}[S_1, \dots, S_n]$ that induces the desired isomorphism of factor algebras $A_1 \to A_2$.

Clearly, the two maps constructed are inverse to each other.

(c) $\leftrightarrow$ (d) See Proposition 1.33. $\square$

Below we explain a method for computing the generating subspace V corresponding to a given pair (A, U) (see [62], Proposition 2.11).

Construction 1.39. Suppose A is a local algebra of dimension m with maximal ideal $\mathfrak{m}$, and a subspace $U \subseteq \mathfrak{m}$ of dimension n generates the algebra A (see Theorem 1.38, (b)). These data define a representation of A as a factor algebra $A = \mathbb{K}[S_1, \dots, S_n]/I$: given a basis $s_1, \dots, s_n$ of the subspace U, let the ideal I be the kernel of the surjective homomorphism $\pi\colon \mathbb{K}[S_1, \dots, S_n] \to A$, $S_i \mapsto s_i$.

For what follows we need a basis of A. Consider a homogeneous lexicographic order on $\mathbb{K}[S_1, \dots, S_n]$. Let $\mu_1, \dots, \mu_k$ be the monomials that are not the leading terms of any polynomials in I. Let us prove that the μ_i form a basis of A. They are linearly independent in A since a linear combination of the μ_i has one of the μ_i as a leading term and cannot belong to I. Further, consider any element of A. It is a linear combination of some monomials; if one of these monomials is not equal to μ_i, then it is the leading term of some $f \in I$ and we can reduce this element using f. In such a way we obtain a representation of the element as a linear combination of the μ_i.

Since $x_1 s_1 + \dots + x_n s_n \in U \subseteq \mathfrak{m}$ is nilpotent for any $x_1, \dots, x_n \in \mathbb{K}$ and the μ_i form a basis of A, we can expand

$$\exp(x_1 s_1 + \dots + x_n s_n) = \sum_{i=1}^{m} f_i(x_1, \dots, x_n)\mu_i.$$

For $g \in \mathbb{K}[S_1, \dots, S_n]$, denote by g_x the same polynomial in the variables $\partial/\partial x_i$. One can easily check that

$$\frac{\partial}{\partial x_i}[\exp(x_1 S_1 + \dots + x_n S_n)] = S_i \exp(x_1 S_1 + \dots + x_n S_n).$$

This leads to the identity

$$g_x[\exp(x_1 S_1 + \dots + x_n S_n)] = g \exp(x_1 S_1 + \dots + x_n S_n).$$

Substituting $S_i = s_i$ into this identity we obtain

$$\sum_{i=1}^{m} g_x[f_i(x_1, \dots, x_n)]\mu_i = \pi(g) \sum_{i=1}^{m} f_i(x_1, \dots, x_n)\mu_i. \tag{1.3}$$

Note that $\{\sum f_i(x_1,\dots,x_n)\mu_i\colon x_i\in\mathbb{K}\}=\exp U$ by definition and $\langle\exp U\rangle=A$ by the proof of the correspondence (b) $\to$ (a) in Theorem 1.38. In particular, the f_i are linearly independent. Then the right-hand side of (1.3) is zero for all $x_i\in\mathbb{K}$ if and only if $\pi(g)=0$ in A, that is, $g\in I$. On the other hand, the left-hand side equals zero for any $x_i\in\mathbb{K}$ if and only if $g_x[f_i]=0$ for all $1\leqslant i\leqslant m$. It follows that $f_i\in V$ for each i, where V is the generating subspace corresponding to the ideal I (see Lemma 1.26). So we obtain the following.

Lemma 1.40. *The polynomials f_i, $1\leqslant i\leqslant m$, form a basis of the generating subspace V corresponding to the given pair (A,U).*

Example 1.41. Consider the local algebra $A=\mathbb{K}[S]/(S^3)$ with maximal ideal $\mathfrak{m}=\langle S,S^2\rangle$.

(i) Take $U=\mathfrak{m}$. In accordance with Construction 1.39, choose a basis $s_1=S+(S^3)$, $s_2=S^2+(S^3)$ of U and let I be the kernel of the projection $\pi\colon\mathbb{K}[S_1,S_2]\to A$, $S_i\mapsto s_i$:

$$I=(S_1^2-S_2,S_1S_2),\qquad A=\mathbb{K}[S_1,S_2]/I,$$
$$s_1=S_1+I,\qquad s_2=S_2+I.$$

We omit '$+I$' for convenience. The elements $\mu_1=1$, $\mu_2=S_1$, and $\mu_3=S_2$ form a basis of A. Since $S_2=S_1^2$ and $S_1^3=0$ in A, it follows that

$$\begin{aligned}\exp(x_1s_1+x_2s_2)&=\exp(x_1S_1+x_2S_1^2)\\&=1+x_1S_1+\biggl(x_2+\frac{x_1^2}{2}\biggr)S_1^2=1+x_1\mu_1+\biggl(x_2+\frac{x_1^2}{2}\biggr)\mu_2,\end{aligned}$$

hence $f_1=1$, $f_2=x_1$, and $f_3=x_2+x_1^2/2$. By Lemma 1.40, $V=\langle 1,x_1,x_2+x_1^2/2\rangle$. This agrees with Example 1.35.

(ii) Take $U=\langle S\rangle$. Its basis $s_1=S+(S^3)$ corresponds to

$$I=(S_1^3)\subseteq\mathbb{K}[S_1],\qquad A=\mathbb{K}[S_1]/I,$$
$$s_1=S_1+I.$$

For $\mu_1=1$, $\mu_2=S_1$, and $\mu_3=S_1^2$ we have

$$\exp(x_1S_1)=1+x_1S_1+\frac{x_1^2}{2}S_1^2,$$

whence $V=\langle 1,x_1,x_1^2\rangle$ in $\mathbb{K}[x_1]$. This agrees with Example 1.27.

Example 1.42. In the same way one can see that the algebra $A=\mathbb{K}[S_1,S_2]/(S_1^2,S_1S_2,S_2^2)$ with $U=\mathfrak{m}=\langle S_1,S_2\rangle$ corresponds to the generating vector space $\langle 1,x_1,x_2\rangle\subseteq\mathbb{K}[x_1,x_2]$, which agrees with Example 1.34. There is no other subspace $U\subseteq\mathfrak{m}$ generating the algebra A.

Now we are going to discuss the duality properties for modules under consideration. In particular, we provide complete proofs for the results mentioned in [62], Remark 2.13. Recall that the generating subspace V contains constants, so the action of $\mathbb{G}_a^n$ on V by translations is linear.

Lemma 1.43. *In the notation of Theorem 1.38 the dual of the representation* $\rho\colon \mathbb{G}_a^n \to \mathrm{GL}_m(\mathbb{K})$ *is equivalent to the representation* $\tau\colon \mathbb{G}_a^n \to \mathrm{GL}(V)$ *by translations.*

Proof. Let $\langle\,\cdot\,|\,\cdot\,\rangle$ be the pairing between $\mathbb{K}[S_1,\dots,S_n]$ and $\mathbb{K}[x_1,\dots,x_n]$ as in Construction 1.19. Note that

$$\big\langle \exp(\beta_1 S_1+\dots+\beta_n S_n)g \mid f(x)\big\rangle = \big\langle g \mid f(x+\beta)\big\rangle \tag{1.4}$$

for any $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{G}_a^n$, $f\in\mathbb{K}[x_1,\dots,x_n]$, and $g\in\mathbb{K}[S_1,\dots,S_n]$. Indeed, the left-hand side equals $\langle g \mid \exp(\beta_1 S_1+\dots+\beta_n S_n)[f(x)]\rangle$, which coincides with $\langle g \mid f(x+\beta)\rangle$ by Taylor's theorem. Since $\langle I_V \mid V\rangle = 0$, we can consider $\langle\,\cdot\,|\,\cdot\,\rangle$ as a pairing between $A=\mathbb{K}[S_1,\dots,S_n]\,/\,I_V$ and $V\subseteq\mathbb{K}[x_1,\dots,x_n]$. According to the proof of Theorem 1.38, we have $\rho\colon \exp U\to\mathrm{GL}(A)$, where $U=\langle S_1,\dots,S_n\rangle$, so equation (1.4) implies that

$$\langle \rho(-\beta)g \mid f\rangle = \langle g \mid \tau(\beta)f\rangle$$

for any $\beta\in\mathbb{G}_a^n$, $f\in V$, and $g\in A$ (we identify $\beta_1S_1+\dots+\beta_nS_n$ with $-\beta$ for $\exp U\cong\mathbb{G}_a^n$). It follows that the representations ρ and τ are dual. □

Example 1.44. Let

$$A=\mathbb{K}[S]/(S^3) \quad\text{and}\quad U=\mathfrak{m}=\langle S,S^2\rangle$$

as in Example 1.41, (i). According to the correspondence (b) → (a) in Theorem 1.38, the corresponding representation $\rho\colon \mathbb{G}_a^2\to\mathrm{GL}_3(\mathbb{K})$ is the representation of $\exp U$ in A via multiplication. For an element $x_1S+x_2S^2$ in U we have

$$\exp(x_1S+x_2S^2)=1+x_1S+\left(x_2+\frac{x_1^2}{2}\right)S^2,$$

so in the basis 1, S, S^2 of A the representation ρ is given by

$$\rho(x_1,x_2)=\begin{pmatrix}1&0&0\\ x_1&1&0\\ x_2+x_1^2/2&x_1&1\end{pmatrix}.$$

For $A=\mathbb{K}[S_1,S_2]/(S_1^2,S_1S_2,S_2^2)$ and $U=\mathfrak{m}=\langle S_1,S_2\rangle$ we obtain

$$\rho(x_1,x_2)=\begin{pmatrix}1&0&0\\ x_1&1&0\\ x_2&0&1\end{pmatrix}.$$

This agrees with Lemma 1.43: the matrices of representations in V in Examples 1.25 and 1.24 are the transposes of the above ones.

In other words, Lemma 1.43 states that A and V are dual $\mathbb{G}_a^n$-modules.

Proposition 1.45. *In the notation of Theorem 1.38 the following conditions are equivalent*:

(a) $\mathbb{G}_a^n$*-modules* A *and* V *are equivalent*;

(b) *the* $\mathbb{G}_a^n$*-module* V *is cyclic*;

(c) *the algebra* A *is Gorenstein.*

Proof. (a) $\Rightarrow$ (b) The module $V \cong A$ is cyclic since the algebra A contains a unity element.

(b) $\Rightarrow$ (a) Since the module structure on V is given by translation operators in $\exp U$, $U = \langle S_1, \dots, S_n\rangle$, and V is cyclic, it follows that there exists a polynomial $f_0 \in V$ such that $V = \langle (\exp U)[f_0]\rangle = (\mathbb{K}[S_1, \dots, S_n])[f_0]$. Hence the kernel of the valuation $\pi\colon \mathbb{K}[S_1, \dots, S_n] \to V$, $g \mapsto g[f_0]$, is equal to

$$\operatorname{Ker}\pi = \{g \in \mathbb{K}[S_1, \dots, S_n]\colon g[f_0] = 0\} = \{g \in \mathbb{K}[S_1, \dots, S_n]\colon g[V] = 0\} = I.$$

Thus π gives an isomorphism between $A = \mathbb{K}[S_1, \dots, S_n]/I$ and V, which is an isomorphism of $\mathbb{G}_a^n$-modules since the module structure on A is given by $\exp U$ as well.

(b) $\Leftrightarrow$ (c) Invariant one-dimensional subspaces $\langle a\rangle$ in A correspond to invariant hyperplanes $\langle a\rangle^\perp$ in the dual module V. Since $\mathbb{G}_a^n$ is unipotent, a one-dimensional vector space is invariant if and only if it consists of fixed points. Notice that $\operatorname{Soc} A$ is the set of fixed points in A. Indeed, $(\exp U)a = a$ if and only if $Ua = 0$, that is, $\mathfrak{m}a = 0$.

If $\dim \operatorname{Soc} A > 1$, then the corresponding invariant hyperplanes cover V. Indeed, any $f \in V$ is contained in $\langle a\rangle^\perp$, where $a \in \operatorname{Soc} A \cap \langle f\rangle^\perp$. So there is no cyclic vector in this case.

If $\dim \operatorname{Soc} A = 1$, then there is a unique invariant hyperplane in V. Let us prove that any vector in the complement of this hyperplane is cyclic. It is sufficient to show that any proper invariant subspace in V is contained in an invariant hyperplane. Indeed, for $W \subsetneq V$ consider the invariant subspace $W^\perp \subseteq A$; by the Lie–Kolchin theorem there exists an invariant one-dimensional subspace $\langle a\rangle \subseteq W^\perp$; it corresponds to the required hyperplane $\langle a\rangle^\perp \supseteq W$.

1.6. The case of additive actions. In this subsection we combine the results of the two previous subsections.

Definition 1.46. A generating subspace $V \subseteq \mathbb{K}[x_1, \dots, x_n]$ is called *basic* if $\dim V = n + 1$.

Basic subspaces are minimal generating subspaces of a polynomial algebra.

Example 1.47. One can check that the following vector subspaces of $\mathbb{K}[x_1, x_2, x_3, x_4]$ are basic:

$$V_1 = \langle 1, x_1, x_2, x_3, x_4\rangle, \qquad V_2 = \left\langle 1, x_1, x_2, x_3 + \frac{x_1^2}{2}\right\rangle,$$

$$V_3 = \langle 1, x_1, x_2, x_3 + x_1x_2\rangle, \qquad V_4 = \left\langle 1, x_1, x_2 + \frac{x_1^2}{2}, x_3 + x_1x_2 + \frac{x_1^3}{6}\right\rangle.$$

The Hassett–Tschinkel correspondence for $m=n+1$ (see [62], Proposition 2.15) or the Knop–Lange theorem for $r=0$ implies a description of additive actions on projective spaces. In view of the correspondence (b) $\to$ (d) in Theorem 1.38, the basic subspace is determined just by the algebra A as we have to set $U=\mathfrak{m}$.

Theorem 1.48. *There are one-to-one correspondences between the following objects*:

(a) *the additive actions on* $\mathbb{P}^n$, *that is, effective actions* $\alpha\colon \mathbb{G}_a^n\times\mathbb{P}^n\to\mathbb{P}^n$ *with open orbit*;

(b) *the faithful cyclic representations* $\rho\colon \mathbb{G}_a^n\to \mathrm{GL}_{n+1}(\mathbb{K})$;

(c) *the local commutative associative unital algebras* A *of dimension* $n+1$;

(d) *the basic subspaces* $V\subseteq\mathbb{K}[x_1,\dots,x_n]$.

These correspondences are considered up to equivalences as in Definitions 1.9, 1.31, 1.36, *and* 1.37.

From Theorem 1.4 we obtain the following statement.

Corollary 1.49. *The projective space* $\mathbb{P}^n$ *admits a finite number of additive actions if and only if* $n\leqslant 5$.

Example 1.50. According to Table 1, there are two local algebras of dimension 3. The corresponding additive $\mathbb{G}_a^2$-actions on $\mathbb{P}^2$ were found in Examples 1.17 and 1.18, and the basic subspaces were given in Examples 1.41, (i), and 1.42. Faithful cyclic representations were written out in Example 1.44. We gather these results in Table 3. In the same way it can be proved that the basic subspaces in Example 1.47 correspond to the four local algebras of dimension 4 in Table 1, and so they are the only basic subspaces in this case.

Table 3

Additive actions	$[z_0:z_1+\alpha z_0:z_2+\beta z_0]$	$[z_0:z_1+\alpha z_0:z_2+\alpha z_1+(\beta+\alpha^2/2)z_0]$
Representations	$\begin{pmatrix}1&0&0\\ \alpha&1&0\\ \beta&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\ \alpha&1&0\\ \beta+\alpha^2/2&\alpha&1\end{pmatrix}$
Local algebras	$\mathbb{K}[S_1,S_2]/(S_1^2,S_1S_2,S_2^2)$	$\mathbb{K}[S]/(S^3)$
Basic vector subspaces	$\langle 1,x_1,x_2\rangle$	$\langle 1,x_1,x_2+x_1^2/2\rangle$

Recall that by Corollary 1.14 there is a unique additive action on $\mathbb{P}^n$ with finitely many orbits; it corresponds to the local algebra $A=\mathbb{K}[S]/(S^{n+1})$. One may look for a generalization of this result. Namely, the *modality* of an action of a connected algebraic group G on a variety X is the maximum value of the smallest codimension of a G-orbit in Y over all irreducible G-invariant subvarieties Y in X. In other words, the modality is the maximum number of parameters in a continuous family of G-orbits on X. In particular, the modality is zero if and only if the number of G-orbits on X is finite.

A classification of additive actions on $\mathbb{P}^n$ of modality 1 was obtained in [12], Theorem 3.1. Such actions correspond to the following 2-generated pairwise non-isomorphic local algebras:

$$\begin{aligned}
A_{a,b} &= \mathbb{K}[S_1,S_2]/(S_1^{a+1},S_2^{b+1},S_1S_2), && a\geqslant b\geqslant 1;\\
B_{a,b} &= \mathbb{K}[S_1,S_2]/(S_1S_2,S_1^a-S_2^b), && a\geqslant b\geqslant 2;\\
C_a &= \mathbb{K}[S_1,S_2]/(S_1^{a+1},S_2^2-S_1^3), && a\geqslant 3;\\
C_a^1 &= \mathbb{K}[S_1,S_2]/(S_1^{a+1},S_2^2-S_1^3,S_1^aS_2), && a\geqslant 3;\\
C_a^2 &= \mathbb{K}[S_1,S_2]/(S_1^{a+1},S_2^2-S_1^3,S_1^{a-1}S_2), && a\geqslant 3;\\
C_a^3 &= \mathbb{K}[S_1,S_2]/(S_2^2-S_1^3,S_1^{a-2}S_2), && a\geqslant 4;\\
D &= \mathbb{K}[S_1,S_2]/(S_1^3,S_2^2);\\
E &= \mathbb{K}[S_1,S_2]/(S_1^3,S_2^2,S_1^2S_2).
\end{aligned}$$

Clearly, the maximum possible value of modality of a non-trivial action of a connected algebraic group on an irreducible n-dimensional algebraic variety is $n-1$. It follows from Corollary 1.13 that this maximum value is attained at a unique additive action on $\mathbb{P}^n$. This action corresponds to the local algebra A with condition $\mathfrak{m}^2=0$, so that $A=\mathbb{K}[S_1,\dots,S_n]/(S_iS_j, 1\leqslant i\leqslant j\leqslant n)$. In this case the hyperplane $\mathbb{P}(\mathfrak{m})$, which is the complement of the open orbit in $\mathbb{P}^n$, consists of $\mathbb{G}_a^n$-fixed points.

In this case we can consider the blowup X of the projective space $\mathbb{P}^n$ along a smooth subvariety contained in $\mathbb{P}(\mathfrak{m})$ and lift the additive action from $\mathbb{P}^n$ to X. This proves the following result, providing many projective varieties admitting an additive action.

Proposition 1.51. *Let X be the blowup of the projective space $\mathbb{P}^n$ along a smooth subvariety contained in a hyperplane in $\mathbb{P}^n$. Then X admits an additive action.*

We finish this section with a characterization of Gorenstein local algebras in terms of the Hassett–Tschinkel correspondence. Any action of an algebraic group G on a variety X has a closed G-orbit. If X is complete, any closed orbit is complete as well. If G is unipotent, such an orbit is a G-fixed point. So, an action of a unipotent group G on a complete variety X has a fixed point.

Proposition 1.52. *In the notation of Theorem 1.48 the following conditions are equivalent*:

(a) *an additive action on $\mathbb{P}^n$ has a unique fixed point*;

(b) *the corresponding local algebra A is Gorenstein.*

Proof. As observed in the proof of (b) $\Leftrightarrow$ (c) in Proposition 1.45, the set of fixed points of the action of $\mathbb{G}_a^n$ on A is $\operatorname{Soc} A$. Since a unipotent group has no non-trivial characters, the set of fixed points of the corresponding additive action on $\mathbb{P}^n=\mathbb{P}(A)$ is $\mathbb{P}(\operatorname{Soc} A)$. So a fixed point is unique if and only if the ideal $\operatorname{Soc} A$ is one-dimensional. By definition this means that the algebra A is Gorenstein. $\square$

2. Generalizations of the Hassett–Tschinkel correspondence

In this section we adapt the method of Hassett and Tschinkel to the study of additive actions on projective varieties X different from projective spaces. We introduce an induced additive action as an additive action on X that can be extended to an ambient projective space. It turns out that every such action comes from an additive action on the projective space via the restriction to a subgroup of the acting vector group. So such actions are given by pairs (A, U), where A is a local algebra defining an additive action on the projective space and U is the subspace in $\mathfrak{m}$ that represents the subgroup.

In § 2.2 we consider the case when the projective subvariety X is a hypersurface and describe a method for writing down explicitly the homogeneous equation for X in terms of the pair (A, U). In particular, the degree of this equation is equal to the maximum number d such that the ideal $\mathfrak{m}^d$ is not contained in U. These results imply that smooth projective hypersurfaces that admit an additive action are precisely hyperplanes and non-degenerate quadrics. Moreover, if a hypersurface in $\mathbb{P}^n$ admits an additive action, then its degree does not exceed n.

In the next three subsections, §§ 2.3–2.5, we apply the methods of multilinear algebra to additive actions on projective hypersurfaces. Namely, we consider the d-linear form on the algebra A that is the polarization of the equation defining the hypersurface X and characterize additive actions on X in terms of this form. This allows us to describe additive actions on non-degenerate and degenerate quadrics, some cubics, and to prove in Theorem 2.32 that any non-degenerate projective hypersurface admits at most one additive action. We also show that induced additive actions on non-degenerate hypersurfaces come from Gorenstein local algebras.

2.1. Additive actions on projective subvarieties. Let X be a closed subvariety of dimension n in a projective space $\mathbb{P}^{m-1}$. Throughout this section we assume that X is not contained in any hyperplane of $\mathbb{P}^{m-1}$, that is, the subvariety X is linearly non-degenerate. In this subsection we introduce the notion of an induced additive action and give a variant of the Hassett–Tschinkel correspondence for induced additive actions.

Definition 2.1. An additive action $\mathbb{G}_a^n \times X \to X$ is *induced* if it can be extended to an action $\mathbb{G}_a^n \times \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$. Two induced additive actions $\alpha_i\colon \mathbb{G}_a^n \times X_i \to X_i$, $X_i \subseteq \mathbb{P}^{m-1}$, $i = 1, 2$, are said to be *equivalent* if there exist automorphisms of groups $\psi\colon \mathbb{G}_a^n \to \mathbb{G}_a^n$ and automorphisms of varieties $\varphi\colon \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$ such that $\varphi(X_1) = X_2$ and $\varphi \circ \alpha_1 = \alpha_2 \circ (\psi \times \varphi)$.

