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ORIGINAL PAPER

On partial descriptions of König graphs for odd paths and all their spanning supergraphs

Dmitry B. Mokeev[1,2] **· Dmitry S. Malyshev**[2]

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Abstract

We consider graphs, which and all induced subgraphs of which possess the following property: the maximum number of disjoint paths on k vertices equals the minimum cardinality of vertex sets, covering all paths on k vertices. We call such graphs König for the k-path and all its spanning supergraphs. For each odd k, we reveal an infinite family of minimal forbidden subgraphs for them. Additionally, for every odd k, we present a procedure for constructing some of such graphs, based on the operations of adding terminal subgraphs and replacement of edges with subgraphs.

Keywords k-path packing · k-path cover · Minimal forbidden subgraph · Constructive description

1 Introduction

Let G be a simple graph and $\mathcal{X}$ be a set of simple graphs. Any subgraph (respectively, induced subgraph) of G, isomorphic to an element of $\mathcal{X}$, is called an $\mathcal{X}$*-subgraph* (respectively, an *induced* $\mathcal{X}$*-subgraph*). An arbitrary set of pairwise vertex-disjoint induced $\mathcal{X}$-subgraphs of G is called an $\mathcal{X}$*-packing* of G. An arbitrary subset of vertices

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✉ Dmitry S. Malyshev
dsmalyshev@rambler.ru

Dmitry B. Mokeev
MokeevDB@gmail.com

1 Lobachevsky State University of Nizhny Novgorod, 23 Gagarina ave., Nizhny Novgorod, Russia 603950

2 Laboratory of Algorithms and Technologies for Networks Analysis, National Research University Higher School of Economics, 136 Rodionova Str., Nizhny Novgorod, Russia 603093

of G, having a nonempty intersection with vertex set of each induced $\mathcal{X}$-subgraph of G, is called an *$\mathcal{X}$-cover* of G.

The maximum number of elements in $\mathcal{X}$-packings of a graph G is denoted by $\mu_{\mathcal{X}}(G)$, the minimum number of vertices in its $\mathcal{X}$-covers is denoted by $\beta_{\mathcal{X}}(G)$. A graph G is called *König for $\mathcal{X}$* if $\mu_{\mathcal{X}}(H) = \beta_{\mathcal{X}}(H)$, for every induced subgraph H of the graph G. The class of all König graphs for $\mathcal{X}$ is denoted by $\mathcal{K}(\mathcal{X})$. For $\mathcal{X} = \{P_2\}$, where P_n is the simple path on n vertices, $\mathcal{X}$-packings are known as *matchings* and $\mathcal{X}$-covers are known as *vertex covers*. The known König theorem claims that, for any bipartite graph G, we have $\mu_{P_2}(G) = \beta_{P_2}(G)$. The converse statement is also true in a certain sense: if this equality holds for a graph G and any its induced subgraph, then G is bipartite. These remarks was the reason to choose the name for graphs in $\mathcal{K}(\mathcal{X})$, firstly introduced in [1].

The class $\mathcal{K}(\mathcal{X})$ is *hereditary* for every $\mathcal{X}$, i.e. $\mathcal{K}(\mathcal{X})$ is closed under vertex removal. It is known that any hereditary class can be defined by the set of its *minimal forbidden induced subgraphs*, i.e. minimal under deletion of vertices graphs not belonging to the class.

The *$\mathcal{X}$-packing* and *$\mathcal{X}$-cover problems*, i.e. the problems for computing $\mu_{\mathcal{X}}(G)$ and $\beta_{\mathcal{X}}(G)$ for a given graph G, are dual as being formulated as integer linear programming problems. Hence, König graphs for $\mathcal{X}$ are exactly instances of these problems, having hereditarily zero duality gap. There is a conjecture claiming that for any $\mathcal{X}$ the $\mathcal{X}$-packing and $\mathcal{X}$-cover problems can be solved in polynomial time on König graphs for $\mathcal{X}$. It is still open, but some advances here give a support for confirmation of this conjecture (see, for example, [2]).

Note that, in literature, an $\mathcal{X}$-cover also often means a set of vertices in G, covering all $\mathcal{X}$-subgraphs of G, not necessarily induced, see, for example, [10,13]. However, any induced $\mathcal{X}$-subgraph is a spanning subgraph of some $\mathcal{X}$-subgraph.

