New approaches to $\mathfrak{gl}_N$ weight system

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Contents

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1 List of Symbols

$\mathfrak{g}, \langle \cdot, \cdot \rangle$	a Lie algebra endowed with a nondegenerate invariant bilinear product
$\mathfrak{gl}_N$	general linear Lie algebra; consists of all $N \times N$ matrices with the commutator serving as the Lie bracket
$\mathfrak{sl}_N$	special linear Lie algebra; consists of all $N \times N$ trace-free matrices with the commutator serving as the Lie bracket
d	dimension of Lie algebra; specifically, for $\mathfrak{gl}_N$, $d = N^2$
D	a chord diagram
n	the number of chords in a chord diagram
K_n	the chord diagram with n chords any two of which intersect one another
π	the projection to the subspace of primitive elements in the Hopf algebra of chord diagrams whose kernel is the subspace of decomposable elements
$C_1, \cdots, C_N$	Casimir elements in $U(\mathfrak{gl}_N)$
w	a weight system
$w_\mathfrak{g}$	the Lie algebra weight system associated to a Lie algebra $\mathfrak{g}$
$\bar{w}_\mathfrak{g}$	$w_\mathfrak{g}(\pi(\cdot))$; the composition of the Lie algebra weight system $w_\mathfrak{g}$ with the projection π to the subspace of primitives
σ	a permutation
m	the number of permutated elements; e.g. for the permutation determined by a chord diagram, $m = 2n$
$G(\sigma)$	the digraph of the permutation σ
$\Lambda^*(N)$	the algebra of shifted symmetric polynomials in N variables
ϕ	the Harish–Chandra projection
$p_1, \cdots, p_N$	shifted power sum polynomials

2 Introduction

In V. A. Vassiliev's theory of finite type knot invariants, a weight system can be associated to each such invariant. A weight system is a function on chord diagrams satisfying so-called 4-term relations.

In the opposite direction, according to a Kontsevich theorem, to each weight system taking values in a field of characteristic 0, a finite type knot invariant can be associated in a canonical way. This makes studying weight systems an important part of knot theory.

There is a number of approaches to constructing weight systems. In particular, a huge class of weight systems can be constructed from metrized finite dimensional Lie algebras. The present paper has been motivated by an aspiration for understanding the weight system corresponding to the Lie algebra $\mathfrak{gl}_N$.

The straightforward approach to computing the values of a Lie algebra weight system on a general chord diagram amounts to elaborating calculations in the noncommutative universal enveloping algebra, in spite of the fact that the result belongs to the center of the latter. This approach is rather inefficient even for the simplest noncommutative Lie algebra $\mathfrak{sl}_2$, whose weight system is associated to the knot invariant known as the colored Jones polynomial. For this Lie algebra, however, there is a recurrence relation due to S. Chmutov and A. Varchenko [3], and numerous computations have been done using it, see e.g. [6, 7, 14]. In particular, recently, values of the $\mathfrak{sl}_2$-weight system have been computed on certain nontrivial infinite families of chord diagrams.

Much less is known about other Lie algebras; for them, explicit answers have been computed only for chord diagrams of very small order or for simple families of chord diagrams, see [13]. In particular, no recurrence similar to the Chmutov–Varchenko one exists (with the exception of the Lie superalgebra $\mathfrak{gl}_{1|1}$, see [5, 2]). The goal of the present paper is to provide two new ways to compute the values of the $\mathfrak{gl}_N$ weight system.

The first approach is based on a suggestion due to M. Kazarian to define an invariant of permutations taking values in the center of the universal enveloping algebra of $\mathfrak{gl}_N$. The restriction of this invariant to involutions without fixed points (such an involution determines a chord diagram) coincides with the value of the $\mathfrak{gl}_N$-weight system on this chord diagram. We describe the recursion allowing one to compute the $\mathfrak{gl}_N$-invariant of permutations and demonstrate how it works in a number of examples.

For $N' < N$, the center of the universal enveloping algebra of $\mathfrak{gl}_{N'}$ is naturally embedded into that of $\mathfrak{gl}_N$, and the $\mathfrak{gl}_N$-weight system is stable: its value on a permutation is a universal polynomial. The recursion we describe allows one to compute this polynomial, simultaneously for all N.

The second approach is based on the Harish-Chandra isomorphism for the Lie algebras $\mathfrak{gl}_N$. This isomorphism identifies the center of the universal enveloping algebra $\mathfrak{gl}_N$ with the ring $\Lambda^*(N)$ of shifted symmetric polynomials in N variables. The Harish-Chandra projection can be applied separately for each monomial in the defining polynomial of the weight system; as a result, the main body of computations can be done in a commutative algebra, rather than noncommutative one.

The paper is organized as follows. In Sec. 3, we recall the construction of Lie algebra weight systems. In Sec. 4, we describe an extension of the $\mathfrak{gl}_N$-weight system to arbitrary permutations and a recursion to computing its values on permutations. In Sec. 5, we apply, for the Lie algebras $\mathfrak{gl}_N$, the Harish-Chandra isomorphism to develop one more algorithm for computing the corresponding weight system. We compare the results with those obtained by the previous method. In Sec. 6, we recall the Hopf algebra structure on the space of chord diagrams modulo 4-term relations, and discuss the behaviour of the $\mathfrak{gl}_N$-weight system with respect to this structure.

The author is grateful to M. Kazarian and G. Olshanskii for valuable suggestions, and to S. Lando for permanent attention.

3 Definition of $\mathfrak{gl}_N$ weight system

Below, we use standard notions from the theory of finite order knot invariants; see, e.g. [4].

A *chord diagram* of order n is an oriented circle (called the *Wilson loop*) endowed with $2n$ pairwise distinct points split into n disjoint pairs, considered up to orientation-preserving diffeomorphisms of the circle.

A *weight system* is a function w on chord diagrams satisfying the 4-term relation; see Fig. 1.