Example 2.2. Consider a cuspidal cubic plane curve

$$X = \{z_0^2 z_3 = z_1^3\} \subseteq \mathbb{P}^2.$$

Let us show that X admits an additive action, but has no induced additive action. We act in the affine chart $\{z_0 \neq 0\}$ by translations on z_1/z_0, that is, an element $a \in \mathbb{G}_a$ acts by the formula

$$[z_0 : z_1 : z_3] = \left[1 : \frac{z_1}{z_0} : \left(\frac{z_1}{z_0}\right)^3\right] \mapsto \left[1 : \frac{z_1}{z_0} + a : \left(\frac{z_1}{z_0} + a\right)^3\right].$$

Notice that we have the identity

$$\frac{z_1}{z_0} + a = \left(z_1\left(\frac{z_1}{z_0} - a\right)\right)^{-1}(z_3 - a^2 z_1),$$

which follows from the equation of X. Substituting in this identity, multiplying the homogeneous coordinates by $(z_1(z_1/z_0 - a))^3$, and using the equation of X we obtain that $a \in \mathbb{G}_a$ acts on $[z_0 : z_1 : z_3]$ by the formula

$$\begin{aligned}
&\left[1 : \frac{z_3 - a^2 z_1}{z_1(z_1/z_0 - a)} : \left(\frac{z_3 - a^2 z_1}{z_1(z_1/z_0 - a)}\right)^3\right] \\
&\quad = \left[z_1^3\left(\frac{z_1}{z_0} - a\right)^3 : z_1^2\left(\frac{z_1}{z_0} - a\right)^2(z_3 - a^2 z_1) : (z_3 - a^2 z_1)^3\right] \\
&\quad = [z_0 z_3^2 - 3a z_1^2 z_3 + 3a^2 z_0 z_1 z_3 - z_1^3 a^3 \\
&\qquad : (z_1 z_3 - 2a z_0 z_3 + z_1^2 a^2)(z_3 - a^2 z_1) : (z_3 - a^2 z_1)^3],
\end{aligned}$$

which is also well defined at the unique point $[0 : 0 : 1] \in X$ not belonging to the affine chart $\{z_0 \neq 0\}$. Thus we obtain an additive action on X. However, by Corollary 2.16 below the degree of a curve admitting an induced additive action on $\mathbb{P}^2$ is at most 2, that is, X has no induced additive action.

This example was also treated in [62], § 4.1. An action on X was constructed there from an additive action on the normalization $\mathbb{P}^1$ of X.

Remark 2.3. We denote the third coordinate by z_3 instead of z_2 for a good reason: see Remark 2.11.

Consider the case of a smooth hypersurface $X \subseteq \mathbb{P}^{m-1}$ of degree d. Denote by $\operatorname{Aut}(X)$ the group of (regular) automorphisms of X and by $\operatorname{Aut}_l(X) \subseteq \operatorname{PGL}_m(X)$ the group of linear automorphisms of X. By Theorem 2 in [87] we have $\operatorname{Aut}(X) = \operatorname{Aut}_l(X)$ if $m \geqslant 5$ or $d \neq m$. Under these assumptions, if X admits an additive action, then it admits an induced additive action. This theorem covers all smooth hypersurfaces except for the cases $(d, m) = (3, 3)$ and $(d, m) = (4, 4)$. For $(d, m) = (3, 3)$ we have a cubic curve with genus 1. It does not admit an additive action since any variety admitting an additive action is rational. By Theorem 4 in [87], in the case $(d, m) = (4, 4)$ the connected component $\operatorname{Aut}(X)^0$ of the automorphism group is trivial, which implies that there is no additive action as well. By Theorem 1 in [87], if $m \geqslant 4$ and $d \geqslant 3$, then the group $\operatorname{Aut}_l(X)$ is finite, so X admits no induced additive action. Thus, we have the following result.

Proposition 2.4. *There is no additive action on smooth hypersurfaces $X \subseteq \mathbb{P}^{m-1}$ of degree d for $m \geqslant 3$ and $d \geqslant 3$.*

As we will see in Theorem 2.25, there is a unique additive action on the non-degenerate quadric of any dimension.

An irreducible subvariety $X \subseteq \mathbb{P}^{m-1}$ is called *linearly normal* if the map $H^0(\mathbb{P}^{m-1}, \mathcal{O}(1)) \to H^0(X, \mathcal{O}(1))$ is surjective, or, equivalently, this subvariety is not contained in any hyperplane, neither is it a linear projection of a subvariety of a larger projective space.

Proposition 2.5 ([10], § 2). *Let X be linearly normal in $\mathbb{P}^{m-1}$ and admit an additive action. Then this action is induced.*

Recall that in § 1 we established the Hassett–Tschinkel correspondence between the faithful cyclic $\mathbb{G}_a^n$-representations, the pairs (A, U) of local algebras and subspaces of these algebras, and ideals and subspaces in polynomial algebras satisfying some conditions up to equivalences; see Theorem 1.38, (a)–(d). Now we are ready to add one more item, (e), to this list; it characterizes induced additive actions on projective subvarieties.

Theorem 2.6. *Let $n, m \in \mathbb{Z}_{\geqslant 0}$. There are one-to-one correspondences between*

(a) *the faithful cyclic representations $\rho\colon \mathbb{G}_a^n \to \mathrm{GL}_m(\mathbb{K})$;*

(b) *the pairs (A, U), where A is a local commutative associative unital algebra of dimension m with maximal ideal $\mathfrak{m}$ and $U \subseteq \mathfrak{m}$ is a subspace of dimension n generating A;*

(c) *the non-degenerate ideals $I \subseteq \mathbb{K}[S_1, \dots, S_n]$ of codimension m supported at the origin;*

(d) *the generating subspaces $V \subseteq \mathbb{K}[x_1, \dots, x_n]$ of dimension m;*

(e) *the classes of equivalence of induced additive actions on projective subvarieties of dimension n in $\mathbb{P}^{m-1}$ that are not contained in a hyperplane.*

Proof. Let us construct the correspondence between (a) and (e). Consider the canonical projections $p\colon \mathbb{K}^m \setminus \{0\} \to \mathbb{P}^{m-1}$ and $\pi\colon \mathrm{GL}_m(\mathbb{K}) \to \mathrm{PGL}_m(\mathbb{K})$.

A faithful representation $\mathbb{G}_a^n \to \mathrm{GL}_m(\mathbb{K})$ defines a subgroup $\mathbb{G}_a^n \subseteq \mathrm{GL}_m(\mathbb{K})$. For the group $\mathbb{K}^\times$ of non-zero scalar matrices we have the direct product $H = \mathbb{K}^\times \times \mathbb{G}_a^n$ in $\mathrm{GL}_m(\mathbb{K})$. Let v be a cyclic vector in $\mathbb{K}^m$ and X be the projectivization of the closure of the orbit $Hv \subseteq \mathbb{K}^m \setminus \{0\}$, that is,

$$X = p(\overline{Hv}) \subseteq \mathbb{P}^{m-1}.$$

Let the effective action on X be given by $\pi(H) \subseteq \mathrm{PGL}_m(\mathbb{K}) = \mathrm{Aut}(\mathbb{P}^{m-1})$. Note that $\pi(H) \cong \mathbb{G}_a^n$ since $\operatorname{Ker} \pi = \mathbb{K}^\times \subseteq H$. Then $p(Hv)$ is an open orbit in X, and X is not contained in any hyperplane since v is a cyclic vector. We will see below that the resulting subvariety X and the additive action on it do not depend on the choice of a cyclic vector v.

Conversely, let the subvariety $X \subseteq \mathbb{P}^{m-1}$ admit an induced additive action. Then X is the closure of an orbit of the effective action $\mathbb{G}_a^n \times \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$. Consider $\mathbb{G}_a^n$ as a subgroup of $\mathrm{PGL}_m(\mathbb{K})$, and let $H = \pi^{-1}(\mathbb{G}_a^n) \subseteq \mathrm{GL}_m(\mathbb{K})$. Then $H \cong \mathbb{K}^\times \times \mathbb{G}_a^n$, where $\mathbb{K}^\times$ is a subgroup of scalar matrices as above and the subgroup $\{1\} \times \mathbb{G}_a^n \subseteq H$ gives the corresponding faithful representation of $\mathbb{G}_a^n$. Let $\langle v \rangle \in \mathbb{P}^{m-1}$ be a point in the open orbit of X for some $v \in \mathbb{K}^m$. Since X is not contained in any hyperplane, the same holds for its open orbit $\mathbb{G}_a^n \langle v \rangle$, so $Hv = p^{-1}(\mathbb{G}_a^n \langle v \rangle) \subseteq \mathbb{K}^m$ is not contained in any hyperplane of $\mathbb{K}^m$ and v is a cyclic vector for $\mathbb{G}_a^n$.

Thus, a subvariety of $\mathbb{P}^{m-1}$ of dimension n with induced additive action is the projectivization of the closure of an orbit of a cyclic vector for a $\mathbb{K}^\times \times \mathbb{G}_a^n$-representation in $\mathbb{K}^m$. In order to show that this construction does not depend on the choice of a cyclic vector we use (b). The representation $\mathbb{G}_a^n \to \mathrm{GL}_m(\mathbb{K})$ corresponding to a pair (A, U) is constructed in the following way: $\exp U \cong \mathbb{G}_a^n$ acts on $A \cong \mathbb{K}^m$ by multiplication in the algebra A. In these terms, the representation of $\mathbb{K}^\times \times \mathbb{G}_a^n$ we are interested in is the representation of the group $\mathbb{K}^\times \exp U$ in A by operators of multiplication, and the orbit of an element $a \in A$ is the set $\mathbb{K}^\times \exp U \cdot a$.

Recall that any element of the maximal ideal $\mathfrak{m}$ in the local algebra A is nilpotent, and any element of $A \setminus \mathfrak{m}$ is invertible. If $a \in \mathfrak{m}$, then a is nilpotent, and all elements in $\mathbb{K}^\times \exp U \cdot a$ are nilpotent as well, so $\mathbb{K}^\times \exp U \cdot a \subseteq \mathfrak{m}$. This implies that $X \subseteq \mathbb{P}(A)$ is contained in the hyperplane $\mathbb{P}(\mathfrak{m})$, so we do not consider this case. If $a \in A \setminus \mathfrak{m}$, then a is invertible, so the orbit $\mathbb{K}^\times \exp U \cdot 1$ is isomorphic to $\mathbb{K}^\times \exp U \cdot a$ via the linear operator L_a of multiplication by a. This isomorphism commutes with the $\mathbb{G}_a^n$-actions on these orbits by the commutativity of multiplication in A. Thus, for any pair (A, U) there is a unique induced additive action corresponding to this pair up to equivalence of induced additive actions; this action does not depend on the choice of a cyclic vector. $\square$

We finish the discussion by the correspondence (b) $\to$ (e) between pairs (A, U) and induced additive actions.

Construction 2.7. Suppose that A is a local commutative associative unital algebra of dimension m with maximal ideal $\mathfrak{m}$, $U \subseteq \mathfrak{m}$ is a subspace of dimension n generating the algebra A, and let $p\colon A \setminus \{0\} \to \mathbb{P}(A) = \mathbb{P}^{m-1}$ be the canonical projection. According to the proof of Theorem 2.6, the corresponding projective subvariety is the projectivization of an orbit of a cyclic vector, that is,

$$X = p(\overline{\mathbb{K}^\times \exp U}),$$

the additive action on X is given by the operators of multiplication by elements of $\mathbb{G}_a^n \cong \exp U \subseteq A$, and the set $p(\mathbb{K}^\times \exp U) = p(\exp U)$ is an open orbit in X. Denote the coordinate along $\mathbb{K}$ in $A = \mathbb{K} \oplus \mathfrak{m}$ by z_0 and consider the affine chart $\{z_0 = 1\} = 1 + \mathfrak{m}$ on the projective space $\mathbb{P}^{m-1} = \mathbb{P}(A)$. Notice that $(\mathbb{K}^\times \exp U) \cap (1 + \mathfrak{m}) = \exp U$ is closed as an orbit of a unipotent group, which implies that $X \cap \{z_0 \neq 0\} = p(\exp U)$ is the open orbit.

Example 2.8. Let $n = 1$, that is, we are interested in the induced $\mathbb{G}_a$-actions on curves in $\mathbb{P}^{m-1}$. By Theorem 2.6 the equivalence classes of such actions are in bijection with the equivalence classes of pairs (A, U), where A is a local commutative algebra of dimension m with maximal ideal $\mathfrak{m}$ and U is a line in $\mathfrak{m}$ generating the algebra A. Then we may assume that $A = \mathbb{K}[S]/(S^m)$ and $U = \langle S \rangle$ up to an automorphism of A. The pair (A, U) corresponds to an additive action on the rational normal curve of degree $m - 1$ in $\mathbb{P}^{m-1}$. In particular, we have a unique class of equivalence of induced additive $\mathbb{G}_a$-actions on curves in $\mathbb{P}^{m-1}$. For explicit computations in the case $m = 4$, see Example 2.10 below.

2.2. The case of projective hypersurfaces: equations. In this subsection we obtain the equation of a projective hypersurface $X \subseteq \mathbb{P}^{n+1}$ admitting an induced additive action in terms of the corresponding pair (A, U) (see Theorem 2.6 and Definition 2.12). First we consider a subvariety that is not necessarily a hypersurface. The following proposition provides a condition that gives the open orbit $\exp U \subseteq 1 + \mathfrak{m}$ (see Construction 2.7).

By ln we mean the standard logarithm series

$$\ln(1 + z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k,$$

which is inverse to the exponential map exp. Applying ln to $1+z$ with nilpotent element z, we obtain a polynomial in z.

Proposition 2.9. *Let A be a local commutative associative unital algebra of dimension m with maximal ideal $\mathfrak{m}$, and let $U \subseteq \mathfrak{m}$ be a subspace of dimension n generating the algebra A. Denote by $\pi\colon \mathfrak{m} \to \mathfrak{m}/U$ the canonical projection of vector spaces. Then $\exp U$ is given in $1+\mathfrak{m}$ by the condition*

$$\pi(\ln(1+z)) = 0 \tag{2.1}$$

for $1+z \in A = \mathbb{K} \oplus \mathfrak{m}$, $z \in \mathfrak{m}$.

Proof. An element $1+z \in 1+\mathfrak{m}$ belongs to $\exp U$ if and only if $\ln(1+z) \in U$.

The above proposition helps us to find the subvariety $X \subseteq \mathbb{P}^{m-1}$ corresponding to a given pair (A,U).

Example 2.10. Let $A = \mathbb{K}[S]/(S^4)$ and $U = \langle S\rangle \subseteq \mathfrak{m} = \langle S, S^2, S^3\rangle$. Let $\pi\colon \mathfrak{m} \to \mathfrak{m}/U$ be the projection onto $\langle S^2, S^3\rangle$ along $\langle S\rangle$. By Proposition 2.9 the set of points $z = z_1S + z_2S^2 + z_3S^3$ that belong to $\exp U \subseteq 1+\mathfrak{m}$ is given by the condition

$$\pi\left(z_1S + z_2S^2 + z_3S^3 - \frac{(z_1S + z_2S^2 + z_3S^3)^2}{2} + \frac{(z_1S + z_2S^2 + z_3S^3)^3}{3} - \cdots\right)$$
$$= \left(z_2 - \frac{z_1^2}{2}\right)S^2 + \left(z_3 - z_1z_2 + \frac{z_1^3}{3}\right)S^3 = 0,$$

that is, the open orbit of X in the affine chart $\{z_0 = 1\}$ is given by the system $z_2 - z_1^2/2 = 0$, $z_3 - z_1z_2 + z_1^3/3 = 0$, or, substituting the first equation into the second, by the parametrization

$$[z_0 : z_1 : z_2 : z_3] = \left[1 : z_1 : \frac{z_1^2}{2} : \frac{z_1^3}{6}\right] \subseteq \mathbb{P}^3.$$

Taking the closure, we add one more (fixed) point $[0:0:0:1]$ and obtain a twisted cubic in $\mathbb{P}^3$.

Notice that the closure of the intersection of hypersurfaces may not be equal to the intersection of their closures, so X may not be given by a system of homogenized equations. For example, in our case, outside the affine chart $\{z_0 = 1\}$ under consideration the system $z_0z_2 - z_1^2/2 = 0$, $z_0^2z_3 - z_0z_1z_2 + z_1^3/3 = 0$ gives the projective line $z_0 = z_1 = 0$, rather than a point.

Remark 2.11. Although the additive action in Example 2.2 is not induced, it is the projection of the induced additive action in Example 2.10 along the coordinate z_2.

Let us apply the above theory to the case of codimension 1. Suppose that $X \subseteq \mathbb{P}^{m-1}$ is a projective hypersurface that is not a hyperplane and (A,U) is the corresponding pair, that is, A is a local commutative associative unital algebra of dimension m with maximal ideal $\mathfrak{m}$ and $U \subseteq \mathfrak{m}$ is a subspace of dimension $m-2$ generating A. The next definition is taken from [12].

Definition 2.12. The *H-pair* corresponding to an induced additive action on a hypersurface $X \subseteq \mathbb{P}^{n+1}$ is the pair (A,U) corresponding to it, where A is a local commutative associative unital algebra of dimension $n+2$ with maximal ideal $\mathfrak{m}$ and $U \subseteq \mathfrak{m}$ is a subspace of dimension n generating the algebra A.

Now we prove the following technical lemma.

Lemma 2.13. *Suppose $\mathfrak{m}$ is the maximal ideal of a local commutative associative algebra A and U is a subspace of $\mathfrak{m}$. Then $\mathfrak{m}^d \subseteq U$ if and only if $z^d \in U$ for all $z \in \mathfrak{m}$.*

Proof. Let

$$f_d(t) = \sum_{k=0}^{d} z_k t^k$$

be a polynomial with coefficients in A and let $f_d(t) \in U$ for all $t \in \mathbb{K}$. We show that $z_0, z_1, \dots, z_d \in U$. First, $z_0 = f_d(0) \in U$. Then for a polynomial $f_{d-1}(t) = \sum_{k=1}^{d} z_k t^{k-1}$ we have $f_{d-1}(t) = \dfrac{f_d(t) - f_d(0)}{t} \in U$ for any $t \in \mathbb{K}^\times$. Note that for $t = 0$ we also have $f_{d-1}(t) \in U$ since the set $\{t \in \mathbb{K} \colon f_{d-1}(t) \in U\}$ is closed in $\mathbb{K}$. So $z_1 = f_{d-1}(0) \in U$. Arguing as above for polynomials $f_{d-1}(t)$ of degree $d-1$, $f_{d-2}(t)$ of degree $d-2$, ..., and $f_0(t)$ of degree 0, we finally obtain $z_0, z_1, \dots, z_d \in U$.

Now let $z^d \in U$ for all $z \in \mathfrak{m}$. Then

$$f(t_1, \dots, t_d) = (t_1 z_1 + \dots + t_d z_d)^d \in U$$

for any $t_1, \dots, t_d \in \mathbb{K}$ and $z_1, \dots, z_d \in \mathfrak{m}$. Fixing arbitrary $t_2, \dots, t_d \in \mathbb{K}$ and applying the above argument we see that all coefficients of f as a polynomial in t_1 belong to U for any $t_2, \dots, t_d \in \mathbb{K}$. We consider these coefficients for fixed $t_3, \dots, t_d$ as polynomials in t_2 and obtain that they also belong to U for any $t_3, \dots, t_d \in \mathbb{K}$. Finally, we see that all the coefficients of f belong to U. In particular, the coefficient $d!\, z_1 \cdots z_d$ of $t_1 \cdots t_d$ is an element of U, so $\mathfrak{m}^d \subseteq U$. The converse is immediate. □

If $f(1+z) = 0$, $z \in \mathfrak{m}$, is an equation of degree d defining the open orbit $\exp U$ in the affine chart $1 + \mathfrak{m}$, then $X \subseteq \mathbb{P}^{n+1}$ is given by the homogenization hf of the polynomial f: $hf(z_0 + z) = z_0^d f(1 + z/z_0)$, $z_0 \in \mathbb{K}$, $z \in \mathfrak{m}$. In particular, the degree of the projective hypersurface X equals the degree d of the affine hypersurface $\exp U \subseteq 1 + \mathfrak{m}$.

The first statement in the theorem below was proved in [12], Theorem 5.1.

Theorem 2.14. *Let $X \subseteq \mathbb{P}^{n+1}$ be a projective hypersurface admitting an induced additive action and (A, U) be the corresponding H-pair. Denote by $\pi \colon \mathfrak{m} \to \mathfrak{m}/U \cong \mathbb{K}$ the canonical projection. Then*

1) *the degree of the hypersurface X equals the largest exponent d such that $\mathfrak{m}^d \not\subseteq U$;*

2) *X is given by the homogeneous equation of degree d*

$$z_0^d\, \pi\left(\ln\left(1 + \frac{z}{z_0}\right)\right) = 0 \tag{2.2}$$

for $z_0 + z \in A = \mathbb{K} \oplus \mathfrak{m}$, $z_0 \in \mathbb{K}$, and $z \in \mathfrak{m}$.

Proof. By Proposition 2.9 the open orbit in the affine chart $\exp U \subseteq 1 + \mathfrak{m}$ is given by the polynomial $f(1+z) = \pi(\ln(1+z))$, $z \in \mathfrak{m}$. By the definition of d we have $\mathfrak{m}^k \subseteq U$ for all $k > d$. It follows that f is of degree at most d since π takes all terms of the logarithm series with exponents greater than d to zero. On the other hand

$\mathfrak{m}^d \nsubseteq U$, so by Lemma 2.13 there exists $z \in \mathfrak{m}$ with $\pi(z^d) \neq 0$. Thus, the degree of the polynomial f equals d.

Let us prove that f is irreducible. Since $\mathfrak{m}^d \nsubseteq U$, we have $\mathfrak{m}^d \cap U \subsetneq \mathfrak{m}^d$. Since the codimension of U in $\mathfrak{m}$ equals 1, the codimension of $\mathfrak{m}^d \cap U$ in $\mathfrak{m}^d$ is at most 1, so by the above we can consider the decomposition $\mathfrak{m}^d = (\mathfrak{m}^d \cap U) \oplus \langle S \rangle$ for some vector $S \in \mathfrak{m}^d$. Since $S \notin U$ and U in $\mathfrak{m}$ is of codimension 1, we also have $\mathfrak{m} = U \oplus \langle S \rangle$ in this case. Let

$$z = z_U + z_{n+1}S, \qquad z_U \in U, \quad z_{n+1} \in \mathbb{K},$$

be the corresponding decomposition of $z \in \mathfrak{m}$. Then

$$\pi(\ln(1+z)) = \pi\biggl(\sum_{k=1}^{d} \frac{(-1)^{k-1}}{k}(z_U + z_{n+1}S)^k\biggr),$$

and one can see that π takes all the z_{n+1} to zero except for the occurrence of z_{n+1} in the term with $k = 1$ since $\mathfrak{m}S \subseteq \mathfrak{m}^{d+1} \subseteq U$. So the variable z_{n+1} appears in the polynomial f only in the linear term and f is irreducible.