Denote by $\langle\mathcal{X}\rangle$ the set of all spanning supergraphs of all graphs in $\mathcal{X}$, i.e. the set of graphs, containing all graphs in $\mathcal{X}$ and all graphs, obtained from them by adding edges. Hereafter, we also use the notation $\langle H\rangle$ (instead of $\langle\{H\}\rangle$) for the set of all spanning supergraphs of a graph H. Any $\langle\mathcal{X}\rangle$-cover is a vertex set, whose removal produces a subgraph, containing no $\mathcal{X}$-subgraphs. Any $\langle\mathcal{X}\rangle$-packing is a set of $\mathcal{X}$-subgraphs, pairwise not containing common vertices.

Several papers on the $\mathcal{X}$-packing and $\mathcal{X}$-cover problems are devoted to their algorithmic aspects (see, for example, [4,9,22]). It is known that the $\mathcal{X}$-packing problem is NP-hard for any $\mathcal{X}$, containing a graph, having a connected component with three or more vertices [11]. It is also known that the P_2-cover problem is NP-hard [8].

Several papers are devoted to algorithmic aspects of the $\mathcal{X}$-packing and $\mathcal{X}$-cover problems both for the induced and general cases, where $\mathcal{X}$ consists of P_k. It is known, in particular, that the $\langle P_k\rangle$-cover problem is NP-hard, for any $k \geq 2$ [3,21], and the $\langle P_k\rangle$-packing problem is polynomial-time solvable, for $k = 2$ [7], NP-hard, for $k \geq 3$ [11,15], and APX-hard, for $k \geq 4$ [6].

It is known, however, that the P_k- and $\langle P_k\rangle$-packing problems as well as the P_k- and $\langle P_k\rangle$-cover problems can be solved in linear time in the class of forests, for any k [3,15]. Moreover, more complex graph classes are known, on which these problems can be solved for various k in polynomial time [5,12,20], including some classes of König graphs and their subclasses (see [2,16,19]).

The present paper continues a series of investigations carried out earlier for the graph classes $\mathcal{K}(\langle P_3\rangle)$ [2] and $\mathcal{K}(\langle P_5\rangle)$ [18]. These classes were appeared to be *strongly hereditary* or *monotone*, i.e. they are closed under deletion of vertices and edges. Any strongly hereditary class can be defined by the set of its *minimal forbidden subgraphs*, i.e. graphs, minimal under deletion of vertices and edges, not belonging to the class.

In the case of general $\mathcal{X}$, the class $\mathcal{K}(\langle\mathcal{X}\rangle)$ is hereditary, but not strongly hereditary. Indeed, let us consider the class $\mathcal{K}(\langle claw\rangle)$, where $claw$ is the complete bipartite graph with one vertex in the first part and three vertices in the second one. The graph B is the graph of a 3-dimensional cube, and the graph B' is obtained from B by deleting all edges of some cycle with 4 vertices of B. Clearly, $B' \notin \mathcal{K}(\langle claw\rangle)$, as $\mu_{\langle claw\rangle}(B') = 1$, $\beta_{\langle claw\rangle}(B') = 2$, and $B \in \mathcal{K}(\langle claw\rangle)$. Therefore, for any $\mathcal{X}$, all minimal forbidden fragments for $\mathcal{K}(\langle\mathcal{X}\rangle)$ are split into two categories: minimal forbidden induced subgraphs and minimal forbidden subgraphs.

In [2,18], for $\mathcal{K}(\langle P_3\rangle)$ and $\mathcal{K}(\langle P_5\rangle)$, complete descriptions of minimal forbidden subgraphs have been obtained as well as constructive descriptions have been presented, i.e. procedures to construct any graph from the classes. A similar result was obtained in [14] for $\mathcal{K}(\langle P_4\rangle)$. In [1,2], for $\mathcal{K}(P_3)$, a complete description of minimal forbidden induced subgraphs have been obtained. In [17], for $\mathcal{K}(P_4)$, a partial description of minimal forbidden induced subgraphs have been obtained.

In the present paper, we give a generalization of the descriptions, mentioned above, for the parametric family $\{\mathcal{K}(\langle P_k\rangle) : \ k \text{ is odd}\}$. In Sect. 3, we reveal some infinite family of minimal forbidden subgraphs for each class from the family. In Sect. 4, for any odd k, we define the class of RT_k-graphs, obtained from pseudographs by applying replacement of edges with subgraphs and adding the so-called terminal subgraphs, and prove that each such a class is contained in $\mathcal{K}(\langle P_k\rangle)$.

2 Notation

2.1 Basic notation

Throughout the paper, k is the main parameter for class $\mathcal{K}(\langle P_k\rangle)$. It is always odd and means the number of vertices in the basic path. Throughout the paper, s equals $\frac{k-1}{2}$, i.e. $k = 2s + 1$.