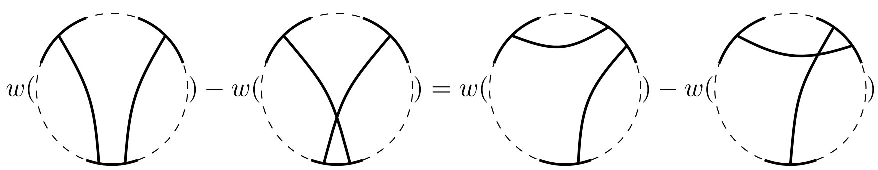
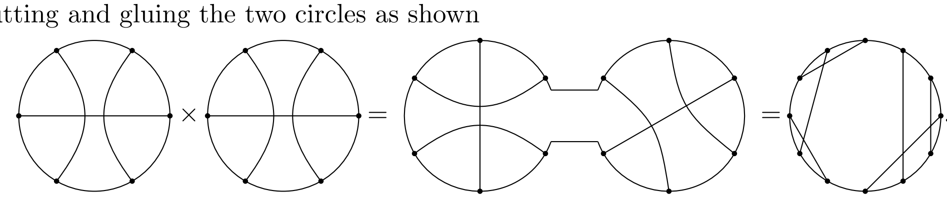


Figure 1: 4-term relation

In figures, the outer circle of the chord diagram is always assumed to be oriented counterclockwise. Dashed arcs may contain ends of arbitrary sets of chords, same for all the four terms in the picture.

Definition 3.1 The *product* of two chord diagrams D_1 and D_2 is defined by cutting and gluing the two circles as shown



Modulo 4-term relations, the product is well-defined.

Given a Lie algebra $\mathfrak{g}$ equipped with a non-degenerate invariant bilinear form, one can construct a weight system with values in the center of its universal enveloping algebra $U(\mathfrak{g})$. This is the form M. Kontsevich [8] gave to a construction due to D. Bar-Natan [1]. Kontsevich's construction proceeds as follows.

Definition 3.2 (Universal Lie algebra weight system) Let $\mathfrak{g}$ be a metrized Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, that is, a Lie algebra with an ad-invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$. Let d denote the dimension of $\mathfrak{g}$. Choose a basis $e_1, \dots, e_d$ of $\mathfrak{g}$ and let $e_1^*, \dots, e_d^*$ be the dual basis with respect to the form $\langle \cdot, \cdot \rangle$, $\langle e_i, e_j^* \rangle = \delta_{ij}$, where δ is the Kronecker delta.

Given a chord diagram D with n chords, we first choose a base point on the circle, away from the ends of the chords of D. This gives a linear order on the endpoints of the chords, increasing in the positive direction of the Wilson loop. Assign to each chord a an index, that is, an integer-valued variable, i_a. The values of i_a will range from 1 to d, the dimension of the Lie algebra. Mark the first endpoint of the chord a with the symbol e_{i_a} and the second endpoint with $e_{i_a}^*$.

Now, write the product of all the e_{i_a} and all the $e_{i_a}^*$, in the order in which they appear on the Wilson loop of D, and take the sum of the d^n elements of the universal enveloping algebra $U(\mathfrak{g})$ obtained by substituting all possible values of the indices i_a into this product. Denote by $w_{\mathfrak{g}}(D)$ the resulting element of $U(\mathfrak{g})$.

Claim 3.3 [8] *The function $w_{\mathfrak{g}} : D \mapsto w_{\mathfrak{g}}(D)$ on chord diagrams has the following properties:*

1. *the element $w_{\mathfrak{g}}(D)$ does not depend on the choice of the base point on the diagram;*
2. *it does not depend on the choice of the basis e_i of the Lie algebra $\mathfrak{g}$;*
3. *its image belongs to the ad-invariant subspace*

$$U(\mathfrak{g})^{\mathfrak{g}} = \{x \in U(\mathfrak{g}) | xy = yx \text{ for all } y \in \mathfrak{g}\} = ZU(\mathfrak{g});$$

4. *it is multiplicative, $w_{\mathfrak{g}}(D_1 D_2) = w_{\mathfrak{g}}(D_1) w_{\mathfrak{g}}(D_2)$ for any pair of chord diagrams D_1, D_2;*
5. *this map from chord diagrams to $ZU(\mathfrak{g})$ satisfies the 4-term relations.*

Consider the Lie algebra $\mathfrak{gl}_N$ of all $N \times N$ matrices. Fix the trace of the product of matrices as the preferred ad-invariant form: $\langle x, y \rangle = \mathrm{Tr}(xy)$. The algebra $\mathfrak{gl}_N$ is linearly spanned by matrix units E_{ij} having 1 on the intersection of i th row with j th column and 0 elsewhere, $i, j = 1, \dots, N$. We have $\langle E_{ij}, E_{kl} \rangle = \delta_{il}\delta_{jk}$. Therefore, the duality between $\mathfrak{gl}_N$ and $\mathfrak{gl}_N^*$ defined by $\langle \cdot, \cdot \rangle$ is given by the formula $E_{ij}^* = E_{ji}$. The commutation relations for $\mathfrak{gl}_N$ have the form

$$[E_{kl}, E_{ji}] = E_{kl}E_{ji} - E_{ji}E_{kl} = \delta_{lj}E_{ki} - \delta_{ik}E_{jl}. \quad (1)$$

Now, the straightforward computation of the value of the $\mathfrak{gl}_N$ weight system looks like follows.

Example 3.4 For a chord diagram K_1 with a single chord, we have

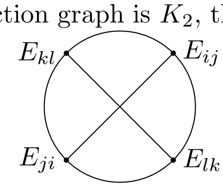
$$w_{\mathfrak{gl}_N}(K_1) = \sum_{i,j=1}^{N} E_{ij}E_{ji}.$$

We denote this element by $C_2 \in ZU(\mathfrak{gl}_N)$ and call the *second Casimir*. Similarly, $\sum_{i=1}^{N} E_{ii} = C_1$, and, more generally,

$$C_k = \sum_{i_1,i_2,\dots,i_k=1}^{N} E_{i_1i_2}E_{i_2i_3}\dots E_{i_ki_1}$$

is the k th *Casimir element* in $ZU(\mathfrak{gl}_N)$. The center $ZU(\mathfrak{gl}_N)$ is the ring of polynomials in the variables $C_1, C_2, \dots, C_N$. The value $w_{\mathfrak{gl}_N}(D)$ of the $\mathfrak{gl}_N$-weight system on a chord diagram D with n chords is a polynomial in $N, C_1, \dots, C_n$, and this polynomial $w_{\mathfrak{gl}_N}(D)$ is the same for all $N \geq n$. Below, we denote this common value of the $w_{\mathfrak{gl}_N}(D)$, for N sufficiently large, by $w_{\mathfrak{gl}}(D)$. This value is an element of the polynomial ring $\mathbb{C}[N, C_1, C_2, \dots]$ in infinitely many variables. For $N' < n$, the polynomial $w_{\mathfrak{gl}_{N'}}(D)$ is obtained from $w_{\mathfrak{gl}_N}(D)$ by setting C_k to its image in $\mathfrak{gl}_{N'}$ for all $k = N'+1, N'+2, \dots, n$.