Thus, $\exp U$ is given by the irreducible polynomial f of degree d. Hence the hypersurface X has degree d and X is given by the homogenization hf as in (2.2). $\square$

Example 2.15. Let

$$A = \mathbb{K}[S_1, S_2, S_3]/(S_1^2, S_2^2, S_1S_3, S_2S_3, S_1S_2 - S_3^3)$$

be 6-dimensional algebra no. 30 from Table 1. Notice that $A = \langle 1, S_1, S_2, S_3, S_3^2, S_3^3 = S_1S_2\rangle$ and consider $U = \langle S_1, S_2, S_3, S_3^2\rangle \subseteq \mathfrak{m}$. Since $\mathfrak{m}^3 = \langle S_3^3\rangle \nsubseteq U$ and $\mathfrak{m}^4 = 0$, the H-pair (A, U) corresponds to an induced additive action on a cubic hypersurface $X \subseteq \mathbb{P}^5$. According to (2.2), for the projection $\pi\colon \mathfrak{m} \to \langle S_3^3\rangle$ along U the left-hand side of the equation of X is

$$\begin{aligned} z_0^3\pi\biggl(\ln\biggl(1 + \frac{z_1}{z_0}S_1 + \frac{z_2}{z_0}S_2 + \frac{z_3}{z_0}S_3 + \frac{z_4}{z_0}S_3^2 + \frac{z_5}{z_0}S_3^3\biggr)\biggr) \\ = z_0^3\biggl(\frac{z_5}{z_0} - \frac{1}{2}\cdot 2\frac{z_3}{z_0}\frac{z_4}{z_0} - \frac{1}{2}\cdot 2\frac{z_1}{z_0}\frac{z_2}{z_0} + \frac{1}{3}\frac{z_3^3}{z_0^3}\biggr), \end{aligned}$$

which gives $X = \{z_0^2z_5 - z_0z_3z_4 - z_0z_1z_2 + \frac{1}{3}z_3^3 = 0\}$.

Corollary 2.16 (see [12], Corollary 5.2). *If $X \subseteq \mathbb{P}^{n+1}$ is a hypersurface of degree d admitting an induced additive action, then $d \leqslant n+1$.*

Proof. Since $\mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \cdots$ and $\dim \mathfrak{m} = n+1$, we have $\mathfrak{m}^{n+2} = 0 \subseteq U$. $\square$

Let us illustrate the method developed by proving a variant of Proposition 2.4.

Corollary 2.17 (see [10], Proposition 4). *If X is a smooth hypersurface admitting an induced additive action, then X is a non-degenerate quadric or a hyperplane.*

Proof. As in the proof of Theorem 2.14, we choose a vector $S \in \mathfrak{m}^d \setminus U$ and obtain the decomposition of vector spaces

$$A = \mathbb{K} \oplus \mathfrak{m} = \mathbb{K} \oplus U \oplus \langle S\rangle.$$

For the compatible coordinates $z_0, \ldots, z_{n+1}$ the variable z_{n+1} appears in the equation $hf(z_0, \ldots, z_{n+1}) = 0$ of X only in the term $z_0^{d-1} z_{n+1}$ since z_{n+1} appears in the polynomial f only in a linear term. Thus, the point $[0 : \cdots : 0 : 1]$ lies on X and is singular, provided that $d \geqslant 3$. It remains to note that the only smooth quadric is a non-degenerate one. □

Corollary 2.18. *If a hypersurface X of degree d admits an induced additive action and (A, U) is the corresponding H-pair, then the complement of the open orbit in $X \subseteq \mathbb{P}(A)$ is*

$$p(\{z \in \mathfrak{m} \colon z^d \in U\}),$$

where $p \colon A \setminus \{0\} \to \mathbb{P}(A)$ is the canonical projection.

Proof. According to Construction 2.7 the complement of the open orbit consists of the points $z \in X$ with zero z_0-coordinate. Substituting $z_0 = 0$ into (2.2) annihilates all terms of the logarithm series, except the last term of degree d, so we obtain the equation $\pi\biggl(\dfrac{(-1)^d}{d!}(0+z)^d\biggr) = 0$, or $z^d \in U$.

2.3. The case of projective hypersurfaces: invariant multilinear forms. It is well known that the quadratic forms $f(z)$ on a vector space V are in one-to-one correspondence with the bilinear maps $F \colon V \times V \to \mathbb{K}$. If X is a quadric given by a quadratic equation $f(z) = 0$, then the corresponding bilinear form F gives a lot of information on X. Recall that in the same way any homogeneous polynomial $f(z)$ of degree d corresponds to a d-linear symmetric form $F \colon \underbrace{V \times \cdots \times V}_{d} \to \mathbb{K}$: any d-linear form F gives a polynomial $f(z) = F(z, \ldots, z)$ and, conversely, $F(z^{(1)}, \ldots, z^{(d)})$ can be found from f as a coefficient of $t_1 \cdots t_d$ in the polynomial $f(t_1 z^{(1)} + \cdots + t_d z^{(d)})$. This fact allows us to study hypersurfaces admitting induced additive actions in terms of multilinear forms.

Suppose (A, U) is an H-pair. Following § 4 in [10], we call a d-linear form $F \colon \underbrace{A \times \cdots \times A}_{d} \to \mathbb{K}$ *invariant* if the following conditions hold:

(i) $F(1, \ldots, 1) = 0$;
(ii) for any $u \in U$, $z^{(1)}, \ldots, z^{(d)} \in A$ we have

$$F(uz^{(1)}, z^{(2)}, \ldots, z^{(d)}) + F(z^{(1)}, uz^{(2)}, \ldots, z^{(d)}) + \cdots + F(z^{(1)}, z^{(2)}, \ldots, uz^{(d)}) = 0. \tag{2.3}$$

Suppose that an H-pair (A, U) corresponds to an induced additive action on a hypersurface $X \subseteq \mathbb{P}^{n+1} = \mathbb{P}(A)$ given by the polynomial f of degree d on A. By the above there is a d-linear form $F \colon \underbrace{A \times \cdots \times A}_{d} \to \mathbb{K}$ corresponding to the polynomial f. It is an invariant d-linear form on (A, U). Indeed, property (i) follows from the construction: we take 1 as a cyclic vector in A, so $F(1, \ldots, 1) = f(1) = 0$. For equation (2.3) notice that $X = \{f(x) = 0\}$ is invariant with respect to the action of the group $\mathbb{G}_a^n \cong \exp U$, that is, the polynomial f is semi-invariant. But the group $\mathbb{G}_a^n$ has no non-trivial character, so f is invariant with respect to $\mathbb{G}_a^n \cong \exp U$ and therefore with respect to the Lie algebra U, that is, we have (2.3). The invariant

d-linear form corresponding to a hypersurface $X \subseteq \mathbb{P}(A)$ is defined up to a scalar. Notice also that the number d is determined by the pair (A, U).

Let F be a d-linear form on a vector space V. Then we define

- $L^{\perp} = \{x \in V : F(x, z^{(2)}, \dots, z^{(d)}) = 0 \ \forall\, z^{(2)}, \dots, z^{(d)} \in L\}$ for a subset $L \subseteq V$;
- the kernel $\operatorname{Ker} F = V^{\perp}$.

Lemma 2.19. *Let* $F\colon \underbrace{A \times \cdots \times A}_{d} \to \mathbb{K}$ *be an invariant d-linear form on an H-pair (A, U). Then*

(a) $U \subseteq 1^{\perp}$;

(b) $\operatorname{Ker} F$ *is the maximal ideal of A contained in U.*

Proof. Assertion (a) follows from (2.3) for $z^{(1)} = z^{(2)} = \cdots = z^{(d)} = 1$.

We prove (b). First we show that $\operatorname{Ker} F$ is an ideal of A. If $z \in \operatorname{Ker} F$ and $u \in U$, then

$$F(uz, z^{(2)}, \dots, z^{(d)}) = -F(z, uz^{(2)}, \dots, z^{(d)}) - \cdots - F(z, z^{(2)}, \dots, uz^{(d)}) = 0$$

for any $z^{(2)}, \dots, z^{(d)} \in A$, so $uz \in \operatorname{Ker} F$ for any $u \in U$. Since U generates A as an algebra, it follows that $Az \subseteq \operatorname{Ker} F$.

Now we prove that $\operatorname{Ker} F \subseteq U$. Since $\operatorname{Ker} F$ is an ideal in A and F is not equal to 0, the kernel $\operatorname{Ker} F$ contains no invertible elements, that is, let $\operatorname{Ker} F \subseteq \mathfrak{m}$. Assume the converse, that is, $\operatorname{Ker} F \nsubseteq U$. Since $\dim \mathfrak{m} = n + 1$ and $\dim U = n$, it follows that $\mathfrak{m} = \operatorname{Ker} F + U$.

Using induction on k we prove that $F(u^{(1)}, \dots, u^{(k)}, 1 \dots, 1) = 0$ for any $u^{(1)}, \dots, u^{(k)} \in U$. For $k = 0$ we have $F(1, \dots, 1) = 0$. Suppose the assertion holds for some k. According to (2.3) we have

$$\sum_{i=1}^{k} F(u^{(1)}, \dots, u^{(k+1)}u^{(i)}, \dots, u^{(k)}, 1, \dots, 1) \\ + \sum_{i=k+1}^{d} F(u^{(1)}, \dots, u^{(k)}, 1, \dots, \underset{i}{u^{(k+1)}}, \dots, 1) = 0.$$

All the $d - k$ terms of the second sum equal $F(u^{(1)}, \dots, u^{(k+1)}, 1, \dots, 1)$ since F is a symmetric form. For a summand of the first sum one can decompose an element $u^{(k+1)}u^{(i)} \in \mathfrak{m}$ as $u^{(k+1)}u^{(i)} = z_i + u_i$, where $z_i \in \operatorname{Ker} F$, $u_i \in U$. Then this summand becomes

$$F(u^{(1)}, \dots, z_i, \dots, u^{(k)}, 1, \dots, 1) + F(u^{(1)}, \dots, u_i, \dots, u^{(k)}, 1, \dots, 1),$$

and the first term is zero by the kernel condition, while the second vanishes by the induction hypothesis. Thus,

$$(d - k)F(u^{(1)}, \dots, u^{(k+1)}, 1, \dots, 1) = 0,$$

which completes the induction.

Since F is multilinear, it follows that $1 \in \langle 1, U\rangle^{\perp}$. Moreover, $1 \in (\operatorname{Ker} F)^{\perp}$ and $\mathfrak{m} = \operatorname{Ker} F + U$, so $1 \in A^{\perp} = \operatorname{Ker} F \subseteq \mathfrak{m}$, which is a contradiction.

It remains to prove maximality. Let $J \subseteq U$ be an ideal of the algebra A. Using induction on k we prove that $F(z^{(1)}, \dots, z^{(k)}, y, 1, \dots, 1) = 0$ for any $y \in J$ and $z^{(1)}, \dots, z^{(k)} \in A$. For $k = 0$, according to (2.3) we have

$$\sum_{k=1}^{d} F(1, \dots, \underset{i}{y}, \dots, 1) = 0$$

since $y \in J \subseteq U$, which gives $F(y, 1, \dots, 1) = 0$ because F is a symmetric form. Suppose that the assertion has been proved for $k-1$. Then

$$\sum_{i=1}^{k} F(z^{(1)}, \dots, yz^{(i)}, \dots, z^{(k)}, 1, \dots, 1) + \sum_{i=k+1}^{d} F(z^{(1)}, \dots, z^{(k)}, 1, \dots, \underset{i}{y}, \dots, 1) = 0.$$

All the $d-k$ terms of the second sum equal $F(z^{(1)}, \dots, z^{(k)}, y, 1, \dots, 1)$ since F is symmetric. Each term of the first sum equals zero by the induction hypothesis since $yz^{(i)} \in J$. Thus, $F(z^{(1)}, \dots, z^{(k)}, y, 1, \dots, 1) = 0$, which completes the induction.

For $k = d-1$ we obtain $F(z^{(1)}, \dots, z^{(d-1)}, y) = 0$ for any $z^{(1)}, \dots, z^{(d-1)} \in A$ and $y \in J$. It follows that $y \in \operatorname{Ker} F$, that is, for any ideal $J \subseteq U$ of A we have $J \subseteq \operatorname{Ker} F$. $\square$

Now let us define the reduction of an induced additive action. First we return to the general case of a projective subvariety which is not necessarily a hypersurface.

Proposition 2.20. *Let a pair (A, U) correspond to an induced additive action on a projective subvariety $X \subseteq \mathbb{P}(A)$. Suppose there is an ideal J of A such that $J \subseteq U$. Then the pair $(A/J, U/J)$ corresponds to an induced additive action on a projective subvariety $X_0 \subseteq \mathbb{P}(A/J)$, and X is the projective cone over X_0. In other words, if coordinates in $\mathbb{P}^{n+1} = \mathbb{P}(A)$ are compatible with the inclusions $A = \mathbb{K} \oplus \mathfrak{m} \supseteq \mathfrak{m} \supseteq J$, then X does not depend on the coordinates in J. Moreover, the additive action on X is coherent with the additive action on X_0, that is, the following diagram, where $\varphi\colon A \to A/J$ is the projection, is commutative*:

$$\begin{tikzcd} \exp U \times A \arrow[r, "\text{mult. in } A"] \arrow[d, "\varphi \times \varphi"] & A \arrow[d, "\varphi"] \\ \exp(U/J) \times (A/J) \arrow[r, "\text{mult. in } A/J"] & A/J \end{tikzcd}$$

Proof. If J is an ideal in a local commutative unital algebra $A = \mathbb{K} \oplus \mathfrak{m}$ with maximal ideal $\mathfrak{m}$, then $A/J = \mathbb{K} \oplus (\mathfrak{m}/J)$ is a local commutative unital algebra with maximal ideal $\mathfrak{m}/J$. Since the subspace $U \subseteq \mathfrak{m}$ generates A, the subspace U/J generates the algebra A/J. Fix some decomposition $\mathfrak{m} = J \oplus \mathfrak{m}'$, and for $z \in \mathfrak{m}$ let $z = z_J + z'$. Let us find the equation of X in accordance with Proposition 2.9. Since J is an ideal of A, we have

$$\ln(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (z_J + z')^k \in \ln(1+z') + J \subseteq \ln(1+z') + U,$$

that is, $\pi(\ln(1+z))$ does not depend on the coordinates in J.

To show coherence, fix a decomposition $U = J \oplus U'$ and let $\varphi\colon A \to A/J$ be the projection onto U' along J. For $u = u_J + u' \in U = J \oplus U'$ notice that

$$\exp(u) = \sum_{k=1}^{\infty} \frac{1}{k!}(u_J + u')^k \in \exp(u') + J,$$

since J is an ideal in A; hence $\varphi(\exp U) = \exp U'$. Consider any $a \in A$. Then

$$\exp(u)a = \sum_{k=1}^{\infty} \frac{1}{k!}(u_J + u')^k a \in \exp(u')a + J = (\exp(u') + J)(a + J),$$

which proves that the diagram is commutative as required. □

Proposition 2.20 motivates the following definition; see [12], § 4.

Definition 2.21. The induced additive action corresponding to a pair (A, U) is *reducible* to the induced additive action corresponding to a pair (A', U') if there exists an algebra homomorphism $\varphi\colon A \to A'$ such that $\varphi(U) = U'$ and $\operatorname{codim}_A U = \operatorname{codim}_{A'} U'$.

In such a case φ is surjective since U' generates A'. So there exists an ideal $J = \operatorname{Ker}\varphi$ of the algebra A such that $J \subseteq U$ and the factorization $A \to A/J \cong A'$ maps U to U', that is, we are in the situation of Proposition 2.20.

Definition 2.22. Suppose a projective hypersurface $X \subseteq \mathbb{P}(V)$ of degree d is given by an equation $f(z_1, \dots, z_n) = 0$ and F is the corresponding d-linear form. Then X is called *non-degenerate* if one of the following equivalent conditions holds:

(a) $\operatorname{Ker} F = 0$;

(b) $\dfrac{\partial f}{\partial z_1}, \dots, \dfrac{\partial f}{\partial z_n}$ are linearly independent $(d-1)$-linear forms;

(c) there is no linear transformation of variables that reduces the number of variables in f.

Corollary 2.23. *Any induced additive action on a hypersurface is reducible to an induced additive action on a non-degenerate hypersurface. More precisely, an induced additive action corresponding to the H-pair (A, U) is reducible to the induced additive action corresponding to the H-pair $(A/\operatorname{Ker} F, U/\operatorname{Ker} F)$, where F is the invariant multilinear form of X.*

Proof. This follows from Proposition 2.20 and Lemma 2.19, (b). □

In [10], Lemma 1, an explicit formula for the form F corresponding to a pair (A, U) was obtained. Since $A = \mathbb{K} \oplus \mathfrak{m}$ and F is multilinear, it is sufficient to define F for arguments that belong to $\mathfrak{m}$ or are equal to 1. Let $\pi\colon \mathfrak{m} \to \mathfrak{m}/U \cong \mathbb{K}$ be the canonical projection. Then

$$F(z^{(1)}, z^{(2)}, \dots, z^{(d)}) = (-1)^k k!\,(d-k-1)!\,\pi(z^{(1)} \cdots z^{(d)}), \tag{2.4}$$

where k is the number of ones among $z^{(1)}, z^{(2)}, \dots, z^{(d)}$, while the other arguments belong to $\mathfrak{m}$, and for $k = d$ we let $F(1, \dots, 1) = 0$. One can check that this agrees with the equation $f(z_0 + z) = z_0^d \pi(\ln(1 + z/z_0)) = 0$ obtained in Theorem 2.14.

Indeed, for a multilinear form F defined by (2.4) and any $z_0 + z \in A = \mathbb{K} \oplus \mathfrak{m}$ we have

$$\begin{aligned} F(z_0+z,\dots,z_0+z) &= \sum_{k=0}^{d} \binom{d}{k} F(\underbrace{z_0,\dots,z_0}_{k},\underbrace{z,\dots,z}_{d-k}) \\ &= \sum_{k=0}^{d} z_0^k \frac{d!}{k!\,(d-k)!}(-1)^k k!\,(d-k-1)!\,\pi(z^{d-k}) \\ &= d!(-1)^d\, z_0^d \pi\biggl(\ln\biggl(1+\frac{z}{z_0}\biggr)\biggr). \end{aligned}$$

2.4. The case of quadrics and cubics. Let us start with the following well-known fact.

Lemma 2.24. *The automorphism group of the non-degenerate quadric* $Q_n \subseteq \mathbb{P}^{n+1}$ *is* $\mathrm{PSO}_{n+2}(\mathbb{K})$.

Proof. Notice that the Picard group of the quadric $\operatorname{Pic} Q_n \cong \mathbb{Z}$ is generated by the line bundle $\mathcal{O}(1)$ for $n \geqslant 3$. Any automorphism of Q_n induces an automorphism of the Picard group, which can take the generator $\mathcal{O}(1)$ either to $\mathcal{O}(1)$ or to $\mathcal{O}(-1)$. The last case is impossible since $\mathcal{O}(-1)$ has no global section, so any hyperplane section of Q_n is mapped to a hyperplane section. The last assertion holds for $n = 1, 2$ as well. Thus, any automorphism of Q_n corresponds to a transformation of $\mathbb{P}(H^0(\mathcal{O}(1))) = (\mathbb{P}^{n+1})^*$. The dual of this transformation is the extension of the initial automorphism of Q_n to an automorphism of $\mathbb{P}^{n+1}$. $\square$

The following theorem answers Question 3.1, (4), in [62].

Theorem 2.25 (see [103], Theorem 4). *Let Q_n be a non-degenerate quadric in $\mathbb{P}^{n+1}$. Then there is a unique additive action on Q_n up to equivalence. It corresponds to the H-pair (A_n, U_n), where*

$$A_n = \mathbb{K}[S_1,\dots,S_n]/(S_i^2 - S_j^2, S_iS_j, i \neq j) \quad \textit{and} \quad U_n = \langle S_1,\dots,S_n\rangle \quad \textit{if } n \geqslant 2;$$
$$A_1 = \mathbb{K}[S_1]/(S_1^3) \quad \textit{and} \quad U_1 = \langle S_1\rangle.$$

Proof. By Lemma 2.24 any additive action on X is induced. Let (A, U) be the corresponding H-pair, so that A is a local algebra of dimension $n+2$ and $U \subseteq \mathfrak{m}$ is a subspace of dimension n generating A. By the first assertion of Theorem 2.14, $\mathfrak{m}^2 \nsubseteq U$ and $\mathfrak{m}^3 \subseteq U$. Since Q_n is non-degenerate, the corresponding multilinear form has a trivial kernel (see Definition 2.22). From Lemma 2.19, (b), it follows that there is no non-zero ideal of A in U. Then $\mathfrak{m}^3 \subseteq U$ implies that $\mathfrak{m}^3 = 0$. Hence $\mathfrak{m}^2 \cap U$ is an ideal in U, so $\mathfrak{m}^2 \cap U = 0$ as well. Since $\mathfrak{m}^2 \nsubseteq U$ implies $\mathfrak{m}^2 \neq 0$, we have $\mathfrak{m} = U \oplus \mathfrak{m}^2$, and $\dim \mathfrak{m}^2 = 1$ for reasons of dimension.

It remains to prove that there is a unique pair (A, U) satisfying the above conditions. Notice that multiplication in the algebra $A = \mathbb{K} \oplus U \oplus \mathfrak{m}^2$ is determined by the restriction $B\colon U \times U \to \mathfrak{m}^2$. Indeed, $U \cdot U \subseteq \mathfrak{m}^2$ since $U \subseteq \mathfrak{m}$, $U \cdot \mathfrak{m}^2 = 0$ and $\mathfrak{m}^2 \cdot \mathfrak{m}^2 = 0$ since $\mathfrak{m}^3 = 0$, and $1 \cdot x = x$ for any $x \in A$. Since $\dim \mathfrak{m}^2 = 1$, it follows that B is a bilinear form on U, and now we are going to prove that this form is non-degenerate. A non-degenerate bilinear form on a vector space is unique up to a linear change of variables, so the non-degeneracy of B will imply the uniqueness of the pair (A, U) and the corresponding additive action.