We use the notation K_n, O_n, P_n, C_n for the complete graphs, empty graphs, simple paths and simple cycles on n vertices, respectively.

An edge, incident to vertices x and y, is denoted by xy. An arbitrary path $(x, v_1, v_2, \ldots, v_t, y)$ is denoted by x--y. To accent that such a path contains a vertex z or an edge z_1z_2, we use the notation x--z--y and x--z_1z_2--y, respectively.

We denote by $V(G)$ the set of vertices of a graph G. The set of vertices, adjacent to a vertex v, is called the *neighbourhood* of v and denoted by $N_G(v)$.

We call a $\langle P_k\rangle$-subgraph of G its *k-tuple*.

We use the notation $Free(\mathcal{Y})$ for the class of all graphs, all induced subgraphs of which are not in $\mathcal{Y}$.

2.2 Some operations with graphs and subgraphs

Let G be a graph, $A \subseteq V(G)$. We denote by $G[A]$ the subgraph, induced by A. We denote by $G \backslash A$ the subgraph, induced by $V(G) \backslash A$. If A consists from one vertex v, then we denote $G \backslash \{v\}$ as $G \backslash v$.

The subgraph of G, obtained by deleting an edge e, is denoted by $G \backslash e$. Let H be an induced subgraph of G and $v \in V(G)$, where $v \in V(G) \backslash V(H)$. The graph $G[V(H) \cup \{v\}]$ is denoted by $H + v$. Let H_1 and H_2 be induced subgraphs of G with $V(H_1) \cap V(H_2) = \emptyset$. The graph $G[V(H_1) \cup V(H_2)]$ is denoted by $H_1 + H_2$.

2.3 Distances, lengths, and critical paths

The distance between vertices u, v is denoted by $d(u, v)$. The length of paths x--y and x--y--z is denoted by $d(x\text{--}y)$ and $d(x\text{--}y\text{--}z)$, respectively.

We call *a critical path* between u and v a simple path u--v of the maximum length. We call the *critical distance* between vertices u, v the length of a critical path u--v and denote it as $d_c(u, v)$, i.e. $d_c(u, v) = \max_{u\text{--}v} d(u\text{--}v)$. Note that the critical distance is not a metric (for example, $d_c(v, v) \neq 0$, for any cyclic vertex v).

2.4 Specific notation

We denote by F_n the set of all connected graphs in $Free(\langle P_n \rangle)$. Its elements will be called *F_n-graphs*. Note that, in each F_n-graph, the critical distance between any pair of its vertices is no more than $n - 2$. Also, note that $F_1 = \emptyset$, $F_2 = \{K_1\}$.

We denote by T_n^* the set of all rooted trees with the following properties:

1. For each leaf l, we have $d(r, l) = n - 1$, where r is the root.
2. For each pair of non-intersected paths a--b and c--d, containing no r, where a, c are leaves, the inequality $d(a, b) + d(c, d) \leq 2n - 3$ is true.

We denote by $\mathbb{Z}_n$ the set $\{0, 1, \ldots, n - 1\}$.

Considering a cycle C_n, we assume that its vertices are labeled along the cycle as $0, 1, \ldots, n - 1$. If $n = km, m \in \mathbb{N}$, then each residue class of vertices modulo k is called a *k-class*. The k-class, which contains a vertex v, is denoted by $Q_k(v)$.

We call the *class distance* $d_k(u, v)$ between vertices u, v the minimum distance between vertices of their k-classes. In other words,

$$d_k(u, v) = \min_{x \in Q_k(u), y \in Q_k(v)} d(x, y).$$

3 Forbidden subgraphs

It was proved in [16] that, for each $k \geq 4$, every forest is a König graph for P_k (this follows from Theorem 1 from [16]). Combining this fact with the fact that any forest has no other $\langle P_k \rangle$-subgraphs, except P_k, we have

Lemma 1 *For every k, each forest is a König graph for $\langle P_k \rangle$.*

For any $i \in \{0, 1, \dots, k-1\}$ and $n \geq 2$, we have

$$\mu_{\langle P_k \rangle}(C_{kn+i}) = \beta_{\langle P_k \rangle}(C_{kn-i}) = n \text{ and } \mu_{\langle P_k \rangle}(C_{k+i}) = \beta_{\langle P_k \rangle}(C_k) = 1.$$

By this fact and by Lemma 1, we have

Theorem 1 *The cycle C_n belongs to $\mathcal{K}(\langle P_k \rangle)$ if $n \leq k$ or k divides n. It is a minimal forbidden subgraph for $\mathcal{K}(\langle P_k \rangle)$, if $n > k$ and k does not divide n.*

3.1 Medusas

Let us consider the family of connected graphs, obtained from a simple cycle by adding a set of simple paths in such a way, that exactly one end vertex of each added path is a vertex of the cycle. We call such graphs *medusas*.