Example 3.5 For the chord diagram, which we denote by K_2, since its intersection graph is K_2, the complete graph on 2 vertices, we have



$$w_{\mathfrak{gl}_N}(K_2) = \sum_{i,j,k,l=1}^{N} E_{ij}E_{kl}E_{ji}E_{lk}.$$

Using the commutation relations (1) we obtain

$$\begin{aligned}
w_{\mathfrak{gl}_N}(K_2) &= \sum_{i,j,k,l=1}^{N} E_{ij}E_{kl}E_{ji}E_{lk} \\
&= \sum_{i,j,k,l=1}^{N} E_{ij}E_{ji}E_{kl}E_{lk} + \sum_{i,j,k,l=1}^{N} \delta_{lj}E_{ij}E_{ki}E_{lk} - \sum_{i,j,k,l=1}^{N} \delta_{ik}E_{ij}E_{jl}E_{lk} \\
&= C_2^2 + \sum_{i,j,k=1}^{N} E_{ij}E_{ki}E_{jk} - \sum_{i,j,l=1}^{N} E_{ij}E_{jl}E_{li} \\
&= C_2^2 + \sum_{i,j,k=1}^{N} E_{ij}[E_{ki}, E_{jk}] \\
&= C_2^2 + \sum_{i,j,k=1}^{N} \delta_{ij}E_{ij}E_{kk} - \sum_{i,j,k=1}^{N} \delta_{kk}E_{ij}E_{ji} \\
&= C_2^2 + C_1^2 - NC_2 \\
&= w_{\mathfrak{gl}}(K_2).
\end{aligned}$$

Even in this simple example, the straightforward computation includes a lot of steps. A much more efficient algorithm is suggested in the next section.

4 The $\mathfrak{gl}$ weight system for permutations

There is no recurrence relation for the weight system $w_{\mathfrak{gl}}$ we know about. Instead, following the suggestion by M. Kazarian, we interpret an arc diagram as an involution without fixed points on the set of its ends and extend the function $w_{\mathfrak{gl}}$ to arbitrary permutations of any number of permutated elements. For permutations, in contrast to chord diagrams, such a recurrence relation could be given.

For a permutation $\sigma \in S_m$, set

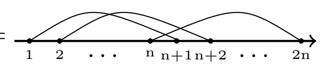
$$w_{\mathfrak{gl}_N}(\sigma) = \sum_{i_1,\cdots,i_m=1}^{N} E_{i_1 i_{\sigma(1)}} E_{i_2 i_{\sigma(2)}} \cdots E_{i_m i_{\sigma(m)}} \in U(\mathfrak{gl}_N).$$

We claim that

- $w_{\mathfrak{gl}_N}$ lies in the center of $U(\mathfrak{gl}_N)$.
- this element is invariant under conjugation by a cyclic permutation: $w_{\mathfrak{gl}_N}(\sigma) = \sum_{i_1,\cdots,i_m=1}^{N} E_{i_2 i_{\sigma(2)}} \cdots E_{i_m i_{\sigma(m)}} E_{i_1 i_{\sigma(1)}}$

For example, the standard generator $C_m = \sum_{i_1,\cdots,i_m=1}^{N} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_{m-1} i_m} E_{i_m i_1}$ corresponds to the cyclic permutation $1 \mapsto 2 \mapsto \cdots \mapsto m \mapsto 1 \in S_m$.

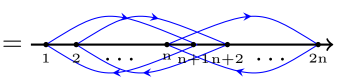
On the other hand, a chord diagram with n chords can be considered as an involution without fixed points on a set of $m = 2n$ elements. The value of $w_{\mathfrak{gl}_N}$ on the corresponding involution is equal to the value of the $\mathfrak{gl}_N$ weight system on the corresponding chord diagram.

Example 4.1 For the chord diagram $K_n =$ 

we have

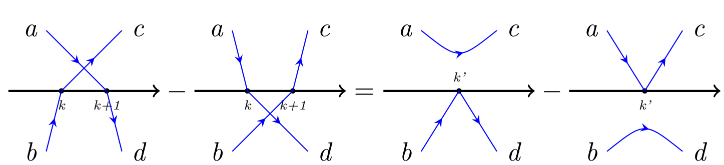
$$w_{\mathfrak{gl}_N}(K_n) = \sum_{i_1,\cdots,i_{2n}=1}^{N} E_{i_1 i_{n+1}} E_{i_2 i_{n+2}} \cdots E_{i_n i_{2n}} E_{i_{n+1} i_1} E_{i_{n+2} i_2} \cdots E_{i_{2n} i_n}$$
$$= w_{\mathfrak{gl}_N}((1\ n+1)(2\ n+2)\cdots(n\ 2n))$$

Definition 4.2 (digraph of the permutation) Let us represent a permutation as an oriented graph. The m vertices of the graph correspond to the permuted elements. They are ordered cyclically and are placed on a unit circle, subsequently connected with horizontal arrows looking right and numbered in the counterclockwise order. The arc arrows show the action of the permutation (so that each vertex is incident with exactly one incoming and one outgoing arc edge). The digraph $G(\sigma)$ of a permutation $\sigma \in S_m$ consists of these m vertices and m oriented edges, for example:

$G((1\ n+1)(2\ n+2)\cdots(n\ 2n)) =$ 

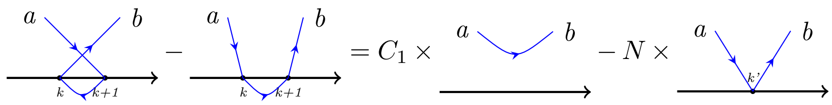
Theorem 4.3 *The value of the $w_{\mathfrak{gl}_N}$ invariant of permutations possesses the following properties:*

- *for an empty graph (with no vertices) the value of $w_{\mathfrak{gl}_N}$ is equal to 1, $w_{\mathfrak{gl}_N}(\bigcirc) = 1$;*
- *$w_{\mathfrak{gl}_N}$ is multiplicative with respect to concatenation of permutations;*
- *for a cyclic permutation (with the cyclic order on the set of permuted elements compatible with the permutation), the value of $w_{\mathfrak{gl}_N}$ is the standard generator $w_{\mathfrak{gl}_N}(1 \mapsto 2 \mapsto \cdots \mapsto k \mapsto 1) = C_k$.*
- (***Recurrence Rule***) *For the graph of an arbitrary permutation σ in S_m, and for any two neighboring elements $k, k{+}1$, of the permuted set $\{1, 2, \ldots, m\}$, we have for the value of the $w_{\mathfrak{gl}}$ weight system*



In the diagrams on the left, two horizontally neighboring vertices and the edges incident to them are depicted, while on the right these two vertices *are replaced with a single one; the other vertices are placed somewhere on the circle and their positions are the same on all diagrams participating in the relations, but the numbers of the vertices to the right of the latter are to be decreased by* 1.