In our situation the left-hand side of equation (2.2) of the quadric Q_n turns to

$$z_0^2\pi\left(\ln\left(1+\frac{z}{z_0}\right)\right) = z_0^2\pi\left(\frac{z}{z_0}-\frac{1}{2}\,\frac{z^2}{z_0^2}\right) = z_0\pi(z)-\frac{1}{2}\pi(z^2),$$

where $\pi\colon \mathfrak{m}\to\mathfrak{m}/U\cong\mathbb{K}$ is a projection. Recall that $\mathfrak{m}=U\oplus\mathfrak{m}^2$, so π can be chosen to be the projection $\pi\colon\mathfrak{m}\to\mathfrak{m}^2$ along U. For $z\in\mathfrak{m}$ set $z=z_U+z_{\mathfrak{m}^2}$, where $z_U\in U$ and $z_{\mathfrak{m}^2}\in\mathfrak{m}^2$. Then $z^2=z_U^2$ since $\mathfrak{m}^3=0$, so the equation $z_0\pi(z)-\frac{1}{2}\pi(z^2)=0$ of Q_n turns to

$$z_0z_{\mathfrak{m}^2}-\frac{1}{2}B(z_U,z_U)=0.$$

It defines a non-degenerate quadric if and only if the form B is non-degenerate, so we arrive at the desired uniqueness.

Now it is easy to calculate the pair (A, U). Denote a non-zero vector in $\mathfrak{m}^2$ by S, and let $S_1,\dots,S_n$ be a basis of U such that $B(z_U,z_U)=(z_1^2+\dots+z_n^2)S$ for $z_U=z_1S_1+\dots+z_nS_n$. From the definition of B it follows that $S_i^2=S_j^2=S$, $S_iS_j=0$ for $i\neq j$, and $\mathfrak{m}S=0$ since $S\in\mathfrak{m}^2$. Thus, the algebra A is isomorphic to $\mathbb{K}[S_1,\dots,S_n]/(S_i^2-S_j^2, S_iS_j,\, i\neq j)$ if $n\geqslant 2$ and to $\mathbb{K}[S_1]/(S_1^3)$ if $n=1$, as required. $\square$

The next result was observed in [12], Proposition 4.2.

Corollary 2.26. *An H-pair (A, U) corresponds to an induced additive action on a quadric if and only if there exists a homomorphism of H-pairs $(A,U)\to(A_n,U_n)$.*

Denote projective quadrics in $\mathbb{P}^{n+1}$ by

$$Q(n,k)=\{[z_0:\dots:z_{n+1}]\mid q(z_0,\dots,z_{n+1})=0\},$$

where q is a quadratic form of rank $k+2$, where $1\leqslant k\leqslant n$. In this notation the non-degenerate quadric $Q_n\subseteq\mathbb{P}^{n+1}$ is $Q(n,n)$.

In [20] the authors obtained a generalization of the Hassett–Tschinkel correspondence for induced actions of commutative linear algebraic groups on non-degenerate quadrics with open orbit. In [20], Theorem 3, it was proved that, apart from the unique additive action from Theorem 2.25, there are only three cases: a $\mathbb{G}_m$-action on Q_1, a $\mathbb{G}_a\times\mathbb{G}_m$-action on Q_2, and a $\mathbb{G}_m^2$-action on Q_2.

At the same time, additive actions on degenerate quadrics are not unique. In particular, there is an infinite family of pairwise non-equivalent induced additive actions on the quadrics $Q(n,n-1)$ for $n\geqslant 4$: see [12], § 4.

Proposition 2.27 ([10], Proposition 7). *The H-pairs corresponding to induced additive actions on quadrics $Q(n,n-1)\subseteq\mathbb{P}^{n+1}$ are:*

(a) $A=\mathbb{K}[S_1,\dots,S_n]\Big/\begin{pmatrix} S_iS_j-\lambda_{ij}S_n,\, S_i^2-S_j^2 & \\ -(\lambda_{ii}-\lambda_{jj})S_n, & 1\leqslant i<j\leqslant n-1 \\ S_lS_n, & 1\leqslant l\leqslant n\end{pmatrix}$, *where $n\geqslant 3$ and the λ_{ij} are the elements of a symmetric $(n-1)\times(n-1)$ block-diagonal*

matrix Λ *such that each block* Λ_l *has the form*

$$\begin{pmatrix} \lambda_l & 1 & 0 & \dots & 0 \\ 1 & \lambda_l & 1 & \ddots & \vdots \\ 0 & 1 & \lambda_l & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & \lambda_l \end{pmatrix} + \begin{pmatrix} 0 & \dots & 0 & \imath/2 & 0 \\ \vdots & \ddots & \imath/2 & 0 & -\imath/2 \\ 0 & \ddots & 0 & -\imath/2 & 0 \\ \imath/2 & \ddots & \ddots & \ddots & \vdots \\ 0 & -\imath/2 & 0 & \dots & 0 \end{pmatrix}, \qquad \imath^2 = -1,$$

and $U = \langle S_1, \dots, S_n \rangle$;

(b) $\mathbb{K}[S_1, S_2]/(S_1^3, S_1S_2, S_2^2)$, $U = \langle S_1, S_2 \rangle$;

(c) $\mathbb{K}[S_1]/(S_1^4)$, $U = \langle S_1, S_1^3 \rangle$.

The matrix Λ *is defined up to a permutation of blocks, scalar multiplication, and addition of a scalar matrix.*

In [12], § 4, an explicit description of the actions for $n = 4$ was also given.

Using the same techniques, a classification of additive actions on quadrics of corank 2 having at least one singular point that is not fixed by the $\mathbb{G}_a^n$-action was presented in [82].

Remark 2.28. In [99], § 5, a classification of additive actions on quadrics of small dimension was presented. It was proved that the surface $Q(2,1)$ admits two additive actions (cf. Proposition 2.27, (b) and (c)) and the 3-folds $Q(3,1)$ and $Q(3,2)$ admit seven and three additive actions, respectively. The number of additive actions on $Q(4,1)$ is finite, but is at least 25. Finally, there are infinitely many additive actions on $Q(4,2)$. They are classified in terms of H-pairs.

In [18] the case of cubic hypersurfaces was studied and the following theorem was proved.

Theorem 2.29. *A cubic hypersurface* $\{f = 0\}$ *in* $\mathbb{P}^{n+1}$ *admits an induced additive action if and only if for some* $1 \leqslant k \leqslant s \leqslant n-k$ *one can choose homogeneous coordinates* $z_0, z_1, \dots, z_s, w_0, w_1, \dots, w_{n-s}$ *in* $\mathbb{P}^{n+1}$ *such that the polynomial* f *has the form*

$$f = z_0^2 w_0 + z_0(z_1 w_1 + \dots + z_k w_k) + z_0(z_{k+1}^2 + \dots + z_s^2) + g(z_1, \dots, z_k),$$

where g *in a non-degenerate cubic form in* k *variables. Moreover, an induced additive action is unique if and only if this hypersurface is non-degenerate, that is,* $k + s = n$.

2.5. Non-degenerate hypersurfaces and Gorenstein algebras. Let A be a Gorenstein local algebra. The socle of A is equal to the one-dimensional ideal $\mathfrak{m}^d$. A hyperplane U in $\mathfrak{m}$ is called *complementary* if $\mathfrak{m} = U \oplus \mathfrak{m}^d$.

Theorem 2.30. *The induced additive actions on non-degenerate hypersurfaces* X *of degree* d *in* $\mathbb{P}^{n+1}$ *are in bijection with the H-pairs* (A, U), *where* A *is a Gorenstein algebra of dimension* $n+2$ *with socle* $\mathfrak{m}^d$ *and* U *is a complementary hyperplane.*

Proof. We observe first that a subspace U of the maximal ideal $\mathfrak{m}$ of a local algebra A contains no non-zero ideal of A if and only if $(\operatorname{Soc} A) \cap U = 0$. The 'only if' part is clear since $(\operatorname{Soc} A) \cap U$ is an ideal of A in U. Conversely, if J is a non-zero ideal of A in U, then, given a maximal s such that $J\mathfrak{m}^s$ is non-zero, we have $J\mathfrak{m}^s \subseteq (\operatorname{Soc} A) \cap J \subseteq (\operatorname{Soc} A) \cap U$.

This shows that if (A, U) is the H-pair corresponding to an induced additive action on a non-degenerate hypersurface in $\mathbb{P}^{n+1}$, then $\mathfrak{m} = U \oplus (\operatorname{Soc} A)$ and A is Gorenstein for reasons of dimension.

Conversely, let A be a Gorenstein local algebra of dimension $n+2$ and U be a complementary hyperplane in $\mathfrak{m}$. Since $A = \mathbb{K} \oplus U \oplus (\operatorname{Soc} A)$, we conclude that U generates A, so (A, U) is an H-pair corresponding to an induced additive action on a hypersurface X in $\mathbb{P}^{n+1}$. Since $(\operatorname{Soc} A) \cap U = 0$, it follows that U contains no non-zero ideal, so the hypersurface X is non-degenerate.

Finally, we notice that by Theorem 2.14 the degree of X is d, where $\operatorname{Soc} A = \mathfrak{m}^d$. □

Remark 2.31. We know of no example where two different complementary hyperplanes U and U' in the maximal ideal $\mathfrak{m}$ of the same Gorenstein local algebra A give rise to additive actions on non-isomorphic hypersurfaces.

As an application of Theorem 2.30, we see from Table 1 that for $n \leqslant 5$ there are induced additive actions on non-degenerate hypersurfaces in $\mathbb{P}^n$ of all degrees from 2 to n. Moreover, there are three types of non-degenerate cubic hypersurfaces in $\mathbb{P}^5$, which come from different Gorenstein algebras.

Now let us prove a generalization of the uniqueness of an additive action on non-degenerate quadrics and cubics.

Theorem 2.32. *Let $X \subseteq \mathbb{P}^{n+1}$ be a non-degenerate hypersurface. Then there is at most one induced additive action on X up to equivalence.*

Proof. Let (A, U) be the H-pair corresponding to an induced additive action on X. The hypersurface X defines the corresponding invariant multilinear form F (see § 2.3). We have to prove that F defines the pair (A, U) uniquely up to equivalence (see Theorem 2.6).

Denote by d the degree of the hypersurface X. Since X is non-degenerate, by Theorem 2.30 the algebra A is Gorenstein, $\mathfrak{m}^{d+1} = 0$, $\mathfrak{m} = U \oplus \mathfrak{m}^d$, and $\dim \mathfrak{m}^d = 1$. Since $1 \cdot a = a$ for any $a \in A$ and $\mathfrak{m}^d \cdot \mathfrak{m} = 0$, it remains to define multiplication of elements in U. Let $\pi \colon \mathfrak{m} \to \mathfrak{m}^d \cong \mathbb{K}$ be the canonical projection along U, and $B \colon U \times U \to \mathfrak{m}^d \cong \mathbb{K}$ be the bilinear map defined by $B(u_1, u_2) = \pi(u_1 u_2)$.

Let us prove that $\operatorname{Ker} B$ is an ideal of A. If $u \in \operatorname{Ker} B$, then $B(u, u_2) = \pi(u u_2) = 0$ for any $u_2 \in U$, that is, $u u_2 \in U$ for any $u_2 \in U$. Moreover, $uU \subseteq U$ implies that $uA \subseteq U$ since $u \cdot 1 \in U$ and $u \cdot \mathfrak{m}^d = 0$. Then for any elements $a \in A$ and $u_2 \in U$ we have $u a u_2 \subseteq uA \subseteq U$, so $B(ua, u_2) = \pi(u a u_2) = 0$. Thus, $ua \in \operatorname{Ker} B$ for any $a \in A$, so $\operatorname{Ker} B$ is an ideal of A. By Lemma 2.19, (b), $\operatorname{Ker} F$ is the maximal ideal of A contained in U, so $\operatorname{Ker} B \subseteq \operatorname{Ker} F$. But F is non-degenerate, so B is non-degenerate as well.

Denote a non-zero vector in $\mathfrak{m}^d$ by S, and let $S_1, \dots, S_n$ be a basis of U such that $B(z_U, z_U) = (z_1^2 + \dots + z_n^2) S$ for $z_U = z_1 S_1 + \dots + z_n S_n$. It follows from the definition of B that $S_i^2 - S \in U$ and $S_i S_j \in U$ for $i \neq j$. Denote $S_i^2 - S = \sum_l \alpha_{il} S_l$ and $S_i S_j = \sum_l \beta_{ijl} S_l$. According to (2.4),

$$F(z^{(1)}, \dots, z^{(d)}) = (-1)^k k! \, (d-k-1)! \, \pi(z^{(1)} \cdots z^{(d)}),$$

where all arguments are either equal to 1 or nilpotent and k is the number of ones. In particular, $F(z^{(1)},z^{(2)},1,\dots,1)=(-1)^{d-2}(d-2)!\,\pi(z^{(1)}z^{(2)})$ for all $z^{(1)},z^{(2)}\in\mathfrak{m}$, so the bilinear form B is uniquely defined by the multilinear form F. Notice that $B(S_i^2-S,S_l)=\alpha_{il}S$ and $B(S_iS_j,S_l)=\beta_{ijl}S$. Hence the coefficients α_{il} and β_{ijl} and, consequently, the products S_i^2 and S_iS_j, $i\neq j$, are uniquely defined by the form F. Thus, F defines multiplication on U. $\square$

Example 2.33. Let $A=\mathbb{K}[S_1,S_2,S_3]/(S_1^2,S_2^2,S_1S_3,S_2S_3,S_1S_2-S_3^3)$ be the 6-dimensional Gorenstein algebra no. 30 from Table 1 and $U=\langle S_1,S_2,S_3,S_3^2\rangle\subseteq\mathfrak{m}$ (see Example 2.15). Recall that $A=\langle 1,S_1,S_2,S_3,S_3^2,S_3^3=S_1S_2\rangle$ and the H-pair (A,U) corresponds to an induced additive action on a cubic hypersurface $X\subseteq\mathbb{P}^5$. Since $\operatorname{Soc}A=\mathfrak{m}^3$ and $\mathfrak{m}=U\oplus\mathfrak{m}^3$, we conclude that X is non-degenerate. By Theorem 2.32 there is a unique induced additive action on X. One can write it down explicitly by Theorem 2.6: identifying $[z_0:z_1:z_2:z_3:z_4:z_5]\in\mathbb{P}^5$ with $z_0+z_1S_1+z_2S_2+z_3S_3+z_4S_3^2+z_5S_3^3\in A$ and multiplying by

$$\begin{aligned}&\exp(\alpha_1S_1+\alpha_2S_2+\alpha_3S_3+\alpha_4S_3^2)\\&\qquad=1+\alpha_1S_1+\alpha_2S_2+\alpha_3S_3+\alpha_4S_3^2+\left(\alpha_1\alpha_2+\alpha_3\alpha_4+\frac{\alpha_3^3}{6}\right)S_3^3,\end{aligned}$$

we obtain that the action of $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in\mathbb{G}_a^4$ is given by

$$\begin{aligned}\Big[&z_0:z_1+\alpha_1z_0:z_2+\alpha_2z_0:z_3+\alpha_3z_0:z_4+\alpha_3z_3+\alpha_4z_0\\&\quad:z_5+\alpha_3z_4+\alpha_4z_3+\alpha_1z_2+\alpha_2z_1+\left(\alpha_1\alpha_2+\alpha_3\alpha_4+\frac{\alpha_3^3}{6}\right)z_0\Big].\end{aligned}$$

Finally, the linear change of coordinates

$$Z_0=-z_0,\quad W_0=z_5,\quad Z_1=z_3,\quad W_1=z_4,\quad Z_2=\frac{\imath(z_2-z_1)}{2},\quad Z_3=\frac{z_2+z_1}{2}$$

takes the equation of X to the form given in Theorem 2.29 with $k=1$, $s=3$, and $n=4$.

3. Additive actions on flag varieties

The main aim of this section is to describe additive actions on flag varieties G/P of a semisimple linear algebraic group G.

We begin with a brief overview of the geometric properties of varieties X admitting an additive action: such a variety is rational and it has a free finitely generated divisor class group that is generated by the classes of boundary divisors. If the variety X is complete, then the classes of boundary divisors generate freely the monoid of classes of effective divisors. Moreover, for smooth X the anti-canonical class $-K_X$ is an integer linear combination of classes of boundary divisors with coefficients at least 2. We also observe that if a commutative subgroup of the automorphism group of a variety X acts on X with open orbit, then it is a maximal commutative subgroup of $\operatorname{Aut}(X)$.

In § 3.2 we classify flag varieties G/P which admit an additive action. It turns out that in this case the parabolic subgroup P is maximal and the existence of an additive action on G/P is almost equivalent to the commutativity of the unipotent radical P_u with few explicit exceptions.

In § 3.3 we discuss a uniqueness result: if a flag variety G/P is not isomorphic to a projective space, then G/P admits at most one additive action.

Finally, in § 3.4 we present a construction due to Feigin that allows one to degenerate an arbitrary flag variety G/P to a projective variety with an additive action.

3.1. Generalities on additive actions on complete varieties. In this subsection we recall briefly some basic geometric properties of varieties admitting an additive action.

Clearly, any variety X with an additive action $\mathbb{G}_a^n \times X \to X$ contains an open $\mathbb{G}_a^n$-orbit that is isomorphic to an affine space. This implies that X is a rational variety.

It is well known that the complement $X \setminus \mathcal{U}$ of an affine open subset $\mathcal{U}$ on an irreducible variety X is a union $D_1 \cup \cdots \cup D_k$ of prime divisors. If X is a normal variety with an additive action and $\mathcal{U}$ is the open orbit, then we call $[D_1], \dots, [D_k]$ the *boundary classes* in $\mathrm{Cl}(X)$.

Proposition 3.1. *Let X be a normal variety and $\mathcal{U} \subseteq X$ be an open subset isomorphic to an affine space. Then any invertible regular function on X is constant and the divisor class group $\mathrm{Cl}(X)$ is a free finitely generated abelian group. Moreover, the boundary classes form a basis in $\mathrm{Cl}(X)$.*

Proof. If f is an invertible regular function on X, then its restriction to $\mathcal{U}$ is invertible and also regular. It follows that f is constant on $\mathcal{U}$ and so on X.

Since all divisors on $\mathcal{U}$ are principal, any divisor on X is linearly equivalent to an integral linear combination of prime divisors in the complement $X \setminus \mathcal{U}$. If such a linear combination is zero in $\mathrm{Cl}(X)$, then it is a principal divisor corresponding to some rational function f on X. The function f has neither zero nor pole on $\mathcal{U}$, so it is an invertible regular function on the affine space. Such a function is a constant, hence the combination is trivial. $\square$

As the next step, let us formulate a result proved in [62], Theorems 2.5 and 2.7.

Theorem 3.2. *Let X be a complete normal variety with an additive action. Then the monoid of classes of effective divisors is generated freely by the boundary classes. Moreover, if X is smooth, then the anti-canonical class $-K_X$ is an integer linear combination of boundary classes, where all coefficients are $\geqslant 2$.*

To obtain the first statement of Theorem 3.2 one should use a linearization of an arbitrary divisor with respect to an action of a unipotent group. Namely, consider an effective divisor D on X. The representation of $\mathbb{G}_a^n$ on the projectivization of the space $H^0(X, \mathcal{O}(D))$ has a fixed point, which corresponds to an effective divisor supported at the boundary and linearly equivalent to D. The proof of the second statement is more delicate: it requires computations with vector fields and related exact sequences.

Remark 3.3. If a complete normal variety X admits an additive action, it need not be projective. Examples of additive actions on smooth non-projective complete toric varieties X in any dimension starting from 3 were constructed in [101].

We make one more observation. We say that a subgroup H of the automorphism group $\mathrm{Aut}(X)$ of an algebraic variety X is *algebraic* if it carries the structure of an algebraic group such that the action $H \times X \to X$ is regular. Note that if the group $\mathrm{Aut}(X)$ itself has the structure of an algebraic group such that the action $\mathrm{Aut}(X) \times X \to X$ is regular, then the notion of an algebraic subgroup coincides with the notion of an algebraic subgroup of an algebraic group.

Proposition 3.4. *Assume that a commutative algebraic group H acts effectively on an irreducible variety with open orbit. Then H is a maximal* (*with respect to inclusion*) *commutative algebraic subgroup of the group* $\mathrm{Aut}(X)$.

Proof. Assume that H is contained in a larger commutative algebraic subgroup F and let $g \in F \setminus H$. Then g permutes H-orbits on X and, in particular, it preserves the open H-orbit $\mathcal{U}$ on X. Take a point $x \in \mathcal{U}$. Then there is an element $h \in H$ such that $gx = hx$. This shows that $h^{-1}g$ fixes x. So $h^{-1}g$ acts identically on $\mathcal{U}$ and on X, which is a contradiction. $\square$

This result shows that each additive action $\mathbb{G}_a^n \times X \to X$ provides a maximal commutative unipotent subgroup of the automorphism group $\mathrm{Aut}(X)$. In particular, the Hassett–Tschinkel correspondence allows one to construct many non-conjugate maximal commutative unipotent subgroups of dimension n in $\mathrm{GL}_{n+1}(\mathbb{K})$. At the same time, there are maximal commutative unipotent subgroups of other dimensions in $\mathrm{GL}_{n+1}(\mathbb{K})$.

In the next subsections we study additive actions on (generalized) flag varieties, that is, on homogeneous spaces G/P of a connected semisimple group G modulo a parabolic subgroup P. The following proposition provides additional motivation to concentrate on varieties of this type.

It is well known that the connected component $\mathrm{Aut}(X)^0$ of the automorphism group of a complete variety is a connected linear algebraic group. In view of the importance of the problem of the existence of a Kähler–Einstein metric, cases when the group $\mathrm{Aut}(X)^0$ is reductive are of particular interest.

Proposition 3.5 (see [10], Proposition 1). *Let X be a complete variety admitting an additive action. Assume that the group* $\mathrm{Aut}(X)^0$ *is a reductive linear algebraic group. Then X is a flag variety G/P for some semisimple group G and some parabolic subgroup P.*

Proof. Let X' be the normalization of X. The action of $\mathrm{Aut}(X)^0$ on X can be lifted to X'. This implies that some commutative unipotent group acts on X' with open orbit. In particular, a maximal unipotent subgroup of the reductive group $\mathrm{Aut}(X)^0$ acts on X' with open orbit. This means that X' is a spherical variety of rank zero (see [107], § 1.5.1, for details). Hence X' is a flag variety G/P (see [107], Proposition 10.1), and $\mathrm{Aut}(X)^0$ acts transitively on X'. This implies that $X = X'$. $\square$

3.2. Existence of an additive action on a flag variety. In this subsection we follow [3] and classify flag varieties that admit an additive action.

Let G be a connected semisimple linear algebraic group of adjoint type over an algebraically closed field of characteristic zero, and P be a parabolic subgroup of G. The homogeneous space G/P is called a (*generalized*) *flag variety*. Recall that G/P is projective and the action of the unipotent radical P_u^- of the opposite parabolic subgroup P^- on G/P by left multiplication has an open orbit. This open orbit $\mathcal{U}$ is called the big Schubert cell on G/P. Since $\mathcal{U}$ is isomorphic to the affine space $\mathbb{A}^n$, where $n = \dim G/P$, every flag variety can be regarded as a completion of an affine space.