Denote by $M_n(m_1, m_2, \dots, m_n)$ the medusa, obtained from the cycle C_n by adding the paths of length $m_1, m_2, \dots, m_n$ to the vertices $1, 2, \dots, n$, respectively. If there is no a path to add to a vertex i, we assume that such a path has one vertex and its length equals 0, i.e. $m_i = 0$ in this case. We call the added paths of non-zero length as the *tentacles* of the medusa.

Theorem 2 *The graph $M_{kn}(m_1, m_2, \dots, m_{kn})$ belongs to $\mathcal{K}(\langle P_k \rangle)$ iff*

$$\exists i \in \mathbb{Z}_k : \forall j \in \mathbb{Z}_{nk} \backslash Q_k(i) : m_j < d_k(i, j). \tag{1}$$

Proof Put $M = M_{kn}(m_1, m_2, \dots, m_{kn})$ and denote the cycle of M by C_M. From the remark before the statement of Theorem 1, C_M contains a $\langle P_k \rangle$-packing of the cardinality n. Similarly, each k-class of C_M is its $\langle P_k \rangle$-cover of the same cardinality. We will formulate a criterion whether a k-class $Q_k(i)$ of C_M is a $\langle P_k \rangle$-cover of the whole graph M.

The graph $M \backslash Q_k(i)$ is a forest. The set $Q_k(i)$ is a $\langle P_k \rangle$-cover of the whole graph M iff each connected component of $M \backslash Q_k(i)$ belongs to F_k. Let T be one of those connected components. It is induced by the vertices $kt + 1, kt + 2 \dots, kt + k - 1$, where $t \in \mathbb{N}$, and by the corresponding tentacles. Let us consider the case $t = 0$. All the cases $t > 0$ are similar.

We have $T \in F_k$ iff $m_{i+p} + m_{i+q} + d(i + p, i + q) \leq k - 2$, for any $p, q \in \{1, 2, \dots, k - 1\}$. Without loss of generality, assume $p < q$. Then,

$$d(i + p, i + q) = q - p, \text{ i.e. } m_{i+p} + m_{i+q} \leq k + p - q - 2.$$

Substituting $p = 1, q = k-1$, we get $m_{i+1} + m_{i+k-1} \leq 0$, i.e. $m_{i+1} = m_{i+k-1} = 0$. Then, substituting $p = 1, q = k-r$, we get $m_{i+k-r} \leq r-1$, substituting $p = r, q = k - 1$, we get $m_{i+r} \leq r - 1$ (in both cases, r is an arbitrary integer from 1 to s). Note that, for the vertices $i + r$ and $i - r$, the distances to the nearest vertices from $Q_k(i)$ equal r, i.e. $d_k(i, i + r) = d_k(i, i - r) = r$. Thus, $m_j < d_k(i, j)$, for every vertex $j \in \{i + 1, i + 2, \dots, i + k - 1\}$.

Thus, $Q_k(i)$ is a $\langle P_k\rangle$-cover of the graph M iff $\forall j \in \mathbb{Z}_{nk}\backslash Q_k(i) : m_j < d_k(i, j)$. In this case, obviously, $\mu_{\langle P_k\rangle}(M) = \beta_{\langle P_k\rangle}(M) = n$. Since each subgraph of M is either a medusa with the same properties or a forest, we, by Lemma 1, conclude that $M \in \mathcal{K}(\langle P_k\rangle)$. Thus, the sufficiency is proved.

Let us prove the necessity. Let now M be a graph, which does not satisfy the condition (1). It means that the following condition holds:

$$\forall i \in \mathbb{Z}_k : \exists j \in \mathbb{Z}_{nk}\backslash Q_k(i) : m_j \geq d_k(i, j). \tag{2}$$

Moreover, if at least one m_j is decreased by 1, then the obtained graph satisfies the condition (1). So, in addition, for M the following condition holds:

$$\begin{aligned}\forall t \in \mathbb{Z}_{nk}, m_t \geq 1 : \exists l(t) \in \mathbb{Z}_k : \forall j \in \mathbb{Z}_{nk}\backslash Q_k(l(t))\backslash\{t\} :\\ m_j < d_k(l(t), j) \text{ and } m_t \leq d_k(l(t), t).\end{aligned} \tag{3}$$

It is followed from (2) that any k-class is a not $\langle P_k\rangle$-cover of the graph M. So, $\beta_{\langle P_k\rangle}(M) \geq n + 1$.