For the special case $\sigma(k+1) = k$*, the recurrence looks like follows:*



These relations are indeed a recursion, that is, they allow one to replace the computation of $w_{\mathfrak{gl}}$ *on a permutation with its computation on simpler permutations.*

Proof. We only need to prove the Recurrence Rule, which is just the graphical explanation of the Lie bracket in $\mathfrak{gl}_N$.

$$\begin{aligned} E_{i_k i_{\sigma(k)}} E_{i_{k+1} i_{\sigma(k+1)}} - E_{i_{k+1} i_{\sigma(k+1)}} E_{i_k i_{\sigma(k)}} &= [E_{i_k i_{\sigma(k)}}, E_{i_{k+1} i_{\sigma(k+1)}}] \\ &= \delta_{i_{\sigma(k)} i_{k+1}} E_{i_k i_{\sigma(k+1)}} - \delta_{i_{\sigma(k+1)} i_k} E_{i_{k+1} i_{\sigma(k)}}. \end{aligned}$$

In the special case, when $\sigma(k+1) = k$, we have

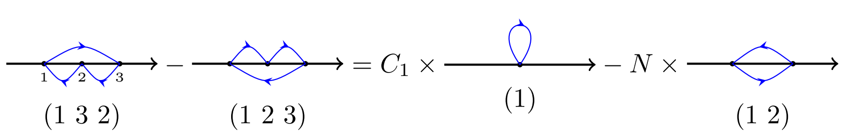
$$E_{i_k i_{\sigma(k)}} E_{i_{k+1} i_k} - E_{i_{k+1} i_k} E_{i_k i_{\sigma(k)}} = [E_{i_k i_{\sigma(k)}}, E_{i_{k+1} i_k}] = \delta_{i_{\sigma(k)} i_{k+1}} E_{i_k i_k} - \delta_{i_k i_k} E_{i_{k+1} i_{\sigma(k)}}.$$

When summing it from $i_1, \cdots, i_m = 1$ to N, we obtain $\sum \delta_{i_{\sigma(k)} i_l} E_{i_k i_k} = C_1 \sum \delta_{i_{\sigma(k)} i_l}$ and $\sum \delta_{i_k i_k} E_{i_l i_{\sigma(k)}} = N \sum E_{i_l i_{\sigma(k)}}$.

The second graph on the left hand side corresponds to a permutation obtained from the first one by a conjugation with a transposition of two neighbouring vertices. Both graphs on the right hand side have smaller number of vertices. Applying these relations, every graph can be reduced to a monomial in the variables C_k (a concatenation of cyclic permutations) modulo terms of smaller degrees. This provides an inductive computation of the invariant $w_{\mathfrak{gl}_N}$. ■

Corollary 4.4 *The value of* $w_{\mathfrak{gl}_N}$ *on a permutation is well defined, is a polynomial in* $N, C_1, C_2, \ldots,$ *and this polynomial is universal.*

Example 4.5 Let us compute the value of $w_{\mathfrak{gl}}$ on the cyclic permutation (1 3 2) by switching the places of node 2 and 3:



$$\begin{aligned} w_{\mathfrak{gl}}((1\ 3\ 2)) &= w_{\mathfrak{gl}}((1\ 2\ 3)) + C_1 \times w_{\mathfrak{gl}}((1)) - N \times w_{\mathfrak{gl}}((1\ 2)) \\ &= C_3 + C_1^2 - NC_2 \end{aligned}$$

The reader will find below a table of values of the $\mathfrak{gl}$ weight system on chord diagrams K_n, which have n chords and each chord crosses each other. These diagrams are chosen because computation of Lie algebra weight systems on them is extremely nontrivial, even for the Lie algebra $\mathfrak{sl}_2$, where we know the Chmutov–Varchenko recurrence relation. In addition, these diagrams generate a Hopf subalgebra of the Hopf algebra of chord diagrams, see Sec. 6, which allows us to compute the $\mathfrak{gl}$ weight system on the projection of K_n to the primitive space.