Our goal is to find all flag varieties G/P that are equivariant completions of $\mathbb{G}_a^n$. Clearly, this is the case when the group P_u^- or, equivalently, the group P_u is commutative.

It is a classical result that the connected component $\widetilde{G}$ of the automorphism group of G/P is a semisimple group of adjoint type and $G/P = \widetilde{G}/Q$ for some parabolic subgroup $Q \subseteq \widetilde{G}$. In most cases $\widetilde{G}$ coincides with G, and all exceptions are well known (see, for example, [92], Theorem 7.1, or [108], p. 118). If $\widetilde{G} \neq G$, then we say that $(\widetilde{G}, Q)$ is the *covering* pair of the *exceptional* pair (G, P). For a simple group G the exceptional pairs are $(\mathrm{PSp}(2r), P_1)$, $(\mathrm{SO}(2r+1), P_r)$, and (G_2, P_1) with the covering pairs $(\mathrm{PSL}(2r), P_1)$, $(\mathrm{PSO}(2r+2), P_{r+1})$, and $(\mathrm{SO}(7), P_1)$, respectively, where PH denotes the quotient of the group H by its centre and P_i is the maximal parabolic subgroup associated with the ith simple root. It turns out that, given a simple group G, the condition $\widetilde{G} \neq G$ implies that the unipotent radical Q_u is commutative and P_u is not. In particular, in this case G/P is an equivariant completion of $\mathbb{G}_a^n$. Our main result states that these are the only possible cases.

Theorem 3.6 (see [3], Theorem 1). *Let G be a connected semisimple group of adjoint type and P be a parabolic subgroup of G. Then the flag variety G/P is an equivariant completion of $\mathbb{G}_a^n$ if and only if for every pair $(G^{(i)}, P^{(i)})$, where $G^{(i)}$ is a simple component of G and $P^{(i)} = G^{(i)} \cap P$, one of the following conditions holds*:

(a) *the unipotent radical $P_u^{(i)}$ is commutative*;

(b) *the pair $(G^{(i)}, P^{(i)})$ is exceptional.*

For the convenience of the reader, we list all pairs (G, P), where G is a simple group (up to local isomorphism) and P is a parabolic subgroup with commutative unipotent radical:

$$(\mathrm{SL}_{r+1}, P_i), \quad i = 1, \dots, r; \qquad (\mathrm{SO}_{2r+1}, P_1); \qquad (\mathrm{Sp}_{2r}, P_r);$$
$$(\mathrm{SO}_{2r}, P_i), \quad i = 1, r-1, r; \qquad (E_6, P_i), \quad i = 1, 6; \qquad (E_7, P_7)$$

(see [97], § 2). Note that the unipotent radical of P_i is commutative if and only if the simple root α_i occurs with coefficient 1 in the highest root ρ (see [97], Lemma 2.2). Another equivalent condition is that the fundamental weight ω_i of the dual group $G^\vee$ is minuscule, that is, the weight system of the simple $G^\vee$-module $V(\omega_i)$ with highest weight ω_i coincides with the orbit $W\omega_i$ of the Weyl group W.

Proof of Theorem 3.6. If the unipotent radical P_u^- is commutative, then the action of P_u^- on G/P by left multiplication is the desired additive action. The same arguments work when for the connected component $\widetilde{G}$ of the automorphism group $\mathrm{Aut}(G/P)$ one has $G/P = \widetilde{G}/Q$ and the unipotent radical Q_u^- is commutative. Since

$$G/P \cong G^{(1)}/P^{(1)} \times \cdots \times G^{(k)}/P^{(k)},$$

where $G^{(1)}, \dots, G^{(k)}$ are the simple components of the group G, $\widetilde{G}$ is isomorphic to the direct product $\widetilde{G^{(1)}} \times \cdots \times \widetilde{G^{(k)}}$. Moreover, $Q_u \cong Q_u^{(1)} \times \cdots \times Q_u^{(k)}$, where $Q^{(i)} = \widetilde{G^{(i)}} \cap Q$. Thus the group Q_u^- is commutative if and only if for every pair $(G^{(i)}, P^{(i)})$ either $P_u^{(i)}$ is commutative or the pair $(G^{(i)}, P^{(i)})$ is exceptional.

Conversely, assume that G/P admits an additive action. We can identify $\mathbb{G}_a^n$ with a commutative unipotent subgroup H of $\widetilde{G}$ and the flag variety G/P with $\widetilde{G}/Q$, where Q is a parabolic subgroup of $\widetilde{G}$.

Let T and B, $T \subseteq B$, be a maximal torus and a Borel subgroup of the group $\widetilde{G}$ such that $B \subseteq Q$. Consider the root system Φ of the tangent algebra $\mathfrak{g} = \mathrm{Lie}(\widetilde{G})$ defined by the torus T, its decomposition $\Phi = \Phi^+ \cup \Phi^-$ into the positive and negative roots associated with B, the set of simple roots $\Delta \subseteq \Phi^+$, $\Delta = \{\alpha_1, \dots, \alpha_r\}$, and the root decomposition

$$\mathfrak{g} = \bigoplus_{\beta \in \Phi^-} \mathfrak{g}_\beta \oplus \mathfrak{t} \oplus \bigoplus_{\beta \in \Phi^+} \mathfrak{g}_\beta,$$

where $\mathfrak{t} = \mathrm{Lie}(T)$ is a Cartan subalgebra of $\mathfrak{g}$ and

$$\mathfrak{g}_\beta = \{x \in \mathfrak{g} \colon [y, x] = \beta(y)x \text{ for all } y \in \mathfrak{t}\}$$

is a root subspace. Set $\mathfrak{q} = \mathrm{Lie}(Q)$ and $\Delta_Q = \{\alpha \in \Delta \colon \mathfrak{g}_{-\alpha} \nsubseteq \mathfrak{q}\}$. For every root $\beta = a_1\alpha_1 + \cdots + a_r\alpha_r$ we set $\deg(\beta) = \sum_{\alpha_i \in \Delta_Q} a_i$. This gives a $\mathbb{Z}$-grading on the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k, \quad \text{where} \quad \mathfrak{t} \subseteq \mathfrak{g}_0 \quad \text{and} \quad \mathfrak{g}_\beta \subseteq \mathfrak{g}_k \quad \text{with} \quad k = \deg \beta.$$

In particular,

$$\mathfrak{q} = \bigoplus_{k \geqslant 0} \mathfrak{g}_k \quad \text{and} \quad \mathfrak{q}_u^- = \bigoplus_{k < 0} \mathfrak{g}_k.$$

Assume that the unipotent radical Q_u^- is not commutative and consider $\mathfrak{g}_\beta \subseteq [\mathfrak{q}_u^-, \mathfrak{q}_u^-]$. For every $x \in \mathfrak{g}_\beta \setminus \{0\}$ there exist $z' \in \mathfrak{g}_{\beta'} \subseteq \mathfrak{q}_u^-$ and $z'' \in \mathfrak{g}_{\beta''} \subseteq \mathfrak{q}_u^-$ such that $x = [z', z'']$. In this case $\deg(z') > \deg(x)$ and $\deg(z'') > \deg(x)$.

Since the subgroup H acts on $\widetilde{G}/Q$ with open orbit, conjugating H we may assume that the H-orbit of the point eQ is open in $\widetilde{G}/Q$. This implies that $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$, where $\mathfrak{h} = \mathrm{Lie}(H)$. On the other hand $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{q}_u^-$. So every element $y \in \mathfrak{h}$ can uniquely be written as $y = y_1 + y_2$, where $y_1 \in \mathfrak{q}$ and $y_2 \in \mathfrak{q}_u^-$, and the linear map $\mathfrak{h} \to \mathfrak{q}_u^-$, $y \mapsto y_2$, is bijective. Take the elements $y, y', y'' \in \mathfrak{h}$ such that $y_2 = x$, $y_2' = z'$, and $y_2'' = z''$. Since the subgroup H is commutative, we have $[y', y''] = 0$. Thus

$$[y_1' + y_2', y_1'' + y_2''] = [y_1', y_1''] + [y_2', y_1''] + [y_1', y_2''] + [y_2', y_2''] = 0.$$

However,

$$[y_2', y_2''] = x \quad \text{and} \quad [y_1', y_1''] + [y_2', y_1''] + [y_1', y_2''] \in \bigoplus_{k > \deg x} \mathfrak{g}_k.$$

This contradiction shows that the group Q_u^- is commutative. As we have seen, the latter condition means that for every pair $(G^{(i)}, P^{(i)})$ either the unipotent radical $P_u^{(i)}$ is commutative, or the pair $(G^{(i)}, P^{(i)})$ is exceptional. □

It is well known that if the ground field $\mathbb{K}$ is the field of complex numbers, then Hermitian symmetric spaces of compact type are precisely homogeneous spaces G/P, where the parabolic subgroup P has a commutative unipotent radical (see [97], for example). So from Theorem 3.6 we deduce the following observation.

Corollary 3.7. *A complete complex homogeneous variety X admits an additive action if and only if X is a Hermitian symmetric space of compact type.*

3.3. Uniqueness results. The following result is a generalization of Theorem 2.25; in the case of Grassmannians it was conjectured in [12], § 6.

Theorem 3.8. *Let G be a connected simple linear algebraic group and P be a parabolic subgroup of G. Assume that the flag variety $X = G/P$ is not isomorphic to the projective space $\mathbb{P}^n$. Then X admits at most one additive action up to equivalence.*

Two different ways to prove this result were obtained independently by Fu and Hwang [51] and Devyatov [41]. We discuss each of these approaches briefly.

Fu–Hwang's proof is based on the study of varieties of minimal rational tangents (VMRT). In [51] the authors proved the following theorem.

Theorem 3.9. *Let X be a smooth Fano variety of dimension n with Picard number 1 that is not isomorphic to $\mathbb{P}^n$. Assume that X has a family of minimal rational curves whose variety of minimal rational tangents $C_x \subseteq \mathbb{P}T_x(X)$ at a general point $x \in X$ is smooth. Then any two additive actions on X are equivalent.*

Corollary 3.10. *Let $X \subseteq \mathbb{P}^N$ be a smooth projective subvariety of Picard number* 1 *such that, given a general point $x \in X$, there exists a line in $\mathbb{P}^N$ that passes through x and lies on X. If X is different from the projective space, then any two additive actions on X are equivalent.*

One can show that when X has a projective embedding satisfying the assumption of Corollary 3.10, some family of lines lying on X gives a family of minimal rational curves, for which the variety of minimal rational tangents C_x at a general point $x \in X$ is smooth (see, for example, [68], Proposition 1.5). Thus Corollary 3.10 follows from Theorem 3.9. This corollary can be applied to smooth quadratic hypersurfaces and Grassmanians because a smooth hyperquadric can be embedded into a projective space so as to have the required property.

Devyatov's approach is completely different. It is based on the representation theory of Lie algebras and the classification of certain multiplications on finite-dimensional spaces and can be regarded as a generalization of the Hassett–Tschinkel correspondence.

Let L be a connected reductive algebraic group, V be a finite-dimensional L-module, and $\mathfrak{l}$ be the Lie algebra of L. By an $\mathfrak{l}$-*compatible multiplication* on V we mean an associative commutative bilinear map $\mu\colon V \times V \to V$ such that for each $v \in V$ the operator $\mu_v\colon V \to V$, $\mu_v(w) = \mu(v, w)$ is nilpotent and for each $v \in V$ there exists $x \in \mathfrak{l}$ such that the operator μ_v coincides with the action of x on V.

A classification of $\mathfrak{l}$-compatible multiplications on an arbitrary module V of a reductive group L was presented in [41], §§ 5 and 6. It may be of independent interest. After a series of reductions to the case of a simple group G and a simple module V, it was proved in [41], Theorem 21, that there exists a non-zero $\mathfrak{l}$-compatible multiplication on V if and only if either L is a group of type A and V is the tautological L-module or its dual, or L is of type C and V is the tautological L-module.

Let $X = G/P$ be a flag variety, where G is a connected simple linear algebraic group and P is a parabolic subgroup of G. We may assume without loss of generality that the pair (G, P) is not exceptional. Then the connected component of the group $\mathrm{Aut}(X)$ coincides with G. By Theorem 3.6 we may assume that the group P_u is commutative and, in particular, the parabolic subgroup P is maximal.

As we know from the previous subsection, additive actions on X correspond to commutative Lie subalgebras $\mathfrak{h}$ that are complementary to $\mathfrak{p} = \mathrm{Lie}(P)$ in $\mathfrak{g} = \mathrm{Lie}(G)$, so that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$. One such subalgebra is $\mathfrak{p}_u^-$, and the uniqueness result means that all other such subalgebras $\mathfrak{h}$ are conjugate to $\mathfrak{p}_u^-$. In [41], Theorem 15, Devyatov established a correspondence between the commutative subalgebras $\mathfrak{h}$ in $\mathfrak{g}$ complementary to $\mathfrak{p}_u$ and the $\mathfrak{l}$-compatible multiplications $\mathfrak{p}_u^- \times \mathfrak{p}_u^- \to \mathfrak{p}_u^-$, where $\mathfrak{l}$ is a Levi subalgebra of the algebra $\mathfrak{p}$. Under this correspondence the subalgebra $\mathfrak{h} = \mathfrak{p}_u^-$ corresponds to zero multiplication. So the classification of compatible multiplications mentioned above shows that commutative subalgebras $\mathfrak{h}$ not conjugate to $\mathfrak{p}_u^-$ and complementary to $\mathfrak{p}_u$ appear only in the case when the flag variety G/P is isomorphic to the projective space $\mathbb{P}^n$. This proves Theorem 3.8.

A more detailed analysis shows that, in terms of the results of Hassett and Tschinkel, in the case of the tautological module V of the group of type A a compatible multiplication is precisely the multiplication $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ on the maximal ideal of the local algebra $A = \mathbb{K} \oplus \mathfrak{m}$ that corresponds to a given additive action on $\mathbb{P}^n$.

Finishing this subsection, let us mention one more related study. In [34] a uniqueness result for equivariant completions of non-commutative unipotent groups by flag varieties was proved. More precisely, let G be a simple linear algebraic group over the field of complex numbers and P be a parabolic subgroup of G. By the Bruhat decomposition, the unipotent radical P_u^- acts on G/P with open orbit isomorphic to P_u^-. Cheong [34] proved that the structure of an equivariant completion of P_u^- on G/P is unique up to isomorphism if G/P is not isomorphic to the projective space and the pair (G, P) is not exceptional in the sense of the preceding subsection. The proof exploits the notion of a smooth variety of minimal rational tangents and mostly follows the proof scheme of Fu and Hwang [51]. The main difference lies in which tool to use in order to obtain an extension of a locally defined map to the whole space, which is the most essential in proving uniqueness: Fu and Hwang used the Cartan–Fubini extension theorem, which is applicable only to smooth Fano varieties of Picard number 1, while Cheong used Yamaguchi's result on the prolongation of simple graded Lie algebras. For exceptional pairs (G, P) the question whether G/P admits a unique equivariant completion structure for P_u^- remains open.

3.4. Degeneration of flag varieties to equivariant completions. By Theorem 3.6 not so many flag varieties admit an additive action. At the same time, in [45] Feigin proposed a construction of a canonical flat degeneration of an arbitrary flag variety to a projective variety with an additive action. This result may be considered as an additive analogue of intensively studied flat degenerations of flag varieties to toric varieties (see [25], [58], and [79]).

Let G be a simple linear algebraic group with Lie algebra $\mathfrak{g}$. Recall that each flag variety G/P can be realized as a G-orbit $G[v_\lambda] \subseteq \mathbb{P}(V(\lambda))$ in the projectivization of a simple G-module $V(\lambda)$ with highest weight vector v_λ. We set $[v_\lambda] = \mathbb{K}v_\lambda$ and let $\mathcal{F}_\lambda := G[v_\lambda]$. Feigin introduced a new family of varieties $\mathcal{F}^a_\lambda$, which are flat degenerations of $\mathcal{F}_\lambda$; the superscript a is for 'abelian'.

The variety $\mathcal{F}^a_\lambda$ is defined as follows. Let $\{F_s,\, s \geqslant 0\}$ be the PBW filtration on $V(\lambda)$:

$$F_s = \operatorname{Span}\{x_1 \dots x_l v_\lambda \colon x_i \in \mathfrak{g},\ l \leqslant s\}.$$

Set $V(\lambda)^a = F_0 \oplus \bigoplus_{s\geqslant 0} F_{s+1}/F_s$. Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^-$ be the Cartan decomposition. The space $V(\lambda)^a$ has the natural structure of a module over the degenerate algebra $\mathfrak{g}^a$, where $\mathfrak{g}^a$ is isomorphic to $\mathfrak{g}$ as a vector space and is a semidirect sum of two subalgebras, the first being the Borel subalgebra $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{t}$ and the second being an abelian ideal $(\mathfrak{n}^-)^a$ that is isomorphic to $\mathfrak{n}$ as a vector space. Here the structure of the $\mathfrak{b}$-module on $(\mathfrak{n}^-)^a$ is given via the identification of $(\mathfrak{n}^-)^a$ with the factor module $\mathfrak{g}/\mathfrak{b}$. The corresponding algebraic group G^a is a semidirect product $B \leftthreetimes \mathbb{G}^n_a$, where n is the dimension of $\mathfrak{n}$. We define the variety $\mathcal{F}^a_\lambda$ as the closure of the $\mathbb{G}^n_a$-orbit of the highest weight vector:

$$\mathcal{F}^a_\lambda := \overline{\mathbb{G}^n_a[v_\lambda]} \subseteq \mathbb{P}(V(\lambda)^a).$$

By definition this variety carries an additive action. Since the highest weight vector v_λ is B-semi-invariant, $\mathcal{F}^a_\lambda$ is invariant under the action of G^a as well. Despite the case of usual flag varieties, here the action of G^a on $\mathcal{F}^a_\lambda$ need not be transitive.

Let $\mathfrak{p}$ be the parabolic subalgebra annihilating the vector v_λ. Assume for a moment that the nilpotent radical $\mathfrak{p}^-_u$ is commutative. Then all root operators in $\mathfrak{n}^- \setminus \mathfrak{p}^-_u$ annihilate the vector v_λ, while operators in $\mathfrak{p}^-_u$ act as pairwise commuting operators on $V(\lambda)$ even before passing to $V(\lambda)^a$. This shows that in this case there is no difference between the original variety $\mathcal{F}_\lambda$ and the degenerate variety $\mathcal{F}^a_\lambda$. By Theorem 3.6 this is precisely the case when the variety $\mathcal{F}_\lambda$ itself admits an additive action.

In the case of a group of type A the varieties $\mathcal{F}_\lambda$ are isomorphic to partial flag varieties. In particular, the ones corresponding to fundamental weights λ are Grassmannians $\operatorname{Gr}(d,n)$. There exist embeddings of partial flags into products of projective spaces, and the image is given by Plücker's relations. These relations describe the coordinate rings on the affine cones over flag varieties.

It was shown in [45] that each degenerate flag variety can be embedded into a product of Grassmannians and thus into a product of projective spaces. Feigin showed that this embedding can be described in terms of an explicit set of multi-homogeneous algebraic equations that are obtained from the Plücker relations by a certain degeneration. He proved that the degeneration $\mathcal{F}_\lambda \to \mathcal{F}^a_\lambda$ is flat.

For further results on the varieties $\mathcal{F}^a_\lambda$, see, for instance, [45], [46], [27], and the references there. The paper [46] is devoted to the study of varieties $\mathcal{F}^a_\lambda$ for groups of type A. The authors prove that in this case the variety $\mathcal{F}^a_\lambda$ has rational singularities and is a normal and locally complete intersection. They construct a desingularization R_λ of $\mathcal{F}^a_\lambda$ explicitly. The variety R_λ can be viewed as a tower of successive $\mathbb{P}^1$-fibrations, thus providing an analogue of the classical Bott–Samelson–Demazure–Hansen desingularization. It was proved that R_λ is Frobenius split. This gives a Frobenius splitting for degenerate flag varieties and allows one to prove Borel–Weil type theorem for $\mathcal{F}^a_\lambda$.

The aim of [27] was to connect degenerate flag varieties with quiver Grassmannians. By definition, quiver Grassmannians are the varieties parametrizing subrepresentations of a quiver representation. It turns out that certain quiver Grassmannians for type A quivers are isomorphic to degenerate flag varieties $\mathcal{F}^a_\lambda$. This leads to considering the class of Grassmannians of subrepresentations of the direct sum of a projective and an injective representation of a Dynkin quiver. It was proved that these are (typically, singular) irreducible normal local complete intersections, which admit a group action with finitely many orbits and a cellular decomposition.

4. Additive actions on toric varieties

In this section we study additive actions on toric varieties. To do this we need to develop new techniques, namely, we consider graded algebras and homogeneous locally nilpotent derivations of such algebras. In order to apply these techniques, we would like to have global coordinates on every toric variety. Such coordinates are provided by Cox rings. In the case of a toric variety X the Cox ring $R(X)$ is a polynomial ring in m variables, where m is the number of prime torus-invariant divisors on X. Moreover, the ring $R(X)$ is graded by the divisor class group $\mathrm{Cl}(X)$, and locally nilpotent derivations, which correspond to $\mathbb{G}_a$-actions on X normalized by the acting torus, are represented by so-called Demazure roots of the corresponding fan.

This allows us to characterize additive actions on X normalized by the acting torus in terms of certain collections of Demazure roots. In particular, we obtain a combinatorial description of the fans of toric varieties admitting a normalized additive action. We show that there is at most one normalized additive action on any toric variety up to equivalence. Further, we prove that if a complete toric variety X admits an additive action, then it admits a normalized additive action. Moreover, this is the case if and only if a maximal unipotent subgroup U of $\mathrm{Aut}(X)$ acts on X with open orbit. A characterization of the polytopes corresponding to projective toric varieties admitting an additive action is obtained.

In § 4.5 we describe additive actions on complete toric surfaces X. It turns out that there are at most two additive actions on X in this case. The last subsection provides a uniqueness criterion for additive actions on a complete toric variety X of arbitrary dimension: if X admits a normalized additive action, then any other additive action on X is equivalent to a normalized one if and only if a maximal unipotent subgroup U in $\mathrm{Aut}(X)$ is commutative. The latter condition can be easily checked in terms of the fan.