Let $m_t \geq 1$. Then, it is followed from (2) and (3) that $m_t = d_k(l(t), t)$. Suppose that m_t is the minimum non-zero number for all $t \in Q_k(t)$. But, $m_{t'} < d_k(l(t), t') = d_k(l(t), t) = m_t$, for each $t' \in Q_k(t)\backslash\{t\}$. So, $m_{t'} = 0$.

Thus, it can be $m_t \geq 1$, for no more than one number t from each k-class. So, we can associate the index of any tentacle with the number of its k-class. Without loss of generality, we can assume that $m_t \geq 1$ only for $t \in \mathbb{Z}_k$. Then, $l(t)$ equals either $t + m_t$ or $t - m_t$.

Denote by S the sum of all m_t in M. For any $i \in \mathbb{Z}_k$, denote by N_i the quantity of numbers $t \neq i$, such that $m_t \geq d_k(i, t)$. Note that, for each t, there are exactly $2m_t$ numbers i, such that $m_t \geq d_k(i, t)$ (they are all belong to $\{t - m_t, \dots t - 1, t + 1, \dots, t + m_t\}$). So, $\sum_{i=1}^{k} N_i = 2S$.

In addition, note that none of the numbers in $\{t - m_t, \dots t - 1, t + 1, \dots, t + m_t\}$ cannot be $l(t')$, for any $t' \in \mathbb{Z}_k$.

Consider the sequence $t_0, t_1, \dots, t_p$, such that $m_{t_j} \geq 1$, for each $j \in \{0, \dots, p\}$, $m_{l(t_p)} = 0$, and $t_j = l(t_{j-1}) = t_{j-1} - m_{t_{j-1}}$, for each $j \in \{1, \dots, p\}$. We prove by induction that there are exactly m_{t_1} numbers i between $l(t_p)$ and $t_1 - 1$, such that $N_i = 1$ and $N_i = 2$, for the others i.

Obviously, $N_i = 1$ holds, for each $i \in \{l(t_p), \dots, t_p - 1\}$, i.e. for m_{t_p} numbers of this set. Thus, if $p = 1$, then the condition is true.

Suppose that $N_i = 1$ holds for m_{t_j} numbers i from the set $\{l(t_p), \dots, t_j - 1\}$. Then, $N_i = 1$ holds, for $i = t_j$ and each $i \in \{t_j + m_{t_j} + 1, \dots, t_{j-1}\}$, and $N_i = 2$ holds, for each $i \in \{t_j + 1, \dots, t_j + m_{t_j}\}$. So, the quantity of numbers i from the set $\{l(t_p), \dots, t_j - 1\}$, where $N_i = 1$, equals

$$m_{t_j} + 1 + t_{j-1} - (t_j + m_{t_j} + 1) = t_{j-1} - t_j = m_{t_{j-1}}.$$

The same statement is true for the sequence $t_0, t_1, \dots, t_p$, such that $m_{t_j} \geq 1$, for each $j \in \{0, \dots, p\}$, $m_{l(t_p)} = 0$, and $t_j = l(t_{j-1}) = t_{j-1} + m_{t_{j-1}}$, for each $j \in \{1, \dots, p\}$.

Let $N_i^{'} = N_i - 2$, for each $i \in \mathbb{Z}_k$. Let $S^{'} = \sum_{i=1}^{k} N_i^{'}$. It is easy to see that $S^{'} = 2(S - k)$.

Consider a number $i \in \mathbb{Z}_k$ and the numbers t_1, t_2, such that $m_{t_j} \geq d_k(i, t_j)$, $j \in \{1, 2\}$ and $0 \geq t_1 - m_{t_1} < t - m_t$, for all $t \neq t_1, m_t \geq d_k(i, t)$, and $t + m_t < t_2 + m_{t_2} \geq k - 1$, for all $t \neq t_2, m_t \geq d_k(i, t)$. Note, that $t_1 + m_{t_1} \leq i \leq t_2 - m_{t_2}$. So, for each $t_1 - m_{t_1} \geq j \geq t_2 + m_{t_2}$ but, maybe, t_1 and t_2, either $m_{t_1} \geq d_k(j, t_1)$ or $m_{t_2} \geq d_k(j, t_2)$. So, if $m_t \geq d_k(i, t)$, for $t \notin \{t_1, t_2\}$, then $l(t)$ equals either t_1 or t_2.