Result

$$
\begin{aligned}
w_{\mathfrak{gl}}(K_2) &= -NC_2 + C_1^2 + C_2^2 \\
w_{\mathfrak{gl}}(K_3) &= 2C_2N^2 + (-2C_1^2 - 3C_2^2)N + C_2^3 + 3C_1^2C_2 \\
w_{\mathfrak{gl}}(K_4) &= -6C_2N^3 + (6C_1^2 + 11C_2^2 - 2C_3)N^2 + (-6C_2^3 - 14C_1^2C_2 + 6C_1C_2 - 2C_2 + 2C_4)N \\
&\quad + 3C_1^4 - 4C_1^3 + 6C_2^2C_1^2 + 2C_1^2 - 8C_3C_1 + C_2^4 + 6C_2^2 \\
w_{\mathfrak{gl}}(K_5) &= 24C_2N^4 + (-24C_1^2 - 50C_2^2 + 24C_3)N^3 \\
&\quad + (35C_2^3 + 70C_1^2C_2 - 72C_1C_2 - 10C_3C_2 + 32C_2 - 24C_4)N^2 \\
&\quad + (-20C_1^4 + 48C_1^3 - 50C_2^2C_1^2 - 32C_1^2 + 30C_2^2C_1 + 96C_3C_1 - 10C_2^4 - 82C_2^2 + 10C_2C_4)N \\
&\quad + C_2^5 + 10C_1^2C_2^3 + 30C_2^3 + 15C_1^4C_2 - 20C_1^3C_2 + 10C_1^2C_2 - 40C_1C_3C_2 \\
w_{\mathfrak{gl}}(K_6) &= -120C_2N^5 + (120C_1^2 + 274C_2^2 - 240C_3)N^4 \\
&\quad + (-225C_2^3 - 404C_1^2C_2 + 720C_1C_2 + 174C_3C_2 - 416C_2 + 224C_4)N^3 \\
&\quad + (130C_1^4 - 480C_1^3 + 375C_2^2C_1^2 - 30C_3C_1^2 + 416C_1^2 - 522C_2^2C_1 \\
&\quad - 896C_3C_1 + 85C_2^4 + 1014C_2^2 - 30C_2^2C_3 - 88C_3 - 174C_2C_4 + 32C_5)N^2 \\
&\quad + (-15C_2^5 - 130C_1^2C_2^3 + 90C_1C_2^3 - 552C_2^3 + 30C_4C_2^2 - 165C_1^4C_2 + 438C_1^3C_2 - 492C_1^2C_2 \\
&\quad + 264C_1C_2 + 696C_1C_3C_2 + 64C_3C_2 - 72C_2 + 30C_1^2C_4 - 160C_1C_4 + 88C_4 - 16C_6)N \\
&\quad + 15C_1^6 - 60C_1^5 + 45C_2^2C_1^4 + 150C_1^4 - 60C_2^2C_1^3 - 120C_3C_1^3 - 176C_1^3 + 15C_2^4C_1^2 \\
&\quad + 120C_2^2C_1^2 + 256C_3C_1^2 + 72C_1^2 - 192C_2^2C_1 - 120C_2^2C_3C_1 - 352C_3C_1 \\
&\quad + 96C_5C_1 + C_2^6 + 90C_2^4 + 264C_2^2 + 160C_3^2 - 240C_2C_4 \\
w_{\mathfrak{gl}}(K_7) &= 720C_2N^6 + (-720C_1^2 - 1764C_2^2 + 2400C_3)N^5 \\
&\quad + (1624C_2^3 + 2688C_1^2C_2 - 7200C_1C_2 - 2324C_3C_2 + 5264C_2 - 1856C_4)N^4 \\
&\quad + (-924C_1^4 + 4800C_1^3 - 2954C_2^2C_1^2 + 644C_3C_1^2 - 5264C_1^2 \\
&\quad + 6972C_2^2C_1 + 7424C_3C_1 - 735C_2^4 - 12892C_2^2 + 714C_2^2C_3 + 3392C_3 + 2212C_2C_4 - 1088C_5)N^3 \\
&\quad + (175C_2^5 + 1365C_1^2C_2^3 - 2142C_1C_2^3 - 70C_3C_2^3 + 8358C_2^3 - 714C_4C_2^2 + 1540C_1^4C_2 \\
&\quad - 6580C_1^3C_2 + 11736C_1^2C_2 - 10176C_1C_2 - 210C_1^2C_3C_2 - 8848C_1C_3C_2 - 2792C_3C_2 + 224C_5C_2 \\
&\quad + 3456C_2 - 644C_1^2C_4 + 5440C_1C_4 - 3392C_4 + 544C_6)N^2 \\
&\quad + (-210C_1^6 + 1288C_1^5 - 735C_2^2C_1^4 - 4412C_1^4 + 2058C_2^2C_1^3 + 2576C_3C_1^3 + 6784C_1^3 - 280C_2^4C_1^2 \\
&\quad - 4704C_2^2C_1^2 - 8704C_3C_1^2 + 210C_2C_4C_1^2 - 3456C_1^2 + 210C_2^4C_1 + 8376C_2^2C_1 + 2856C_2^2C_3C_1 \\
&\quad + 13568C_3C_1 - 1120C_2C_4C_1 - 3264C_5C_1 - 21C_2^6 - 2212C_2^4 - 10680C_2^2 - 4096C_3^2 + 448C_2^2C_3 \\
&\quad + 70C_2^3C_4 + 7432C_2C_4 - 112C_2C_6)N \\
&\quad + 504C_1^2C_2 - 1232C_1^3C_2 + 1050C_1^4C_2 - 420C_1^5C_2 + 105C_1^6C_2 + 3192C_2^3 - 1344C_1C_2^3 \\
&\quad + 700C_1^2C_2^3 - 140C_1^3C_2^3 + 105C_1^4C_2^3 + 210C_2^5 + 21C_1^2C_2^5 + C_2^7 - 5152C_1C_2C_3 + 1792C_1^2C_2C_3 \\
&\quad - 840C_1^3C_2C_3 - 280C_1C_2^3C_3 + 1120C_2C_3^2 + 1344C_1^2C_4 - 1680C_2^2C_4 + 672C_1C_2C_5
\end{aligned}
$$

Remark The Lie algebra $\mathfrak{gl}_N$ is not simple. Instead, it is a direct sum of a commutative one-dimensional Lie algebra and a simple Lie algebra $\mathfrak{sl}_N$. The one-dimensional commutative Lie subalgebra in $\mathfrak{gl}_N$ consists of scalar matrices, which are $\mathbb{C}$-multiples of the identity matrix. Therefore, the center $ZU(\mathfrak{gl}_N)$ of the universal enveloping algebra of $\mathfrak{gl}_N$ is the tensor product of the center of the universal enveloping algebra of $\mathbb{C}$ and that of $\mathfrak{sl}_N$, whence the ring of polynomials in the first Casimir C_1 with coefficients in $ZU(\mathfrak{sl}_N)$. Therefore, the values of the weight system $w_{\mathfrak{sl}_N}$ can be computed from that of $w_{\mathfrak{gl}_N}$ by setting $C_1 = 0$. In the result, $C_2, C_3, \dots$ denote the projections of the corresponding Casimir elements in $ZU(\mathfrak{gl}_N)$ to $ZU(\mathfrak{sl}_N)$.

5 Symmetric functions and Harish–Chandra isomorphism

In this section, we make use of the Harish-Chandra isomorphism for the Lie algebras $\mathfrak{gl}_N$ to compute the corresponding weight systems.

Definition 5.1 (algebra of shifted symmetric polynomials)
For a positive integer N, the algebra $\Lambda^*(N)$ of shifted symmetric polynomials in N variables $x_1, x_2, \cdots, x_N$ consists of polynomials that are invariant under changes of variables

$$(x_1, \cdots, x_i, x_{i+1}, \cdots, x_N) \mapsto (x_1, \cdots, x_{i+1} - 1, x_i + 1, \cdots, x_N),$$

for all $i = 1, \cdots, N-1$. Equivalently, this is the algebra of symmetric polynomials in the shifted variables $(x_1 - 1, x_2 - 2, \dots, x_N - N)$.