4.1. Graded algebras and locally nilpotent derivations. In this subsection we follow the presentation in [11]. Consider an irreducible affine variety X with an effective action of an algebraic torus T. Let M be the character lattice of T and $N = \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice of one-parameter subgroups of T. Let $B = \mathbb{K}[X]$ be the algebra of regular functions on X. It is well known that there is a bijection between the faithful T-actions on X and the effective M-gradings on B. In fact, the algebra B is graded by the semigroup of lattice points in a convex polyhedral cone $\omega \subseteq M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. We have

$$B = \bigoplus_{m \in \omega_M} B_m \chi^m,$$

where $\omega_M = \omega \cap M$ and χ^m is the character of the torus T corresponding to a point $m \in M$.

A derivation ∂ of an algebra B is said to be *locally nilpotent* (an LND for short) if for every $f \in B$ there exists $k \in \mathbb{Z}_{>0}$ such that $\partial^k(f) = 0$. For any LND ∂ on B the map $\varphi_\partial \colon \mathbb{G}_a \times B \to B$, $\varphi_\partial(s, f) = \exp(s\partial)(f)$, defines the structure of a rational $\mathbb{G}_a$-algebra on B. This induces a regular action $\mathbb{G}_a \times X \to X$, where $X = \operatorname{Spec} B$. In fact, any regular $\mathbb{G}_a$-action on X arises in this way (see [49], § 1.5). A derivation ∂ on B is said to be *homogeneous* if it respects the M-grading, that is, ∂ sends homogeneous elements to homogeneous ones. If $f, h \in B \setminus \ker \partial$ are homogeneous, then $\partial(fh) = f\partial(h) + \partial(f)h$ is homogeneous as well and $\deg \partial(f) - \deg f = \deg \partial(h) - \deg h$. So any homogeneous derivation ∂ has a well-defined *degree* given by $\deg \partial = \deg \partial(f) - \deg f$ for any homogeneous $f \in B \setminus \ker \partial$. It is easy to see that an LND on B is homogeneous if and only if the corresponding $\mathbb{G}_a$-action is normalized by the torus T in the automorphism group $\mathrm{Aut}(X)$ (cf. [49], § 3.7).

Let X be an affine toric variety, that is, a normal affine variety with effective action of a torus T with open orbit. In this case

$$B = \bigoplus_{m \in \omega_M} \mathbb{K}\chi^m = \mathbb{K}[\omega_M]$$

is the semigroup algebra. Recall that, given a cone $\omega \subseteq M_{\mathbb{Q}}$, its *dual cone* $\sigma \subseteq N_{\mathbb{Q}}$ is defined by

$$\sigma = \{p \in N_{\mathbb{Q}} \colon \langle p, v \rangle \geqslant 0 \ \forall v \in \omega\},$$

where $\langle \cdot\,, \cdot \rangle$ is the pairing $N_{\mathbb{Q}} \times M_{\mathbb{Q}} \to \mathbb{Q}$ between the dual spaces $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$. Let $\sigma(1)$ be the set of rays of the cone σ and p_ρ be the primitive lattice vector on the ray ρ. For $\rho \in \sigma(1)$ we set

$$\mathfrak{R}_\rho := \{e \in M \colon \langle p_\rho, e \rangle = -1 \quad \text{and} \quad \langle p_{\rho'}, e \rangle \geqslant 0 \ \forall \rho' \in \sigma(1),\ \rho' \neq \rho\}.$$

One checks easily that the set $\mathfrak{R}_\rho$ is infinite for each $\rho \in \sigma(1)$, provided that the cone σ has dimension at least 2. Elements of the set $\mathfrak{R} := \bigsqcup_{\rho \in \sigma(1)} \mathfrak{R}_\rho$ are called *Demazure roots* of the cone σ. Let $e \in \mathfrak{R}_\rho$. Then ρ is the *distinguished ray* of the root e. One can define a homogeneous LND on the algebra B by

$$\partial_e(\chi^m) = \langle p_\rho, m \rangle \chi^{m+e}.$$

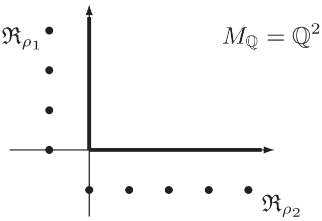

Figure 1

In fact, every homogeneous LND on B has the form $\alpha\partial_e$ for some $\alpha \in \mathbb{K}$ and $e \in \mathfrak{R}$ (see [80], Theorem 2.7). In other words, the $\mathbb{G}_a$-actions on X normalized by the acting torus are in bijection with the Demazure roots of the cone σ.

Example 4.1. Consider $X = \mathbb{K}^n$ with the standard action of the torus $(\mathbb{K}^\times)^n$. It is a toric variety with cone $\sigma = \mathbb{Q}^n_{\geqslant 0}$, which has the rays $\rho_i = \langle p_i \rangle_{\mathbb{Q}_{\geqslant 0}}$, where

$$p_1 = (1, 0, \dots, 0), \quad \dots, \quad p_n = (0, \dots, 0, 1).$$

The dual cone ω is $\mathbb{Q}^n_{\geqslant 0}$ as well. In this case we have

$$\mathfrak{R}_{\rho_i} = \{(c_1, \dots, c_{i-1}, -1, c_{i+1}, \dots, c_n) \colon c_j \in \mathbb{Z}_{\geqslant 0}\},$$

where $c_j = \langle p_j, e \rangle$ (see Fig. 1). Set $x_1 = \chi^{(1,0,\dots,0)}, \dots, x_n = \chi^{(0,\dots,0,1)}$. Then $\mathbb{K}[X] = \mathbb{K}[x_1, \dots, x_n]$. Consider the monomial

$$x^e := x_1^{c_1} \cdots x_{i-1}^{c_{i-1}} x_{i+1}^{c_{i+1}} \cdots x_n^{c_n}.$$

It is easy to see that the homogeneous LND corresponding to the root $e = (c_1, \dots, c_n) \in \mathfrak{R}_{\rho_i}$ is

$$\partial_e = x^e \, \frac{\partial}{\partial x_i}\,.$$

This LND gives rise to the $\mathbb{G}_a$-action

$$x_i \mapsto x_i + s x^e, \qquad x_j \mapsto x_j \quad \text{for } j \neq i, \qquad s \in \mathbb{G}_a.$$

4.2. Cox rings and Demazure roots. We keep the notation of the previous subsection and continue to follow [11]. Let X be a toric variety of dimension n with an acting torus T. This time we do not assume that X is affine, and so X is represented by a fan Σ of convex polyhedral cones in $N_{\mathbb{Q}}$ (see [55] or [37] for details).

Let $\Sigma(1)$ be the set of rays of the fan Σ and p_ρ be the primitive lattice vector on the ray ρ. For $\rho \in \Sigma(1)$ we consider the set $\mathfrak{R}_\rho$ of all vectors $e \in M$ such that

(a) $\langle p_\rho, e \rangle = -1$ and $\langle p_{\rho'}, e \rangle \geqslant 0$ for all $\rho' \in \Sigma(1)$ and $\rho' \neq \rho$;

(b) if σ is a cone in Σ and $\langle v, e \rangle = 0$ for all $v \in \sigma$, then the cone generated by σ and ρ is in Σ as well.

Note that (a) implies (b) if Σ is a fan with convex support. This is the case when X is affine or complete.

Elements of $\mathfrak{R} := \bigsqcup_{\rho\in\Sigma(1)} \mathfrak{R}_\rho$ are called *Demazure roots* of the fan Σ (cf. [38], Définition 4, and [91], § 3.4). Again, the elements $e \in \mathfrak{R}$ are in bijection with the $\mathbb{G}_a$-actions on X normalized by the acting torus (see [38], Théorème 3, and [91], Proposition 3.14). If X is affine, then the $\mathbb{G}_a$-action given by a Demazure root e coincides with the action corresponding to the locally nilpotent derivation ∂_e of the algebra $\mathbb{K}[X]$, which was defined in the previous subsection. Let H_e denote the image in $\mathrm{Aut}(X)$ of the group $\mathbb{G}_a$ under this action. Thus, H_e is a one-parameter unipotent subgroup normalized by T in $\mathrm{Aut}(X)$.

We recall some basic facts from toric geometry. There is a bijection between the cones $\sigma \in \Sigma$ and the T-orbits $\mathcal{O}_\sigma$ on X such that $\sigma_1 \subseteq \sigma_2$ if and only if $\mathcal{O}_{\sigma_2} \subseteq \overline{\mathcal{O}_{\sigma_1}}$. Here $\dim \mathcal{O}_\sigma = n - \dim\langle\sigma\rangle$. Moreover, each cone $\sigma \in \Sigma$ defines an open affine T-invariant subset U_σ on X such that $\mathcal{O}_\sigma$ is the unique closed T-orbit on U_σ, and $\sigma_1 \subseteq \sigma_2$ if and only if $U_{\sigma_1} \subseteq U_{\sigma_2}$.

Let ρ_e be the distinguished ray corresponding to a root e, p_e be the primitive lattice vector on ρ_e, and R_e be the one-parameter subgroup of T corresponding to p_e. Denote the set of H_e-fixed points on X by X^{H_e}.

We aim to describe the action of H_e on X.

Proposition 4.2 (see [11], Proposition 1). *For every point $x \in X \setminus X^{H_e}$ the orbit $H_e x$ meets exactly two T-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ on X, where* $\dim \mathcal{O}_1 = \dim \mathcal{O}_2 + 1$. *The intersection $\mathcal{O}_2 \cap H_e x$ consists of a single point, while*

$$\mathcal{O}_1 \cap H_e x = R_e y \quad \textit{for any } y \in \mathcal{O}_1 \cap H_e x.$$

Proof. It follows from the proof of Proposition 3.14 in [91] that the affine charts U_σ, where $\sigma \in \Sigma$ is a cone containing ρ_e, are H_e-invariant, and the complement of their union is contained in X^{H_e}. This reduces the proof to the case when X is affine. In this case the assertion was proved in [13], Proposition 2.1. □

A pair of T-orbits $(\mathcal{O}_1, \mathcal{O}_2)$ on X is said to be *H_e-connected* if $H_e x \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ for some $x \in X \setminus X^{H_e}$. By Proposition 4.2, $\mathcal{O}_2 \subseteq \overline{\mathcal{O}_1}$, for such a pair (up to permutation) and $\dim \mathcal{O}_1 = \dim \mathcal{O}_2 + 1$. Since the torus normalizes the subgroup H_e, any point in $\mathcal{O}_1 \cup \mathcal{O}_2$ can actually serve as x.

Lemma 4.3 (see [11], Lemma 1). *A pair of T-orbits $(\mathcal{O}_{\sigma_1}, \mathcal{O}_{\sigma_2})$ is H_e-connected if and only if $e\big|_{\sigma_2} \leqslant 0$ and σ_1 is a facet of σ_2 given by the equation $\langle v, e\rangle = 0$.*

The proof reduces again to the affine case, when the assertion in question is Lemma 2.2 in [13].

Now we recall the basic ingredients of Cox's construction; see [6], Chap. 1 for more details. Let X be a normal variety with free finitely generated divisor class group $\mathrm{Cl}(X)$ and only constant invertible regular functions. Denote by $\mathrm{WDiv}(X)$ the group of Weil divisors on X and fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ that maps onto $\mathrm{Cl}(X)$ isomorphically. The *Cox ring* of the variety X is defined by

$$R(X) = \bigoplus_{D\in K} H^0(X, D),$$

where $H^0(X, D) = \{f \in \mathbb{K}(X)^\times :\ \mathrm{div}(f) + D \geqslant 0\} \cup \{0\}$ and multiplication on homogeneous components coincides with multiplication in the field of rational functions

$\mathbb{K}(X)$ and extends to $R(X)$ by linearity. It is easy to see that, up to isomorphism the graded ring $R(X)$ does not depend on the choice of the subgroup K.

It was proved in [36] that if X is toric, then $R(X)$ is the polynomial algebra $\mathbb{K}[x_1, \ldots, x_m]$, where the variables x_i correspond to T-invariant prime divisors D_i on X or, equivalently, to rays ρ_i of the fan Σ_X. The $\mathrm{Cl}(X)$-grading on $R(X)$ is given by $\deg(x_i) = [D_i]$.

Suppose that the Cox ring $R(X)$ is finitely generated. Then $\overline{X} := \operatorname{Spec} R(X)$ is called the *total coordinate space* of the variety X. It is an affine factorial variety with an action of the torus $H_X := \operatorname{Spec} \mathbb{K}[\mathrm{Cl}(X)]$. There is an open H_X-invariant subset $\widehat{X} \subseteq \overline{X}$ such that the complement $\overline{X} \setminus \widehat{X}$ is of codimension at least 2 in $\overline{X}$, there exists a good quotient $p_X \colon \widehat{X} \to \widehat{X}/\!/H_X$, and the quotient space $\widehat{X}/\!/H_X$ is isomorphic to X (see [6], Construction 1.6.3.1). If X is smooth, then the quotient map $p_X \colon \widehat{X} \to \widehat{X}/\!/H_X$ is called the *universal torsor* over X. So we have the diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, "i"] \arrow[d, "/\!/H_X"] & \overline{X} = \operatorname{Spec} R(X) \\ X & \end{tikzcd}$$

If X is toric, then $\overline{X}$ is isomorphic to $\mathbb{K}^m$ and $\overline{X} \setminus \widehat{X}$ is a union of some coordinate planes in $\mathbb{K}^m$ of codimension at least 2 (see [36]).

By Theorem 4.2.3.2 in [6] any $\mathbb{G}_a$-action on X can be lifted to a $\mathbb{G}_a$-action on $\overline{X}$ commuting with the action of the torus H_X.

If X is toric and a $\mathbb{G}_a$-action is normalized by the acting torus T, then the lifted $\mathbb{G}_a$-action on $\mathbb{K}^m$ is normalized by the diagonal torus $(\mathbb{K}^\times)^m$. Conversely, any $\mathbb{G}_a$-action on $\mathbb{K}^m$ normalized by the torus $(\mathbb{K}^\times)^m$ and commuting with the subtorus H_X induces a $\mathbb{G}_a$-action on X. This shows that the $\mathbb{G}_a$-actions on X normalized by T are in bijection with the locally nilpotent derivations of the Cox ring $\mathbb{K}[x_1, \ldots, x_m]$ that are homogeneous with respect to the standard grading by the lattice $\mathbb{Z}^m$ and have degree zero with respect to the $\mathrm{Cl}(X)$-grading.

4.3. Normalized additive actions. Let X be a normal variety admitting an additive action with open orbit $\mathcal{U}$. By Proposition 3.1 any invertible regular function on X is a constant and the divisor class group $\mathrm{Cl}(X)$ is freely generated by the classes $[D_1], \ldots, [D_l]$ of prime divisors such that $X \setminus \mathcal{U} = D_1 \cup \cdots \cup D_l$. In particular, the Cox ring $R(X)$ is well defined for such a variety X.

Now we assume that X is toric and an additive action $\mathbb{G}_a^n \times X \to X$ is normalized by the acting torus T. Then the group $\mathbb{G}_a^n$ is the direct product of n subgroups $\mathbb{G}_a$, normalized by T each. They correspond to pairwise commuting homogeneous locally nilpotent derivations on the Cox ring $\mathbb{K}[x_1, \ldots, x_m]$ which have degree zero with respect to the $\mathrm{Cl}(X)$-grading. In turn, such derivations up to scalars are in bijection with the Demazure roots of the fan Σ_X.

Consider the set of homogeneous derivations ∂_e of the polynomial algebra $\mathbb{K}[x_1, \ldots, x_m]$ which correspond to some Demazure roots e of the fan Σ_X.

Lemma 4.4 ([11], Lemma 2). *Two derivations ∂_e and $\partial_{e'}$ commute if and only if either $\rho_e = \rho_{e'}$ or $\langle p_e, e' \rangle = \langle p_{e'}, e \rangle = 0$.*

Now we arrive at the key definition.

Definition 4.5. A set $e_1, \dots, e_n$ of Demazure roots of a fan Σ of dimension n is called a *complete collection* if $\langle p_i, e_j \rangle = -\delta_{ij}$ for all $1 \leqslant i, j \leqslant n$, where δ_{ij} is the Kronecker delta.

In this case the vectors $p_1, \dots, p_n$ form a basis of the lattice N, and $-e_1, \dots, -e_n$ is the dual basis of the dual lattice M.

The following result can be considered as a combinatorial description of normalized additive actions on toric varieties.

Theorem 4.6 ([11], Theorem 1). *Let X be a toric variety. Then the additive actions on X normalized by the acting torus T are in bijection with the complete collections of Demazure roots of the fan Σ_X.*

As we have seen, a normalized additive action on X determines n pairwise commuting homogeneous locally nilpotent derivations of the Cox ring $\mathbb{K}[x_1, \dots, x_m]$. They have the form ∂_e for some Demazure roots e. So Theorem 4.6 follows directly from the next lemma.

Lemma 4.7 (see [11], Lemma 3). *The homogeneous locally nilpotent derivations $\partial_1, \dots, \partial_n$ of the Cox ring $\mathbb{K}[x_1, \dots, x_m]$ corresponding to the Demazure roots $e_1, \dots, e_n$ define a normalized additive action on X if and only if $e_1, \dots, e_n$ form a complete collection.*

Proof. First assume that the derivations $\partial_1, \dots, \partial_n$ give rise to an additive action $\mathbb{G}_a^n \times X \to X$. If some roots e_i and e_j with $i \neq j$ correspond to the same ray of the fan Σ_X, then the $\mathbb{G}_a^n$-action changes at most $n-1$ coordinates of the ring $\mathbb{K}[x_1, \dots, x_m]$, and no $\mathbb{G}_a^n$-orbit on X can be n-dimensional. Then Lemma 4.4 implies that $\langle p_i, e_j \rangle = 0$ for $i \neq j$. By definition we have $\langle p_i, e_i \rangle = -1$, and thus $e_1, \dots, e_n$ form a complete collection.

Conversely, if $e_1, \dots, e_n$ is a complete collection, then the corresponding homogeneous locally nilpotent derivations commute. They define a $\mathbb{G}_a^n$-action on $\mathbb{K}[x_1, \dots, x_m]$ and therefore on $\mathbb{K}^m$. This action descends to X. We have to show that the $\mathbb{G}_a^n$-action on X has an open orbit. For this purpose it suffices to check that the group $\mathbb{G}_a^n \times H_X$ acts on $\mathbb{K}^m$ with open orbit.

By construction, the group $\mathbb{G}_a^n$ changes exactly n of the coordinates $x_1, \dots, x_m$, while the weights of the remaining $m-n$ coordinates with respect to the $\mathrm{Cl}(X)$-grading form a basis of the lattice of characters of the torus H_X. This shows that the stabilizer of the point $(1, \dots, 1) \in \mathbb{K}^m$ in the group $\mathbb{G}_a^n \times H_X$ is trivial. Since $\dim(\mathbb{G}_a^n \times H_X) = n+m-n = m$, we conclude that the $(\mathbb{G}_a^n \times H_X)$-orbit of $(1, \dots, 1)$ is open in $\mathbb{K}^m$. □

Corollary 4.8. *A toric variety X admits a normalized additive action if and only if there is a complete collection of Demazure roots of the fan Σ_X.*

Remark 4.9. Another application of Demazure roots to the theory of equivariant completions of commutative linear algebraic groups was presented in [8].

The following theorem shows that a normalized additive action on a toric variety is essentially unique.

Theorem 4.10 ([11], Theorem 2). *Any two normalized additive actions on a toric variety are equivalent.*

Let X be a complete toric variety with acting torus T. It is well known that the automorphism group $\operatorname{Aut}(X)$ is a linear algebraic group with T as a maximal torus (see [38] and [36]). In particular, $\operatorname{Aut}(X)$ contains a maximal unipotent subgroup U, and all such subgroups are conjugate in $\operatorname{Aut}(X)$.

Theorem 4.11 ([11], Theorem 3). *Let X be a complete toric variety with acting torus T. Then the following conditions are equivalent*:

1) *there exists an additive action on X normalized by the acting torus T*;
2) *there exists an additive action on X*;
3) *a maximal unipotent subgroup U of the automorphism group* $\operatorname{Aut}(X)$ *acts on X with an open orbit.*

We conclude that Corollary 4.8 characterizes complete toric varieties admitting some additive action.

Corollary 4.12. *A complete toric variety X admits an additive action if and only if there is a complete collection of Demazure roots of the fan Σ_X.*

In terms of the fan Σ_X this condition means that, up to renumbering, the first n primitive vectors $p_1,\dots,p_n$ on the rays of Σ_X form a basis of the lattice N and the remaining primitive vectors $p_{n+1},\dots,p_m$ have non-positive coordinates in this basis.

4.4. Projective toric varieties and polytopes. It is well known that there is a correspondence between the convex lattice polytopes and the projective toric varieties. This subsection aims to characterize the polytopes corresponding to toric varieties that admit an additive action.

We begin with preliminary results; see [37], Chap. 2, and [55], § 1.5, for more details. Let M be a lattice of rank n and P be a full-dimensional convex polytope in the space $M_{\mathbb{Q}}$. We say that P is a *lattice polytope* if all of its vertices are in M.

A subsemigroup $S\subseteq M$ is called *saturated* if S coincides with the intersection of the group $\mathbb{Z}S$ and the cone $\mathbb{Q}_{\geqslant 0}S$ it generates. A lattice polytope P is *very ample* if for every vertex $v\in P$ the semigroup $S_{P,v}:=\mathbb{Z}_{\geqslant 0}(P\cap M-v)$ is saturated. It is known that for every lattice polytope P and every $k\geqslant n-1$ the polytope kP is very ample (see [37], Corollary 2.2.19).

Let us regard M as the lattice of characters of a torus T. Let $P\subseteq M_{\mathbb{Q}}$ be a very ample polytope and let $P\cap M=\{m_1,\dots,m_s\}$. We consider the map

$$T\to\mathbb{P}^{s-1},\quad t\mapsto[\chi^{m_1}(t):\dots:\chi^{m_s}(t)]$$

and define a variety X_P to be the closure of the image of this map in $\mathbb{P}^{s-1}$. It is known that X_P is a projective toric variety with acting torus T, and any projective toric variety appears in this way.

Definition 4.13. We say that a very ample polytope P is *inscribed in a rectangle* (see Fig. 2) if there is a vertex $v_0\in P$ such that

1) the primitive vectors on the edges of P containing v_0 form a basis $e_1,\dots,e_n$ of the lattice M;

2) for every inequality $\langle p, x\rangle \leqslant a$ on P that corresponds to a facet of P not passing through v_0 we have $\langle p, e_i\rangle \geqslant 0$ for all $i = 1, \ldots, n$.