Suppose that $N_i = 4$ (note that it cannot be more). Then, there are 4 numbers t_1, t_2, t_3, t_4, such that

$$t_1 < t_3 < t_4 < t_2 \text{ and } l(t_1) < t_1, l(t_2) > t_2, l(t_3) = t_1, l(t_4) = t_2.$$

The number of vertices between $l(t_1)$ and $l(t_2)$ equals

$$m_{t_1} + m_{t_3} + (t_4 - t_3 + 1) + m_{t_4} + m_{t_2},$$

which is not less than $m_{t_1} + m_{t_2} + m_{t_3} + m_{t_4} + 2$. Moreover, for each i from t_1 to t_2, the inequality $m_t \geq d_k(i, t)$ can be true only for $t \in \{t_1, t_2, t_3, t_4\}$. Since $l(t_3) = t_1$ and $l(t_4) = t_2$, we can consider two sequences $t_1, t_1^{'}, \ldots, t_p^{'}$ with the property (3.1) and $t_2, t_1^{''}, \ldots, t_q^{''}$ with the property (3.1). The number of vertices between $l(t_p^{'})$ and $l(t_1) - 1$ equals $m_{t_p^{'}} + m_{t_{p-1}^{'}} + \cdots + m_{t_1}$. The number of vertices between $l(t_2)$ and $l(t_q^{''}) - 1$ equals $m_{t_q^{''}} + m_{t_{q-1}^{''}} + \cdots + m_{t_2}$ Thus,

$$\sum_{l(t_p^{'})}^{l(t_q^{''})} N_i^{'} = \sum_{l(t_p^{'})}^{l(t_1)-1} N_i^{'} + \sum_{l(t_1)}^{t_1-1} N_i^{'} + \sum_{t_1}^{t_2} N_i^{'} + \sum_{t_2}^{l(t_2)} N_i^{'} + \sum_{l(t_2)+1}^{l(t_q^{''})} N_i^{'}$$
$$\leq m_{t_1} + m_{t_1} + m_{t_2} + 2m_{t_3} + 2m_{t_4} + m_{t_2} - 2(m_{t_1} + m_{t_2} + m_{t_3} + m_{t_4} + 2) = -4$$

Suppose that $N_i = 3$ and, for each j between t_1 and t_2, we have $N_j \leq 3$. Then, without loss of generality, there are 3 numbers t_1, t_2, t_3, such that

$$t_1 < t_3 < t_2, l(t_1) < t_1, l(t_2) > t_2, l(t_3) = t_1.$$

If $m_{l(t_2)} \geq 1$, then, similarly to the previous reasonings, we have

$$\sum_{l(t_p^{'})}^{l(t_q^{''})} N_i^{'} \leq -4.$$

Otherwise, for each i from t_1 to $l(t_2)$, the inequality $m_t \geq d_k(i, t)$ can be true only for $t \in \{t_1, t_2, t_3\}$. The quantity of vertices between $l(t_1)$ and $l(t_2)$ equals $m_{t_1} + m_{t_3} + (t_2 - t_3 + 1) + m_{t_2}$, which is not less than $m_{t_1} + m_{t_2} + m_{t_3} + 2$. Since $l(t_3) = t_1$, we can consider a sequence $t_1, t_1^{'}, \ldots, t_p^{'}$ with the property (3.1) and the quantity of vertices $m_{t_p^{'}} + m_{t_{p-1}^{'}} + \cdots + m_{t_1}$. Thus,

$$\sum_{l(t_p^{'})}^{l(t_2)} N_i^{'} = \sum_{l(t_p^{'})}^{l(t_1)-1} N_i^{'} + \sum_{l(t_1)}^{t_1-1} N_i^{'} + \sum_{t_1}^{l(t_2)} N_i^{'}$$
$$\leq m_{t_1} + m_{t_1} + 2m_{t_2} + 2m_{t_3} - 2(m_{t_1} + m_{t_2} + m_{t_3} + 2) = -4.$$

For the other numbers i, N_i equals either 1 or 2, i.e. $N_i^{'} \leq 0$. So, by the arguments above, if there exists i, for which $N_i \geq 3$, then $S^{'} \leq -4$.

Suppose that for each i, N_i equals either 1 or 2. For each t, where $m_t \geq 1$, $N_{l(t)} = 1$ holds and, therefore, $N_{l(t)}^{'} = -1$. Since there are at least two tentacles in M, the total sum of all $N_i^{'}$ is less or equal than -2.