The universal enveloping algebra $U(\mathfrak{gl}_N)$ of the Lie algebra $\mathfrak{gl}_N$ admits the direct sum decomposition

$$U(\mathfrak{gl}_N) = (\mathfrak{n}_- U(\mathfrak{gl}_N) + U(\mathfrak{gl}_N)\mathfrak{n}_+) \oplus U(\mathfrak{h}), \tag{2}$$

where $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are the nilpotent subalgebras of, respectively, upper and lower triangular matrices in $\mathfrak{gl}_N$, and $\mathfrak{h}$ is the subalgebra of diagonal matrices.

Definition 5.2 (Harish–Chandra projection in $U(\mathfrak{gl}_N)$)
The Harish–Chandra projection for $U(\mathfrak{gl}_N)$ is the projection to the second summand in (2)

$$\phi : U(\mathfrak{gl}_N) \to U(\mathfrak{h}) = \mathbb{C}[E_{11}, \cdots, E_{NN}],$$

where $E_{11}, \cdots, E_{NN}$ are the diagonal matrix units in $\mathfrak{gl}_N$; they commute with one another.

Theorem 5.3 (Harish–Chandra isomorphism [10]) *The Harish–Chandra projection, when restricted to the centre $ZU(\mathfrak{gl}_N)$, is an algebra isomorphism to the algebra $\Lambda^*(N) \subset U(\mathfrak{h})$ of shifted symmetric polynomials in $E_{11}, \cdots, E_{NN}$.*

Thus, the computation of the value of the $\mathfrak{gl}_N$ weight system on a chord diagram can be elaborated by applying the Harish-Chandra projection to each monomial of the polynomial. For such a monomial, the projection can be computed by moving variables E_{ij} with $i > j$ to the left, and/or variables E_{ij} with $i < j$ to the right by means of applying the commutator relations. If, in the process, we obtain monomials in $\mathfrak{n}_- U(\mathfrak{gl}_N)$ or $U(\mathfrak{gl}_N)\mathfrak{n}_+$, then we replace such a monomial with 0. A monomial in the (mutually commuting) variables E_{ii} cannot be simplified, and its projection to $U(\mathfrak{h})$ coincides with itself. The resulting polynomial in $E_{11}, \cdots, E_{NN}$ will be automatically shifted symmetric.

Example 5.4 Let's compute the projection of the quadratic Casimir element

$$C_2 = \sum_{i,j} E_{ij}E_{ji} \in ZU(\mathfrak{gl}_N)$$

to $U(\mathfrak{h})$. We have

$$\begin{aligned} C_2 &= \sum_i E_{ii}^2 + \sum_{i<j} E_{ij}E_{ji} + \sum_{i>j} E_{ij}E_{ji} \\ &= \sum_i E_{ii}^2 + 2\sum_{i>j} E_{ij}E_{ji} + \sum_{i<j} [E_{ij}, E_{ji}] \\ &= \sum_i E_{ii}^2 + 2\sum_{i>j} E_{ij}E_{ji} + \sum_{i<j} (E_{ii} - E_{jj}). \end{aligned}$$

In this expression, the first and the third summand depend on the diagonal unit elements E_{ii} only, while the second summand is in $\mathfrak{n}_- U(\mathfrak{gl}_N) + U(\mathfrak{gl}_N)\mathfrak{n}_+$, whence the image under the projection is

$$\begin{aligned} \phi(C_2) &= \sum_i E_{ii}^2 + \sum_{i<j} (E_{ii} - E_{jj}) \\ &= \sum_i (E_{ii}^2 + (N + 1 - 2i)E_{ii}). \end{aligned}$$

Similarly to the ring of ordinary symmetric functions, the ring $\Lambda^*(N)$ of shifted symmetric functions in N variables is isomorphic to a polynomial ring in N variables. There is a variety of convenient N-tuples of generators in $\Lambda^*(N)$. One of them is the tuple of shifted power sum polynomials

$$p_k = \sum_i \left(\left(E_{ii} + \frac{N+1}{2} - i \right)^k - \left(\frac{N+1}{2} - i \right)^k \right).$$

Representing $\phi(C_2)$ in the form

$$\phi(C_2) = \sum_i \left(\left(E_{ii} + \frac{N+1}{2} - i \right)^2 - \left(\frac{N+1}{2} - i \right)^2 \right),$$

we see that it is just p_2.

Remark Since the Harish–Chandra isomorphism can be applied to arbitrary elements of $ZU(\mathfrak{gl}_N)$, we can also apply it to the values of $w_{\mathfrak{gl}}$ on permutations.

For $k > 2$, the expression for $\phi(C_k)$ is not reduced to just linear combinations of power sums. If fact, we have the following explicit formula, which follows from[15, § 61] and [11, Remark 2.1.20],

$$\begin{aligned} 1 - Nu - \sum_{k=1}^{\infty} \phi(C_k) u^{k+1} &= \prod_{i=1}^{N} \frac{1-(E_{ii}+1)u}{1-E_{ii}u} \\ &= (1-Nu) e^{\sum_{k=1}^{\infty} \frac{(1-\frac{N-1}{2}u)^{-k} - (1-\frac{N+1}{2}u)^{-k}}{k} u^k p_k} \end{aligned}$$

This provides an expression for the image $\phi(C_k)$ of C_k as a polynomial in $p_1, p_2, \ldots$, which is valid for all N. The projections of the Casimir elements $C_1, \cdots, C_N$ to $U(\mathfrak{h})$ can be expressed in shifted power sums $p_1, \cdots, p_N$ in the following way:

$$\begin{aligned} &\phi(C_1) = p_1 \\ &\phi(C_2) = p_2 \\ &\phi(C_3) = -\frac{1}{4}N^2 p_1 + \frac{Np_2}{2} + \frac{p_1}{4} + p_3 - \frac{p_1^2}{2} \\ &\phi(C_4) = -\frac{1}{4}N^3 p_1 + N\left(-\frac{p_1^2}{2} + \frac{p_1}{4} + p_3\right) - p_1 p_2 + \frac{p_2}{2} + p_4 \\ &\quad \ldots \end{aligned}$$

For the values of the $\mathfrak{gl}$ weight system on the chord diagrams K_n, this yields