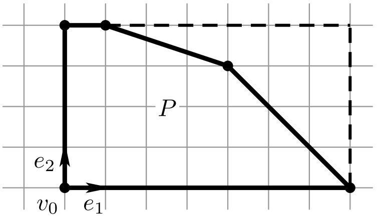

Figure 2

Theorem 4.14 (see [11], Theorem 4). *Let P be a very ample polytope and X_P be the corresponding projective toric variety. Then X_P admits an additive action if and only if P is inscribed in a rectangle.*

Proof. By Corollary 4.12 a toric variety X admits an additive action if and only if the fan Σ_X admits a complete collection of Demazure roots. By Proposition 3.1.6 in [37] the fan Σ_{X_P} of the toric variety X_P corresponding to the polytope P coincides with the normal fan Σ_P of the polytope P. It is straightforward to check that the two conditions in Definition 4.13 are precisely the conditions for $-e_1, \ldots, -e_n$ to form the a complete collection of Demazure roots of the fan Σ_P.

Remark 4.15. The result of Theorem 4.14 can also be obtained using the language of polytopal linear groups developed in [23].

Let us illustrate this approach with two examples.

The closed interval $P = [0, d]$ in $\mathbb{Q}^1$ with $d \in \mathbb{Z}_{\geqslant 1}$ is a polytope inscribed in a rectangle, and the variety

$$X_P = \overline{\{[1 : a : \cdots : a^d] \mid a \in \mathbb{K}\}} \subseteq \mathbb{P}^d$$

is a rational normal curve of degree d.

Further, the polytope shown in Fig. 3 defines the surface

$$X_P = \overline{\{[1 : a : a^2 : b : ab : a^2b : b^2 : ab^2 : b^3] \mid a, b \in \mathbb{K}\}} \subseteq \mathbb{P}^8,$$

which is isomorphic to the Hirzebruch surface $\mathbb{F}_1$.

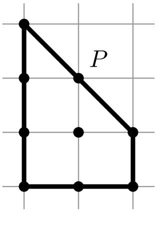

Figure 3

Now we give some explicit formulae for additive actions on toric varieties in terms of Cox rings.

Example 4.16. The fan of the projective space $X = \mathbb{P}^n$ is generated by a basis $p_1, \dots, p_n$ of the lattice $\mathbb{Z}^n$ and the vector $p_0 = -p_1 - \dots - p_n$. The complete collection of Demazure roots e_i, $1 \leqslant i \leqslant n$, which consists of the vectors opposite to the dual basis of $p_1, \dots, p_n$, corresponds to the pairwise commuting locally nilpotent derivations $\partial_{e_i} = x_0\, \partial/\partial x_i$, $1 \leqslant i \leqslant n$, on the Cox ring $R(X) = \mathbb{K}[x_0, \dots, x_n]$. They define the $\mathbb{G}_a^n$-action

$$(x_0, x_1, \dots, x_n) \mapsto (x_0, x_1 + s_1 x_0, \dots, x_n + s_n x_0)$$

on the total coordinate space $\overline{X} = \mathbb{A}^{n+1}$ (see Example 4.1). This action is normalized by the diagonal torus $(\mathbb{K}^\times)^{n+1}$ and commutes with the action of the one-dimensional torus $H_X = \mathbb{G}_m$ since locally nilpotent derivations are homogeneous with respect to the standard grading by $\mathbb{Z}^{n+1}$ and have degree zero with respect to the grading by $\mathrm{Cl}(X) = \mathbb{Z}$, when $\deg x_i = 1$, $0 \leqslant i \leqslant n$. Thus, this action induces the normalized additive action on the projective space $\mathbb{P}^n$:

$$[z_0 : z_1 : \dots : z_n] \mapsto [z_0 : z_1 + s_1 z_0 : \dots : z_n + s_n z_0]. \tag{4.1}$$

The hyperplane $\{z_0 = 0\}$ consists of $\mathbb{G}_a^n$-fixed points and thus for $n \geqslant 2$ the number of $\mathbb{G}_a^n$-orbits on $\mathbb{P}^n$ is infinite.

Consider the case $n = 2$. A maximal unipotent subgroup of the automorphism group $\mathrm{Aut}(\mathbb{P}^2)$ is isomorphic to the unitriangular matrix subgroup of $\mathrm{GL}_3(\mathbb{K})$ and consists of the automorphisms

$$[z_0 : z_1 : z_2] \mapsto [z_0 : z_1 + a_{12} z_0 : z_2 + a_{23} z_1 + a_{13} z_0], \qquad a_{12}, a_{23}, a_{13} \in \mathbb{K}.$$

Two-dimensional (commutative) subgroups of this group have the form

$$H_{[\alpha:\beta]} = \left\{ \begin{pmatrix} 1 & \alpha a & b \\ 0 & 1 & \beta a \\ 0 & 0 & 1 \end{pmatrix},\ a, b \in \mathbb{K} \right\}$$

for $[\alpha : \beta] \in \mathbb{P}^1$. For $[\alpha : \beta] = [0 : 1]$ the corresponding action of the group $\mathbb{G}_a^2$ has no open orbit, for $[\alpha : \beta] = [1 : 0]$ we obtain the normalized additive action (4.1), and the points $[\alpha : \beta] \in \mathbb{P}^1 \setminus \{0, \infty\}$ define pairwise isomorphic non-normalized additive actions with three orbits (see Example 1.50).

Example 4.17. In the same way as in Example 4.16 one can check that the normalized additive action on the product $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$ and the corresponding action on the total coordinate space $\mathbb{A}^{2n}$ are given by

$$([z_1 : z_2], \dots, [z_{2n-1} : z_{2n}]) \mapsto ([z_1 : z_2 + s_1 z_1], \dots, [z_{2n-1} : z_{2n} + s_n z_{2n-1}])$$

and

$$(x_1, x_2, \dots, x_{2n-1}, x_{2n}) \mapsto (x_1, x_2 + s_1 x_1, \dots, x_{2n-1}, x_{2n} + s_n x_{2n-1}).$$

This shows that the number of $\mathbb{G}_a^n$-orbits on $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$ is 2^n.

Example 4.18. Let X be the Hirzebruch surface $\mathbb{F}_d$. Its fan is generated by the vectors

$$p_1 = (1,0), \quad p_2 = (0,1), \quad p_3 = (-1,d), \quad p_4 = (0,-1).$$

The Cox ring $\mathbb{K}[x_1,x_2,x_3,x_4]$ carries a $\mathbb{Z}^2$-grading given by

$$\deg x_1 = (1,0), \quad \deg x_2 = (0,1), \quad \deg x_3 = (1,0), \quad \deg x_4 = (d,1).$$

Moreover, X is obtained as the geometric quotient of

$$\widehat{X} = \mathbb{K}^4 \setminus (\{x_1 = x_3 = 0\} \cup \{x_2 = x_4 = 0\})$$

by the action of the torus $H_X = (\mathbb{K}^\times)^2$ that is given by the $\mathbb{Z}^2$-grading.

In this case the Demazure roots are $(1,0)$, $(-1,0)$, and $(k,1)$, where $0 \leqslant k \leqslant d$, and the corresponding homogeneous locally nilpotent derivations are

$$x_1 \frac{\partial}{\partial x_3}, \quad x_3 \frac{\partial}{\partial x_1}, \quad \text{and} \quad x_1^k x_2 x_3^{d-k} \frac{\partial}{\partial x_4}.$$

There are two complete collections of Demazure roots, namely, $(1,0)$, $(d,1)$ and $(-1,0)$, $(0,1)$. They define two normalized additive actions on X, which are defined on $\widehat{X}$ by

$$(x_1,x_2,x_3,x_4) \mapsto (x_1,x_2,x_3+s_1x_1,x_4+s_2x_1^d x_2)$$

and

$$(x_1,x_2,x_3,x_4) \mapsto (x_1+s_1x_3,x_2,x_3,x_4+s_2x_2x_3^d) \tag{4.2}$$

and are interchanged by the automorphism $(x_1,x_2,x_3,x_4) \mapsto (x_3,x_2,x_1,x_4)$.

According to the results in [38] or [36], the automorphism group $\mathrm{Aut}(X)$ is isomorphic to $\mathbb{K}^\times \cdot \mathrm{PSL}(2) \leftthreetimes \mathbb{G}_a^{d+1}$. Consider the case $d = 1$. Then a maximal unipotent subgroup of $\mathrm{Aut}(X)$ is isomorphic to unitriangular matrix subgroup of $\mathrm{GL}_3(\mathbb{K})$ and acts in the following way:

$$(x_1,x_2,x_3,x_4) \mapsto (x_1+a_{12}x_3,x_2,x_3,x_4+a_{23}x_1x_2+a_{13}x_2x_3),$$

where $a_{12},a_{23},a_{13} \in \mathbb{K}$. Its two-dimensional (commutative) subgroups have the form $H_{[\alpha:\beta]}$ (see Example 4.16). For $[\alpha:\beta] = [0:1]$ we obtain an action of $\mathbb{G}_a^2$ on $\mathbb{F}_1$ with one-parameter family of one-dimensional orbits, for $[\alpha:\beta] = [1:0]$ we obtain normalized additive action (4.2), and points $[\alpha:\beta] \in \mathbb{P}^1 \setminus \{0,\infty\}$ define pairwise isomorphic non-normalized additive actions on $\mathbb{F}_1$. Thus, there are exactly two equivalence classes of additive actions on $\mathbb{F}_1$. This result was obtained in [62], Proposition 5.5, by means of geometric arguments.

Let us mention one more recent result on additive actions on toric varieties. In [99] Shafarevich classified toric projective hypersurfaces admitting an additive action. Every toric hypersurface of dimension n can be represented by a lattice polytope in the lattice M such that the number of lattice points inside this polytope is $n+2$. So the question is when such a polytope is inscribed in a rectangle. In [99], Proposition 1, all such polytopes were found. It turns out that, apart from the projective space, there are two toric projective hypersurfaces admitting an additive action in every dimension $n \geqslant 2$; they are the quadrics of rank 3 and 4 (see [99], Theorem 2).

Using the results of [36] Shafarevich computed the automorphism groups of these quadrics. Then he applied the correspondence between additive actions on projective hypersurfaces and pairs (A, U), where A is a local algebra and U is a hyperplane in the maximal ideal of A that generates A (see Theorem 2.6), and found in [99], § 5, the number of non-equivalent additive actions on quadrics of rank 3 and 4 in dimensions 2 to 4 (see Remark 2.28).

4.5. Additive actions on complete toric surfaces. We observe that by blowing up fixed points repeatedly one can obtain infinitely many different (smooth) complete toric surfaces that admit an additive action. In this subsection we discuss a result of Dzhunusov [43] which clarifies how many additive actions we can have on a complete toric surface.

Let X_Σ be a complete toric variety of dimension n admitting an additive action and Σ be the corresponding fan. We begin with some results on the structure of the set of Demazure roots of the cone Σ following [43], § 5.

Denote primitive vectors on the rays of the fan Σ by p_i, where $1 \leqslant i \leqslant m$. It follows from Theorem 4.6 that we can order the vectors p_i in such a way that the first n vectors form a basis of the lattice N and the remaining vectors p_j $(n < j \leqslant m)$ are equal to $\sum_{i=1}^{n} -\alpha_{ji} p_i$ for some non-negative integers α_{ji}.

Let us denote the dual basis of the basis $p_1, \dots, p_n$ by $p_1^*, \dots, p_n^*$ and let $\mathfrak{R}_i = \mathfrak{R}_{\rho_i}$.

Lemma 4.19 ([43], Lemma 2). *For $1 \leqslant i \leqslant n$ the set $\mathfrak{R}_i$ is a subset of the set $-p_i^* + \sum_{l=1, l\neq i}^{n} \mathbb{Z}_{\geqslant 0} p_j^*$, and the vector $-p_i^*$ belongs to $\mathfrak{R}_i$.*

Now we divide the set of Demazure roots $\mathfrak{R}$ into two classes:

$$\mathfrak{S} = \mathfrak{R} \cap -\mathfrak{R} \quad \text{and} \quad \mathfrak{U} = \mathfrak{R} \setminus \mathfrak{S}.$$

Roots in $\mathfrak{S}$ and $\mathfrak{U}$ are called *semisimple* and *unipotent*, respectively.

Consider the set

$$\operatorname{Reg}(\mathfrak{S}) = \{u \in N \colon \langle u, e\rangle \neq 0 \ \forall\, e \in \mathfrak{S}\}.$$

Any element u of $\operatorname{Reg}(\mathfrak{S})$ divides the set of semisimple roots $\mathfrak{S}$ into two classes as follows:

$$\mathfrak{S}_u^+ = \{e \in \mathfrak{S} \colon v = \langle u, e\rangle > 0\} \quad \text{and} \quad \mathfrak{S}_u^- = \{e \in \mathfrak{S} \colon v = \langle u, e\rangle < 0\}.$$

Any element of $\mathfrak{S}_u^+$ is said to be *positive* and any element of $\mathfrak{S}_u^-$ is said to be *negative*.

Lemma 4.20 ([43], Proposition 2). *Let X_Σ be a complete toric variety admitting an additive action. Then*

1) *any element in $\mathfrak{R}_j$, $j > n$, is equal to $p_{i_j}^*$ for some $1 \leqslant i_j \leqslant n$;*
2) *all unipotent Demazure roots lie in the set $\bigcup_{i=1}^{n} \mathfrak{R}_i$;*
3) *there exists a vector $u \in \operatorname{Reg}(\mathfrak{S})$ such that $\mathfrak{S}_u^+ \subseteq \bigcup_{i=1}^{n} \mathfrak{R}_i$.*

Now we consider a complete toric surface X_Σ with fan Σ that admits an additive action. We assume as before that p_1 and p_2 form a basis of the lattice N and the remaining vectors $p_3, \dots, p_m$ are non-positive integer combinations of p_1 and p_2. We say that a fan Σ is *wide* if there exist indices $2 < i_1,\ i_2 \leqslant m$ such that $\alpha_{i_1 1} > \alpha_{i_1 2}$ and $\alpha_{i_2 1} < \alpha_{i_2 2}$. One can check that this condition means that $\mathfrak{R}_1 = \{-p_1^*\}$ and $\mathfrak{R}_2 = \{-p_2^*\}$.

Now we are ready to formulate the main result.

Theorem 4.21 (see [43], Theorem 3). *Let X_Σ be a complete toric surface admitting an additive action. Then there is only one additive action on X_Σ if and only if the fan Σ is wide*; *otherwise there exist exactly two non-equivalent additive actions*, *one of which is normalized and the other is not.*

Lemmas 4.19 and 4.20 show that for a wide fan Σ a maximal unipotent subgroup of the linear algebraic group $\mathrm{Aut}(X_\Sigma)$ has dimension 2, and so it is the only candidate for a commutative unipotent group acting on X_Σ with open orbit. To treat the case of a non-wide fan, Dzhunusov classified pairs of commuting homogeneous LNDs on the Cox ring of the variety X_Σ and showed that precisely one equivalence class of such pairs corresponds to non-normalized additive actions on X_Σ.

4.6. Uniqueness criterion. This subsection contains a criterion of uniqueness for an additive action on a complete toric variety of arbitrary dimension proved by Dzhunusov [42]. This result is also based on Lemmas 4.19 and 4.20. We keep the notation of the previous subsection.

Theorem 4.22 (see [42], Theorem 4). *Let X_Σ be a complete toric variety admitting an additive action. Then any additive action on X is equivalent to the normalized additive action if and only if for every $1 \leqslant i \leqslant n$ the set $\mathfrak{R}_i$ is equal to $\{-p_i^*\}$.*

In the proof Dzhunusov showed that the second claim of the theorem means that a maximal unipotent subgroup of the linear algebraic group $\mathrm{Aut}(X_\Sigma)$ is a commutative group of dimension n, and it is again the only candidate for a commutative unipotent group acting on X_Σ with open orbit. If this condition does not hold, then an n-tuple of pairwise commuting homogeneous LNDs of the Cox ring of the variety X_Σ was constructed in [42], § 6, and it was proved that this n-tuple defines an additive action on X_Σ that is not equivalent to the normalized one. More precisely, if the normalized additive action corresponds to the tuple of homogeneous LNDs

$$(\partial_{-p_1^*}, \partial_{-p_2^*}, \partial_{-p_3^*}, \dots, \partial_{-p_n^*}),$$

then the second tuple is given by

$$(\partial_{-p_1^*}, \partial_{-p_2^*} + \partial_{-p_1^*+dp_2^*}, \partial_{-p_3^*}, \dots, \partial_{-p_n^*})$$

for some positive integer d.

Corollary 4.23. *Let X_Σ be a complete toric variety admitting an additive action. If the rank of the divisor class group $\mathrm{Cl}(X_\Sigma)$ is 1, then there are at least two non-equivalent additive actions on X_Σ.*

Corollary 4.23 covers the case of weighted projective spaces. By Proposition 2 in [11] the weighted projective space $\mathbb{P}(a_0, a_1, \dots, a_n)$, $a_0 \leqslant a_1 \leqslant \dots \leqslant a_n$, admits an additive action if and only if $a_0 = 1$. So there are at least two non-equivalent additive actions on the weighted projective space $\mathbb{P}(1, a_1, \dots, a_n)$.

An explicit description of additive actions on weighted projective planes was given in [5], Proposition 7. It turns out that, as in the case of the projective plane $\mathbb{P}^2$, every weighted projective plane $\mathbb{P}(1, a_1, a_2)$ admits precisely two non-equivalent additive actions.

5. Further results and questions on equivariant completions

The aim of § 5.1 is to collect recent geometric and classificational results on additive actions on Fano varieties. We begin with the surface case and list all singular and generalized del Pezzo surfaces admitting an additive action. In dimension three we recall a classification, due to Hassett and Tschinkel, of smooth projective 3-folds with irreducible boundary divisor that admit an additive action. The next step is a classification of smooth Fano 3-folds of Picard number at least 2 that admit an additive action. There are 17 varieties satisfying all these conditions. In dimensions starting from four the corresponding classifications are possible only under a restriction on the index of a Fano variety. These results are due to Fu and Montero.

In § 5.2 we present a short discussion of so-called Euler-symmetric varieties. Such a variety is defined by the condition that for a generic point P there is a one-dimensional torus $\mathbb{G}_m$ in the automorphism group such that P is an isolated fixed point for $\mathbb{G}_m$ and $\mathbb{G}_m$ acts by scalar multiplication on the tangent space at P. It is known that any Euler-symmetric variety admits an additive action, and there is a conjecture that this is a way to describe smooth Fano varieties admitting an additive action.

In § 5.3 we formulate several open problems and conjectures on equivariant completions of affine spaces. They concern all subjects discussed in this paper.

5.1. Classification results on additive actions on Fano varieties. We begin this subsection with the case of surfaces. In [39] a classification of del Pezzo surfaces that are equivariant compactifications of the group $\mathbb{G}_a^2$ was presented. Recall that del Pezzo surfaces are defined as smooth projective surfaces X whose anticanonical class $-K_X$ is ample. A singular del Pezzo surface is a normal singular projective surface with only ADE-singularities whose anticanonical class is ample. A generalized del Pezzo surface is either a smooth del Pezzo surface or a minimal desingularization of a singular del Pezzo surface. The main result of [39] claims that if S is a (possibly singular or generalized) del Pezzo surface of degree d defined over a field K of characteristic zero, then S admits an additive action precisely in the following cases:

(i) S has a non-singular rational K-point and is of the form of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface $\mathbb{F}_2$, or the corresponding singular del Pezzo surface;

(ii) S is of the form of $\mathrm{Bl}_1(\mathbb{P}^2)$ or $\mathrm{Bl}_2(\mathbb{P}^2)$;

(iii) $d = 7$ and S is of type A_1;

(iv) $d = 6$ and S is of type A_1 (with three lines), $2A_1$, A_2, or $A_2 + A_1$;

(v) $d = 5$ and S is of type A_3 or A_4;

(vi) $d = 4$ and S is of type D_5.

More generally, in [40] the authors determined all (possibly singular) del Pezzo surfaces that are equivariant compactifications of homogeneous spaces of two-dimensional linear algebraic groups. It is well known that, apart from the torus $\mathbb{G}_m^2$ and the vector group $\mathbb{G}_a^2$, the only connected two-dimensional linear algebraic groups are semidirect products $\mathbb{G}_m \leftthreetimes \mathbb{G}_a$. The classification result claims that a del Pezzo surface S of degree d, possibly singular with rational double points, is an equivariant compactification of some semi-direct product $\mathbb{G}_m \leftthreetimes \mathbb{G}_a$ if and only if it has one of the following types:

(i) $d \geqslant 7$: all types;

(ii) $d = 6$: types $A_2 + A_1$, A_2, $2A_1$, or A_1 (with three or four lines);

(iii) $d = 5$: types A_3, $A_2 + A_1$, or A_2;

(iv) $d = 4$: types $A_3 + 2A_1$, D_4, or $A_3 + A_1$.

Additionally, precisely the following types are equivariant compactifications of a homogeneous space for some semi-direct product $\mathbb{G}_m \leftthreetimes \mathbb{G}_a$:

(i) $d = 5$: type A_4;

(ii) $d = 4$: types D_5 or A_4;

(iii) $d = 3$: types E_6 or $A_5 + A_1$.

As we know already, the structure of a torus compactification on a toric variety is unique up to isomorphism, while even the projective plane $\mathbb{P}^2$ admits two different additive actions. It was proved in [40], Theorem 3.3, that if $\mathbb{G}_m \leftthreetimes \mathbb{G}_a$ is not the direct product $\mathbb{G}_m \times \mathbb{G}_a$ then up to equivalence $\mathbb{P}^2$ admits precisely two different structures of an equivariant compactification of $\mathbb{G}_m \leftthreetimes \mathbb{G}_a$. Moreover, it was shown there that $\mathbb{P}^2$ admits infinitely many different structures of an equivariant compactification of a homogeneous space for each $\mathbb{G}_m \leftthreetimes \mathbb{G}_a$.

A characterization of complete $\mathbb{G}_m$-surfaces admitting an additive action was obtained in [63], Proposition 13.17.

Now we go over from surfaces to dimension 3. In [62] a classification of smooth projective 3-folds of Picard number 1 admitting an additive action was given.

Theorem 5.1 ([62], Theorem 6.1). *Let X be a smooth projective 3-fold admitting an additive action with irreducible boundary divisor D. Then X is one of the following*:

(i) *$\mathbb{P}^3$ with D a hyperplane*;

(ii) *$Q_3 \subseteq \mathbb{P}^4$ a smooth quadric with D a tangent hyperplane section.*

Recall that for a Fano variety X of dimension n, its *index* i_X is the greatest integer such that $-K_X = i_X H$ for some divisor H on X. In the proof of Theorem 6.1 in [62] the authors observed that $-K_X = rD$, where $r \geqslant 2$. Therefore, X is a rational Fano variety of index $r \geqslant 2$. They considered the cases $r > 2$ and $r = 2$ separately and used Furushima's classification of non-equivariant compactifications of affine 3-space.