Thus, in any case, $S^{'} = 2(S - k) \leq -2$ and, therefore, $S \leq k - 1$. It means that M has no more than $k(n + 1) - 1$ vertices, and, therefore, $\mu_{\langle P_k\rangle}(M) \leq n < \beta_{\langle P_k\rangle}(M)$. □

Corollary 1 *The graph $M_{kn}(m_1, m_2, \ldots, m_{kn})$ is a minimal forbidden subgraph for $\mathcal{K}(\langle P_k\rangle)$ iff the following conditions simultaneously hold:*

1. $\forall i \in \mathbb{Z}_k : \exists j \in \mathbb{Z}_{nk} \backslash Q_k(i) : m_j \geq d_k(i, j)$.
2. $\forall t, m_t \geq 1 : \exists i \in \mathbb{Z}_k : \forall j \in \mathbb{Z}_{nk} \backslash Q_k(i) \backslash \{t\} : m_j < d_k(i, j)$ *and* $m_t = d_k(i, t)$.

4 RT_k-graphs

In this Section, we describe the procedure of RT_k-extention of pseudographs and the class RT_k. We prove here that any RT_k-extention of every pseudograph is a König graph for $\langle P_k\rangle$.

Definition 1 We call a connected subgraph H of a graph G *terminal* if there exists a vertex $c \in V(G) \backslash V(H)$, such that c is adjacent to one or more vertices of H and H is a connected component of graph $G \backslash c$. We call c as a *contact vertex* of H.

By *adding a terminal subgraph H to a vertex $v \in V(G)$*, we mean that we add H to G and connect some vertices of H with v by new edges.

Definition 2 Let us define the recursive procedure of "cascade" adding some terminal F_i-subgraphs. We denote this procedure as $AddTF(v, i)$, where v is a vertex of a graph G and i is a positive integer.

If $i = 1$, then we cannot add any subgraph and have to stop the procedure.

If $i \geq 2$, then we choose some positive integers $j_1, j_2, \ldots, j_m$, each no more than i, and add to v some arbitrary terminal subgraphs $H_1, H_2, \ldots, H_m$, where $H_t \in F_{j_t}$, for each $t \in \{1, \ldots, m\}$. After this we can (not necessary) choose a vertex v_t in H_t and apply the procedure $AddTF(v_t, i - j_t + 1)$, for each $t \in \{1, \ldots, m\}$.

Lemma 2 *Let H be a terminal subgraph, added by the procedure $AddTF(v, i)$ to a vertex v of a graph G. Then, $d_c(v, x) \leq i - 1$, for each vertex x of H.*

Proof The proof is by induction on the number of applications for the procedure $AddTF$.

Assume that $AddTF(v, i)$ have been applied one time, i.e. there is no another procedure $AddTF$ inside. Then, $H \in F_i$ and $d(x, y) \leq i-2$, for each pair of vertices x, y of H. Thus, $d_c(v, x) \leq i - 1$, for each vertex x of H.

Now, suppose that the assertion is true for the procedure $AddTF$, having no more than k recursive calls. Let us consider the procedure $AddTF$, having $k + 1$ calls.

The first call of $AddTF$ is adding to v some terminal subgraphs $H_1, H_2, \dots, H_m$ and choosing vertices $v_1, \dots, v_m$ in them. Consider $t \in \{1, \dots, m\}$. By the definition, $j_t \leq i$ and H_t is a F_{j_t}-graph. Thus, $d_c(v, x) \leq j_t - 1 \leq i - 1$, for each vertex x of H_t.

The second call is applying the procedure $AddTF(v_t, i - j_t + 1)$. Let $H^{'}$ be a terminal subgraph, added by this procedure. By the inductive assumption $d_c(v_t, x) \leq i - j_t$, for each vertex x of $H^{'}$. Hence, we have

$$d_c(v, x) = d_c(v, v_t) + d_c(v_t, x) \leq j_t - 1 + i - j_t = i - 1.$$

□

Definition 3 Let G be a graph and xy be an edge of G. Define the procedure, which will be called the *(A, F_k)-extension of the edge xy*.

1. Delete xy from G.
2. Add to G all vertices and edges of two arbitrary T_s^*-graphs H_x and H_y. Let r_x, r_y be the roots and V_x, V_y be the sets of leaves of H_x and H_y, respectively.
3. Add the edge $r_x r_y$.
   All the following steps are described for H_x. The steps for H_y are similar.
4. Add the edge $u_x x$, for each $u_x \in V_x$.
5. For each vertex v of H_x, put

$$h(v) := \min\{d(v, x), \min_{\substack{v\text{--}x\text{--}w, \\ r_x \notin v\text{--}x\text{--}w}} \{k - d(v\text{--}x\text{--}w) - 1\}\}.$$

6. Perform all the steps of the following procedure:
   {
   while(true)
   (a) Chose an arbitrary vertex $v \in \{v : h(v) \geq 2\}$.
   (b) Apply the procedure $AddTF(v, h(v))$.
   (c) For each vertex w, where $r_x \notin v\text{--}x\text{--}w$, change

$$h(w) := \min\{h(w), k - d(v\text{--}x\text{--}w) - h(v)\}.$$

   }
   Hereafter, we denote by H_x^a and H_y^a the subgraphs, obtained from H_x and H_y after applying all the $AddTF$ procedures. Put $H^a = H_x^a + H_y^a$.