Result

$$
\begin{aligned}
w_{\mathfrak{gl}}(K_2) &= -Np_2 + p_1^2 + p_2^2\\
w_{\mathfrak{gl}}(K_3) &= 2N^2p_2 + N\left(-2p_1^2 - 3p_2^2\right) + p_2^3 + 3p_1^2p_2\\
w_{\mathfrak{gl}}(K_4) &= -7N^3p_2 + N^2\left(8p_1^2 + 11p_2^2\right) + N\left(-6p_2^3 - 14p_1^2p_2 - p_2 + 2p_4\right)\\
&\quad + 3p_1^4 + 6p_2^2p_1^2 - 8p_3p_1 + p_2^4 + 6p_2^2\\
w_{\mathfrak{gl}}(K_5) &= 36N^4p_2 + N^3\left(-48p_1^2 - 55p_2^2\right) + N^2\left(35p_2^3 + 80p_1^2p_2 + 20p_2\right)\\
&\quad + N\left(-20p_1^4 - 50p_2^2p_1^2 - 8p_1^2 - 10p_2^4 - 77p_2^2 + 10p_2p_4 - 24p_4\right)\\
&\quad + p_2^5 + 10p_1^2p_2^3 + 30p_2^3 + 15p_1^4p_2 - 40p_1p_3p_2 + 96p_1p_3\\
w_{\mathfrak{gl}}(K_6) &= -243N^5p_2 + N^4\left(376p_1^2 + 361p_2^2\right) + N^3\left(-240p_2^3 - 593p_1^2p_2 - 334p_2 + 252p_4\right)\\
&\quad + N^2\left(160p_1^4 + 405p_2^2p_1^2 + 232p_1^2 - 1088p_3p_1 + 85p_2^4 + 999p_2^2 - 174p_2p_4\right)\\
&\quad + N(-15p_2^5 - 130p_1^2p_2^3 - 537p_2^3 + 30p_4p_2^2 - 165p_1^4p_2 - 159p_1^2p_2\\
&\quad + 696p_1p_3p_2 - 31p_2 + 30p_1^2p_4 + 68p_4 - 16p_6)\\
&\quad + 15p_1^6 + 45p_2^2p_1^4 + 48p_1^4 - 120p_3p_1^3 + 15p_2^4p_1^2 + 90p_2^2p_1^2\\
&\quad - 120p_2^2p_3p_1 - 192p_3p_1 + 96p_5p_1 + p_2^6 + 90p_2^4 + 144p_2^2 + 160p_3^2 - 240p_2p_4\\
w_{\mathfrak{gl}}(K_7) &= 2022N^6p_2 + N^5\left(-3580p_1^2 - 2947p_2^2\right) + N^4\left(1981p_2^3 + 5446p_1^2p_2 + 5556p_2 - 2808p_4\right)\\
&\quad + N^3\left(-1568p_1^4 - 3773p_2^2p_1^2 - 5336p_1^2 + 13280p_3p_1 - 770p_2^4 - 14108p_2^2 + 2408p_2p_4\right)\\
&\quad + N^2(175p_2^5 + 1435p_1^2p_2^3 + 8505p_2^3 - 714p_4p_2^2 + 1750p_1^4p_2 + 5258p_1^2p_2\\
&\quad - 10192p_1p_3p_2 + 1862p_2 - 644p_1^2p_4 - 2712p_4 + 544p_6)\\
&\quad + N(-210p_1^6 - 735p_2^2p_1^4 - 1656p_1^4 + 2576p_3p_1^3 - 280p_2^4p_1^2 - 2877p_2^2p_1^2\\
&\quad + 210p_2p_4p_1^2 - 524p_1^2 + 2856p_2^2p_3p_1 + 8800p_3p_1 - 3264p_5p_1 - 21p_2^6 - 2177p_2^4\\
&\quad - 6985p_2^2 - 4096p_3^2 + 70p_2^3p_4 + 7292p_2p_4 - 112p_2p_6)\\
&\quad + p_2^7 + 21p_1^2p_2^5 + 210p_2^5 + 105p_1^4p_2^3 + 630p_1^2p_2^3 - 280p_1p_3p_2^3 + 2352p_2^3 - 1680p_4p_2^2\\
&\quad + 105p_1^6p_2 + 336p_1^4p_2 + 1120p_3^2p_2 - 840p_1^3p_3p_2 - 4032p_1p_3p_2 + 672p_1p_5p_2 + 1344p_1^2p_4
\end{aligned}
$$

6 Hopf algebra structure and projection to primitives

Multiplicative weight systems often become simpler when restricted to primitive elements in the Hopf algebra of chord diagrams. This is true, in particular, for the weight systems associated to metrized Lie algebras. The degree of the value of such a weight system $w_{\mathfrak{g}}$ on a chord diagram with n chords is $2n$, while for the projection of the chord diagram to primitives it is at most n, see [3]. In many cases, knowing the value of a weight system on projections to primitives allows one to understand its structure.

In this section, we recall the Hopf algebra structure on the algebra of chord diagrams modulo 4-term relations, and discuss the values of $w_{\mathfrak{gl}}$ on projections of chord diagrams to primitives.

Definition 6.1 The *coproduct* Δ of a chord diagram D is defined by

$$\Delta(D) := \sum_{J \subseteq [D]} D_J \otimes D_{\bar{J}},$$

where the summation is taken over all subsets J of the set $[D]$ of chords of D. Here D_J is the chord subdiagram of D consisting of the chords that belong to J and $\bar{J} = [D] \setminus J$ is the complementary subset of chords.

Claim 6.2 *The algebra of chord diagrams modulo* 4*-term relations endowed with the above coproduct is a graded commutative, cocommutative and connected Hopf algebra.*

Definition 6.3 An element p of a Hopf algebra is called *primitive* if $\Delta(p) = 1 \otimes p + p \otimes 1$.

The Milnor–Moore theorem, when applied to the Hopf algebra of chord diagrams, asserts that this Hopf algebra admits a decomposition into the direct sum of the subspace of primitive elements and the subspace of decomposable elements (polynomials in primitive elements of smaller degree). There exists, therefore, a natural projection from the space of chord diagrams to the subspace of primitive elements, whose kernel is the subspace of decomposable elements. We denote this projection by π.