A classification of all smooth Fano 3-folds of Picard number at least 2 that admit additive action was presented in [65]. This classification includes 17 varieties. The authors considered the case of smooth toric Fano 3-folds first. They used the classification due to Batyrev and Watanabe–Watanabe and applied to it the criterion of the existence of an additive action on a toric variety from Theorem 4.6. In this way they obtained 13 smooth toric Fano 3-folds admitting an additive action. In the non-toric case they went through the classification due to Mori and Mukai and checked the existence of an additive action in each case. This analysis resulted in four smooth non-toric Fano 3-folds with an additive action.

In higher dimensions, a classification of smooth Fano varieties admitting an additive action is available only for varieties with high index. It is well known that the index i_X of a smooth Fano variety of dimension n does not exceed $n+1$. A classification of smooth Fano varieties of index $i_X \geqslant n-2$ was obtained by Fujita, Mella, Mukai, and Wisniewski. On the basis of this classification, a complete list of smooth Fano varieties of dimension n and index $i_X \geqslant n-2$ that admit an additive action was obtained in [54].

We begin with the case of Picard number 1.

Theorem 5.2 ([54], Theorem 1.1). *Let X be an n-dimensional smooth projective variety of Picard number* 1 *that admits an additive action. Assume that $i_X \geqslant n-2$. Then X is isomorphic to one of the following varieties*:

(1) *six homogeneous varieties of algebraic groups*: $\mathbb{P}^n$, Q_n, $\mathrm{Gr}(2,5)$, $\mathrm{Gr}(2,6)$, $\mathbb{S}_5$, *and* $\mathrm{Lag}(6)$;

(2) *five non-homogeneous varieties*:

   (2-a) *smooth linear sections of* $\mathrm{Gr}(2,5)$ *of codimension* 1 *or* 2;

   (2-b) $\mathbb{P}^4$*-general linear sections of* $\mathbb{S}_5$ *of codimension* 1, 2, *or* 3.

Here $\mathbb{S}_5$ *and* $\mathrm{Lag}(6)$ *are the* 10*-dimensional spinor variety and the* 6*-dimensional Lagrangian Grassmannian*, *respectively*.

A classification of smooth n-dimensional Fano varieties with $i_X \geqslant n-2$ of Picard number at least 2 that admit an additive action was obtained in [54], § 3. This result is based on Wisniewski's classifications of smooth Fano n-dimensional varieties of index $\geqslant (n+1)/2$ with Picard number at least 2 and of Mukai 4-folds with Picard number at least 2.

5.2. Euler-symmetric varieties. In this subsection we discuss a general construction of varieties with an additive action due to Fu and Hwang [53]. We work over the field of complex numbers.

Definition 5.3. Let $Z \subseteq \mathbb{P}(V)$ be a projective variety. For a non-singular point $x \in Z$ a $\mathbb{G}_m$-action on Z coming from a multiplicative subgroup of $\mathrm{GL}(V)$ is said to be of *Euler type* at x if x is an isolated fixed point of the restricted $\mathbb{G}_m$-action on Z and the induced $\mathbb{G}_m$-action on the tangent space $T_x(Z)$ is by scalar multiplication. A non-singular point $x \in Z$ is said to be *Euler* if there is a $\mathbb{G}_m$-action on Z which is of Euler type at x. We say that $Z \subseteq \mathbb{P}(V)$ is *Euler symmetric* if there is an open dense subset W in Z consisting of Euler points.

Remark 5.4. The condition on the action of $\mathbb{G}_m$ on the tangent space $T_x(Z)$ implies that the $\mathbb{G}_m$-fixed point x is isolated. Indeed, since the action of $\mathbb{G}_m$ on $\mathbb{P}(V)$ is diagonalizable, the point x is contained in a $\mathbb{G}_m$-invariant open affine chart X on Z. Since x is a non-singular point in X, by Theorem 6.4 in [95] it is also non-singular in the subvariety of $\mathbb{G}_m$-fixed points $X^{\mathbb{G}_m}$ and $T_x(X^{\mathbb{G}_m}) = T_x(X)^{\mathbb{G}_m}$. But the action of $\mathbb{G}_m$ on $T_x(X)$ is by scalar multiplication, so x is an isolated fixed point in X and therefore in Z.

It was proved in [53], Proposition 2.3, that for an Euler-symmetric projective variety $Z \subseteq \mathbb{P}(V)$ the group $\mathrm{Aut}_l(Z) \subseteq \mathrm{PGL}(V)$ of linear automorphisms preserving Z acts on Z with open orbit. In fact, Theorem 5.5 below provides a more concrete version of this result.

In [53], Theorem 3.7, the authors showed that Euler-symmetric varieties are classified in terms of certain algebraic data called symbol systems. Such a description makes this class of varieties accessible to investigation. Our interest in these varieties is explained by the following result; it opens the way for the systematic study of equivariant completions of affine space (see, for example, Conjecture 5.20 below).

Theorem 5.5 (see [53], Theorem 3.7, (i)). *Every Euler-symmetric variety admits an additive action.*

Let us sketch the proof of Theorem 5.5. Let $Z \subseteq \mathbb{P}(V)$ be an Euler-symmetric variety of dimension n and $x \in Z$ be an Euler point. Choose homogeneous coordinates in $\mathbb{P}(V) = \mathbb{P}^m$ in such a way that x has coordinates $[1 : 0 : \dots : 0]$. Since the $\mathbb{G}_m$-action of Euler type at x is linear on $\mathbb{P}^m$, we may assume that $\mathbb{G}_m$ acts diagonally in these coordinates.

Let $y_i = z_i/z_0$, $1 \leqslant i \leqslant m$, be coordinates on the affine chart $U_0 = \{z_0 \neq 0\}$ in $\mathbb{P}^m$. We may assume that the tangent space $T_x(Z)$ is given by the equations $y_{n+1} = \dots = y_m = 0$. It follows that the torus $\mathbb{G}_m$ acts on $y_1, \dots, y_n$ by scalar multiplication.

By the analytic implicit function theorem there exists a neighbourhood of x in which $y_1, \dots, y_n$ are coordinates on Z and Z is given by a system of equations

$$y_{n+i} = h_i(y_1, \dots, y_n), \qquad 1 \leqslant i \leqslant k = m - n, \tag{5.1}$$

with some holomorphic functions h_i. Each function h_i has a Taylor series in $y_1, \dots, y_n$ at x. Denote by $h_1^0, \dots, h_k^0$ the sums of the non-zero terms of lowest degrees $d_1, \dots, d_k$ in the Taylor series of $h_1, \dots, h_k$, respectively. Since the functions $y_{n+1}, \dots, y_m$ are homogeneous with respect to the torus $\mathbb{G}_m$, we conclude that $h_i = h_i^0$ for all $1 \leqslant i \leqslant k$. In particular, the functions h_i are homogeneous polynomials.

Remark 5.6. It was shown in [100], Proposition 5, that the variety Z is toric if and only if the functions h_i are monomials corresponding to lattice points in some very ample polytope inscribed in a rectangle (see Definition 4.13).

Since Z is an irreducible variety, the intersection $Z \cap U_0$ is an irreducible affine variety in U_0. On the other hand, the system of equations (5.1) defines an irreducible affine variety Z' in U_0 that is isomorphic to the affine space $\mathbb{A}^n$ with coordinates $y_1, \dots, y_n$. Since the irreducible varieties Z' and $Z \cap U_0$ coincide in some neighborhood, they are equal, so $Z \cap U_0$ is given by the system of equations $y_{n+i} = h_i(y_1, \dots, y_n)$, $1 \leqslant i \leqslant k$. In particular, the variety $Z \cap U_0$ is isomorphic to $\mathbb{A}^n$.

We consider the functions h_i as elements of $\mathrm{Sym}^{d_i} T_x(Z)^*$. Then the space

$$F_x = \mathbb{K} \oplus T_x(Z)^* \oplus \langle h_1^0, \dots, h_k^0 \rangle$$

is called the *fundamental form* of Z at the point x. It is a subspace of the direct sum $\bigoplus_{l \geqslant 0} \mathrm{Sym}^l T_x(Z)^*$.

Below we will need the following classical result (see, for instance, [53], Theorem 3.3).

Theorem 5.7 (É. Cartan). *Let Z be a projective variety. Then there exists an open subset $W' \subseteq Z$ such that for any point $x \in W'$ the fundamental form F_x is a symbol system, that is, for any $h \in F_x$ and any $v \in T_x(Z)$ the derivation of h by v belongs to F_x.*

Thus, for an Euler-symmetric variety Z we have two open subsets W and W' in Z (see Definition 5.3). Such subsets have a non-empty intersection. So we may assume that x is an Euler point and the fundamental form F_x is a symbol system.

Since $Z \cap U_0$ is isomorphic to $\mathbb{A}^n$, we have an additive action by parallel translations on $Z \cap U_0$. This action can be extended to an action on $\mathbb{P}^m$, provided it can be extended to an action by affine transformations on U_0.

Let us show that any $\mathbb{G}_a$-subgroup H of this action of $\mathbb{G}_a^n$ on $Z \cap U_0$ can be extended to a $\mathbb{G}_a$-subgroup of affine transformations of U_0. Let ∂ be the locally nilpotent derivation on $\mathbb{K}[Z \cap U_0]$ corresponding to H. Since F_x is a symbol system, the result of the derivation ∂ applied to h_i belongs to F_x as well. On the other hand, $F_x = \mathbb{K} \oplus T_x(Z)^* \oplus \langle h_1, \dots, h_k \rangle = \langle 1, y_1, \dots, y_m \rangle$ since x is an Euler point. Then for any $1 \leqslant i \leqslant k$ we have

$$\partial(y_{n+i}) = \partial(h_i(y_1, \dots, y_n)) = \ell_i(1, y_1, \dots, y_m),$$

where ℓ_i is a linear form. So the action of $s \in \mathbb{G}_a$ given by $\exp s\partial$ is an action by affine transformations:

$$y_{n+i} \mapsto y_{n+i} + s\ell_{i,1}(1, y_1, \dots, y_m) + \frac{s^2}{2}\ell_{i,2}(1, y_1, \dots, y_m) + \cdots,$$

where all the $\ell_{i,j}$ are linear forms. Finally, the group H acts on $y_1, \dots, y_n$ by shifts, hence also by affine transformations.

We conclude that the additive action on $Z \cap U_0$ extends to an action on $\mathbb{P}^m$ and so it induces an additive action on Z since Z is the closure of $Z \cap U_0$. This completes the proof of Theorem 5.5.

Remark 5.8. There are many arguments showing that $\mathbb{G}_m$- and $\mathbb{G}_a$-actions are of completely different nature. At the same time one can prove that the existence of $\mathbb{G}_m$-actions of certain type implies the existence of $\mathbb{G}_a$-actions. For example, if an affine variety X admits two actions of the torus $\mathbb{G}_m$ that do not commute, then X admits a non-trivial $\mathbb{G}_a$-action (see [47], § 3, and [7], the proof of Theorem 2.1). Further, it was shown in [4], Theorem 1, that the existence of a $\mathbb{G}_m$-action of parabolic type on a normal affine variety implies the existence of a non-trivial $\mathbb{G}_a$-action. Theorem 5.5 can also be regarded as a result of this form.

It turns out that the condition to be Euler symmetric is a criterion of the existence of an additive action for wide classes of projective varieties. Let us start with the toric case.

Theorem 5.9 ([100], Theorem 3). *Let X be a projective toric variety. Then the following conditions are equivalent:*

(i) *the variety X is Euler symmetric with respect to some embedding into a projective space*;
(ii) *the variety X is Euler symmetric with respect to any linearly non-degenerate, linearly normal embedding into a projective space*;
(iii) *the variety X admits an additive action.*

We sketch the proof of this theorem. In order to obtain the implication (i) $\Rightarrow$ (ii) one uses the linearizability of a (very ample) line bundle on a normal variety with respect to a torus action (see [75], Proposition 2.4). This allows one to extend a $\mathbb{G}_m$-action of Euler type from X to the ambient projective space. This implication does not use that X is toric. The implication (ii) $\Rightarrow$ (i) is trivial.

The implication (i) $\Rightarrow$ (iii) follows from Theorem 5.5. An alternative proof of this implication that uses the specifics of the toric case, namely, the description of orbits of the automorphism group on a complete toric variety due to Bazhov [17] and Corollary 4.8, was given in [100], Proposition 4.

The proof of the implication (iii) $\Rightarrow$ (i) is split into three steps. At the first step it was checked in [100], Proposition 2, that every non-singular T-fixed point x_0 on a projective toric variety X is Euler with respect to some linearly non-degenerate, linearly normal projective embedding. The second step is the claim that a point $x \in X$ is Euler if and only if x can be moved to a non-singular T-fixed point x_0 on X by an automorphism of X (see [100], Proposition 3). Finally, if X admits an additive action, then X admits an additive action normalized by the acting torus T (see Theorem 4.11). The open orbit $\mathcal{U}$ of this additive action is T-invariant and isomorphic to an affine space. This implies that $\mathcal{U}$ contains a (non-singular) T-fixed point x_0. We know that x_0 is Euler with respect to some linearly non-degenerate, linearly normal projective embedding. Using Proposition 2.4 in [75] again, one can extend the additive action on X to a $\mathbb{G}_a^n$-action on the ambient projective space. This implies that all points in $\mathcal{U}$ are Euler on X, and so X is Euler-symmetric.

Remark 5.10. The proof of the implication (i) $\Rightarrow$ (ii) shows that for a normal projective variety Z the property to be Euler-symmetric can be defined in intrinsic terms, without involving an embedding into a projective space. Namely, for a non-singular point $x \in Z$ a $\mathbb{G}_m$-action on Z is said to be of *Euler type* at x if x is an isolated fixed point of this action and the induced $\mathbb{G}_m$-action on the tangent space $T_x(Z)$ is by scalar multiplication. A non-singular point $x \in Z$ is said to be *Euler* if there is a $\mathbb{G}_m$-action on Z that is of Euler type at x. We say that a normal projective variety Z is *Euler symmetric* if there is an open dense subset of Z consisting of Euler points.

In this case, for any linearly non-degenerate linearly normal embedding $Z \subseteq \mathbb{P}(V)$ we can extend $\mathbb{G}_m$-actions of Euler type at all Euler points on Z to actions on $\mathbb{P}(V)$, and so any such embedding is Euler symmetric in the sense of Definition 5.3.

It is an interesting problem to prove Theorem 5.5 without involving an embedding of Z into a projective space. At the moment we do not know such a proof.

Remark 5.11. Shafarevich gave some examples illustrating the properties of the set of Euler points on a projective toric variety X. In particular, such points may not form one orbit of the group $\mathrm{Aut}(X)$ (see [100], Example 1), and not every point on a smooth Euler-symmetric projective variety is Euler (see [100], Example 2).

The next result concerns flag varieties. It was mentioned in [53], Example 3.13; we give a direct proof below.

Theorem 5.12. *A flag variety G/P is Euler symmetric if and only if it admits an additive action.*

Proof. We use the notation and results of § 3.2. We assume that G is a connected simple linear algebraic group, P is a maximal parabolic subgroup of G corresponding to a simple root $\alpha_i \in \Delta$, and G is the identity component of the automorphism group $\mathrm{Aut}(G/P)$. In view of Theorem 3.6 it suffices to prove that G/P is Euler symmetric if and only if the unipotent radical P_u^- is commutative. The latter is equivalent to the commutativity of the tangent algebra $\mathfrak{p}_u^-$.

We know that $\mathfrak{p}_u^- = \bigoplus_{\alpha\in\Phi_i^-}\mathfrak{g}_\alpha$, where Φ_i^- is the set of negative roots whose decompositions into linear combinations of simple roots contain the root α_i. Since the variety G/P is homogeneous, it is Euler symmetric if and only if there is a $\mathbb{G}_m$-action of Euler type at the point $x = eP$. The subgroup $\mathbb{G}_m$ is contained in P, and we may assume that, up to conjugation, $\mathbb{G}_m$ is a subgroup of the maximal torus T.

Since the action of T on G by conjugation descends to G/P as the action by left translations, its differential acts on the tangent space $T_x(G/P) = \mathfrak{p}_u^-$ by endomorphisms of the Lie algebra structure.

If all operators of scalar multiplication preserve the Lie bracket, then the Lie bracket is zero. Conversely, assume that the Lie algebra $\mathfrak{p}_u^-$ is commutative. Then any root in Φ_i^- contains α_i with coefficient -1; otherwise it is a sum of two roots from Φ_i^-, and then $\mathfrak{p}_u^-$ is not commutative. In this case the $\mathbb{G}_m$-subgroup in T given by the equations $\alpha_j(t) = 1$ for all $\alpha_j \in \Delta$, $j \neq i$, acts on $\mathfrak{p}_u^-$ by scalar multiplication, and so G/P is Euler symmetric.

On the other hand, if a projective hypersurface X admits an additive action, then X need not be Euler symmetric. Indeed, it was proved in [52], Example 3.14, that a non-degenerate hypersurface is Euler symmetric if and only if it is a smooth quadric. By Proposition 2.4 these are the only smooth hypersurfaces admitting an additive action. However, we have non-degenerate singular hypersurfaces admitting additive actions (see Theorem 2.29, for example).

At the same time, in [99] one can find examples of additive actions on degenerate toric quadrics. By Theorem 5.9 such quadrics are Euler symmetric.

The question whether Euler-symmetric varieties are complete intersections in projective spaces is considered in a recent preprint [85].

We finish this subsection with further examples of Euler-symmetric varieties. A smooth Euler-symmetric projective surface is the result of successive blow-ups of the projective plane $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$ along fixed points of the $\mathbb{G}_a^2$-action (see [53], Example 3.12). In higher dimensions, it was shown in [53], Example 2.2, that scalar multiplications of $\mathbb{A}^n$ can be extended to $\mathbb{G}_m$-actions of Euler type on the blow-up of a smooth subvariety in $\mathbb{P}^n \setminus \mathbb{A}^n$. This proves that such blow-ups are Euler symmetric (cf. Proposition 1.51).

Finally, in [52] a complete classification of Euler-symmetric varieties of rank 2 was obtained; the rank was defined there in terms of fundamental forms. Such varieties are also called quadratically symmetric. Fu and Hwang observed that if one considers Euler-symmetric varieties as quasi-homogeneous generalizations of Hermitian symmetric spaces, then quadratically symmetric varieties are quasi-homogeneous generalizations of Hermitian symmetric spaces of rank 2.

5.3. Open problems. In this subsection we formulate some open problems and conjectures on additive actions. We hope that they will stimulate further progress in this area.

It is well known that a complete normal algebraic variety X is toric if and only if the Cox ring $R(X)$ is a polynomial ring.

Problem 5.13. *Characterize complete normal algebraic varieties admitting an additive action in terms of their Cox rings.*

Applying results of § 4 it is easy to construct two complete toric varieties X_{Σ_1} and X_{Σ_2} such that the fans Σ_1 and Σ_2 have the same number of rays, and X_{Σ_1} admits an additive action, but X_{Σ_2} does not. This shows that the existence of an additive action cannot be characterized in terms of $R(X)$ as an abstract ring. But the ring $R(X)$ is graded by the group $\mathrm{Cl}(X)$, and we believe that there is a characterization in terms of this grading.

If we are going to study additive actions via lifting the action to the total coordinate space, the solution of the following problem may be very helpful.

Problem 5.14. *Fix positive integers r and n, and let $d = r+n$. Describe all affine factorial varieties X of dimension d equipped with an effective action of the group $\mathbb{G}_m^r \times \mathbb{G}_a^n$ with open orbit.*

The case $n = 0$ corresponds to affine factorial toric varieties that are known to be direct products of a torus and an affine space. The case $n = 1$ also corresponds to affine factorial toric varieties: see [8]. In turn, for $r = 0$ X is an affine space with a transitive action of the group $\mathbb{G}_a^n$. All other cases remain open.

As we know from Corollary 1.49, the projective space $\mathbb{P}^n$ with $n \geqslant 6$ admits infinitely many non-equivalent additive actions. At the same time the results on the uniqueness of additive actions on smooth projective quadrics and, more generally, on flag varieties suggest that the situation with projective spaces can be exceptional in a certain sense. This motivates the following problem.

Problem 5.15. *Describe all complete toric varieties that admit infinitely many additive actions.*

It is natural to ask for a description of all additive action on concrete complete toric varieties.

Problem 5.16. *Describe all additive actions on the weighted projective space $\mathbb{P}(1, a_1, \dots, a_n)$.*

The case of weighted projective planes suggests an idea that the number of additive actions on $\mathbb{P}(1, a_1, \dots, a_n)$ depends only on n, rather than on the values of $a_1, \dots, a_n$. One can expect that there is a description of additive actions on $\mathbb{P}(1, a_1, \dots, a_n)$ in terms of some 'weighted Hassett–Tschinkel correspondence'.

In Proposition 3.4 we observed a connection between additive actions on a given complete variety X and maximal commutative unipotent subgroups in the automorphism group $\mathrm{Aut}(X)$. It is interesting to make this correspondence more precise.

Problem 5.17. *Is it true that any maximal commutative unipotent subgroup of dimension n in the group $\mathrm{GL}_{n+1}(\mathbb{K})$ acts on the projective space $\mathbb{P}^n$ with open orbit?*

Let us show that in the case of the action of the group $\mathrm{SO}_{n+2}(\mathbb{K})$ on the quadric $Q_n \subseteq \mathbb{P}^{n+1}$ the same question has a negative answer. By Theorem 2.25 there is just one conjugacy class of maximal commutative unipotent subgroups of dimension n in $\mathrm{SO}_{n+2}(\mathbb{K})$ that acts on the quadric Q_n with open orbit. At the same time, in [103], § 6, an example of a maximal commutative unipotent subgroup of dimension n from another conjugacy class in $\mathrm{SO}_{n+2}(\mathbb{K})$ was presented. Such a subgroup corresponds to a so-called *free-rowed* maximal commutative nilpotent subalgebra of the Lie algebra $\mathfrak{so}_{n+2}(\mathbb{K})$ for $n \geqslant 6$ (see [67] for details).

Problem 5.18. *Let G be a connected linear algebraic group and H be a commutative unipotent subgroup of G that is maximal in the class of commutative subgroups of G. Does there exist a G-variety X such that the induced action of H on X has an open orbit?*

The next conjecture is about additive actions on projective hypersurfaces. It can be a good complement to the result of Theorem 2.32.

Conjecture 5.19. *Let $X \subseteq \mathbb{P}^{n+1}$ be a degenerate hypersurface admitting an induced additive action. Then there are at least two induced additive actions on X up to equivalence.*

Finally, we give the following conjectural characterization of a class of varieties with an additive action. In [53], Conjecture 5.1, the following was formulated.

Conjecture 5.20. *Let X be a smooth Fano variety of Picard number 1 that is an equivariant compactification of a vector group. Then X can be realized as an Euler-symmetric projective variety under a suitable projective embedding.*

Partial positive results on this conjecture can be found in [53], § 5.

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