Theorem 6.4 ([9, 12]) *The projection $\pi(D)$ of a chord diagram D to the subspace of primitive elements is given by the formula*

$$\begin{aligned}\pi(D) &= D - 1! \sum_{[D_1]\sqcup[D_2]=[D]} D_1 \cdot D_2 + 2! \sum_{[D_1]\sqcup[D_2]\sqcup[D_3]=[D]} D_1 \cdot D_2 \cdot D_3 \cdots \\ &= D - \sum_{i=2}^{|[D]|} (-1)^i (i-1)! \sum_{\substack{\bigsqcup_{j=1}^{i} [D_j]=[D] \\ [D_j]\neq\emptyset}} \prod_{j=1}^{i} D_j\end{aligned}$$

where the sum is taken over all unordered splittings of the set of chords of D into $2, 3$, etc nonempty subsets.

In particular, the chord diagrams K_n generate a graded Hopf subalgebra in the Hopf algebra of chord diagrams (since any chord subdiagram of K_n is K_k, for some k). Rewriting the formula for the projection for the exponential generating series $1 + \sum_{n=1}^{\infty} K_n \frac{x^n}{n!}$ we obtain

Corollary 6.5 *The generating series for the projections $\pi(K_n)$ to the subspace of primitive elements is given by the formula*

$$\sum_{n=1} \pi(K_n) \frac{x^n}{n!} = \log\left(1 + \sum_{n=1} K_n \frac{x^n}{n!}\right) \tag{3}$$

Now, knowing the values of the $\mathfrak{gl}$ weight system on the diagrams K_n for $n = 1, 2, \ldots, 7$, we easily obtain the values $\bar{w}_{\mathfrak{gl}} = w_{\mathfrak{gl}} \circ \pi$ on their projections to primitives:

Result

$$\bar{w}_{\mathfrak{gl}}(K_2) = -NC_2 + C_1^2$$
$$\bar{w}_{\mathfrak{gl}}(K_3) = 2N^2C_2 - 2NC_1^2$$
$$\bar{w}_{\mathfrak{gl}}(K_4) = -6C_2N^3 + (6C_1^2 - 2C_3)N^2 + (6C_1C_2 - 2C_2 + 2C_4)N - 4C_1^3 + 2C_1^2 + 6C_2^2 - 8C_1C_3$$
$$\begin{aligned}\bar{w}_{\mathfrak{gl}}(K_5) = {}& 24C_2N^4 + (24C_3 - 24C_1^2)N^3 + (-72C_1C_2 + 32C_2 - 24C_4)N^2 \\ & + (48C_1^3 - 32C_1^2 + 96C_3C_1 - 72C_2^2)N\end{aligned}$$
$$\begin{aligned}\bar{w}_{\mathfrak{gl}}(K_6) = {}& -120C_2N^5 + (120C_1^2 - 240C_3)N^4 + (720C_1C_2 - 416C_2 + 224C_4)N^3 \\ & + (-480C_1^3 + 416C_1^2 - 896C_3C_1 + 792C_2^2 - 88C_3 + 32C_5)N^2 \\ & + (-240C_2C_1^2 + 264C_2C_1 - 160C_4C_1 - 72C_2 + 64C_2C_3 + 88C_4 - 16C_6)N \\ & + 120C_1^4 - 176C_1^3 + 72C_1^2 - 192C_1C_2^2 + 264C_2^2 + 160C_3^2 + 256C_1^2C_3 \\ & - 352C_1C_3 - 240C_2C_4 + 96C_1C_5\end{aligned}$$
$$\begin{aligned}\bar{w}_{\mathfrak{gl}}(K_7) = {}& 720C_2N^6 + (2400C_3 - 720C_1^2)N^5 + (-7200C_1C_2 + 5264C_2 - 1856C_4)N^4 \\ & + (4800C_1^3 - 5264C_1^2 + 7424C_3C_1 - 9168C_2^2 + 3392C_3 - 1088C_5)N^3 \\ & + (7200C_2C_1^2 - 10176C_2C_1 + 5440C_4C_1 + 3456C_2 - 2176C_2C_3 - 3392C_4 + 544C_6)N^2 \\ & + (-3600C_1^4 + 6784C_1^3 - 8704C_3C_1^2 - 3456C_1^2 + 6528C_2^2C_1 + 13568C_3C_1 \\ & - 3264C_5C_1 - 10176C_2^2 - 4096C_3^2 + 6816C_2C_4)N + 1344C_2^3 - 2688C_1C_2C_3 + 1344C_1^2C_4\end{aligned}$$

In the basis $p_1, p_2, \ldots$ of shifted power series, these formulas look simpler:

Result

$$\bar{w}_{\mathfrak{gl}}(K_2) = -Np_2 + p_1^2$$
$$\bar{w}_{\mathfrak{gl}}(K_3) = 2N^2p_2 - 2Np_1^2$$
$$\bar{w}_{\mathfrak{gl}}(K_4) = -7N^3p_2 + 8N^2p_1^2 + N(2p_4 - p_2) + 6p_2^2 - 8p_1p_3$$
$$\bar{w}_{\mathfrak{gl}}(K_5) = 36N^4p_2 - 48N^3p_1^2 + 20N^2p_2 + N\left(-8p_1^2 - 72p_2^2 - 24p_4\right) + 96p_1p_3$$
$$\begin{aligned}\bar{w}_{\mathfrak{gl}}(K_6) = {}& -243N^5p_2 + 376N^4p_1^2 + N^3(252p_4 - 334p_2) + N^2\left(232p_1^2 - 1088p_3p_1 + 864p_2^2\right) \\ & + N\left(-96p_2p_1^2 - 31p_2 + 68p_4 - 16p_6\right) + 48p_1^4 + 144p_2^2 + 160p_3^2 - 192p_1p_3 - 240p_2p_4 + 96p_1p_5\end{aligned}$$
$$\begin{aligned}\bar{w}_{\mathfrak{gl}}(K_7) = {}& 2022N^6p_2 - 3580N^5p_1^2 + N^4(5556p_2 - 2808p_4) + N^3\left(-5336p_1^2 + 13280p_3p_1 - 11280p_2^2\right) \\ & + N^2\left(2976p_2p_1^2 + 1862p_2 - 2712p_4 + 544p_6\right) \\ & + N\left(-1488p_1^4 - 524p_1^2 + 8800p_3p_1 - 3264p_5p_1 - 6768p_2^2 - 4096p_3^2 + 6816p_2p_4\right) \\ & + 1344p_2^3 - 2688p_1p_2p_3 + 1344p_1^2p_4\end{aligned}$$

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