Lemma 3 *The graph H^a is a F_k-graph.*

Proof Since H_x and H_y are T_s^*-graphs, each maximum path in $H_x + H_y$ connects two vertices $u_x \in V_x, u_y \in V_y$ and

$$\begin{aligned} d_c(u_x, u_y) &= d(u_x, u_y) = d(u_x, r_x) + 1 + d(r_y, u_y) \\ &= s - 1 + 1 + s - 1 = 2s - 1 = k - 2. \end{aligned}$$

It means that any maximum path in $H_x + H_y$ consists of $k - 1$ vertices.

Let v be a vertex of H_x and u be a vertex from V_x, which is nearest to v. It is easy to see that $h(v) \leq d(v, u) + 1$ at every iteration of the step 6. Let $H^{''}$ be a terminal subgraph, added by the procedure $AddTF(v, h(v))$. By Lemma 2, we have $d_c(v, w) \leq h(v) - 1$, for each vertex w of $H^{''}$. Thus, $d_c(v, w) \leq d(v, u)$. But, every maximum path, beginning at w, contains v. So, it length cannot be more than $k - 2$. □

Corollary 2 *Both graphs H_x^a and H_y^a are F_k-graphs.*

Lemma 4 *Both graphs $H_x^a + x\backslash r_x$ and $H_y^a + y\backslash r_y$ are F_k-graphs.*

Proof We will prove it only for $H_x^a + x$.

If $|V_x| = 1$, then $H_x \cong P_s$. In this case, we have $h(v) = d(v, x)$, for each $v \in V(H_x)$. Let $v_1, v_2 \in V(H_x)$, such that $d(v_1, x) < d(v_2, x)$, $H_1^{''}$ and $H_2^{''}$ be terminal subgraphs, added by the procedures $AddTF(v_1, h(v_1))$ and $AddTF(v_2, h(v_2))$, respectively, w_1 and w_2 be vertices of $H_1^{''}$ and $H_2^{''}$, respectively. In the graph $H_x^{'} + x\backslash r_x$, we have

$$d_c(w_1, w_2) = d_c(w_1, v_1) + d_c(v_1, v_2) + d_c(v_2, w_2).$$

By Lemma 2, we have

$$\begin{aligned} d_c(w_1, w_2) &\leq h(v_1) - 1 + d(v_1, v_2) + h(v_2) - 1 \\ &\leq d(v_1, x) + d(v_1, v_2) + d(v_2, x) - 2. \end{aligned}$$

Since $d(v_1, x) < d(v_2, x)$, we have $d(v_2, x) = d(v_1, x) + d(v_1, v_2)$. Therefore,

$$d_c(w_1, w_2) \leq 2d(v_2, x) - 2 \leq 2s - 2 = k - 3.$$

Similarly, $d_c(w_1, x) \leq 2s - 1 = k - 2$ and $d_c(w_2, x) \leq 2s - 1 = k - 2$, i.e. $H_x^a + x\backslash r_x \in F_k$.

Now, assume that $|V_x| \geq 2$. As $H_x \in T_s^*$, any maximum path of $H_x + x\backslash r_x$ consists of the edges ax, cx, and two non-intersected paths a--b and c--d, where $a, c \in V_x$. Then, by the properties of T_s^*, we have

$$d_c(a, b) + d_c(c, d) = d(a, b) + d(c, d) \leq 2s - 3.$$

So, the maximum length of paths of $H_x + x\backslash r_x$ is $2s - 1 = k - 2$.

By the description of the step 5, we have $h(v) \leq k - d(v\text{--}x\text{--}w) - 1$, where $v\text{--}x\text{--}w$ has the maximum length among such paths containing no r_x.

Let H_x^i be the graph obtained from $H_x + x \backslash r_x$ after i iterations of the step 6. Now, we prove the two following statements for the graph H_x^i by induction on i.

1. The inequality $h(v) \leq k - d(v\text{--}x\text{--}w) - 1$ is true, for any vertices $v, w \in V(H_x^i)$.
2. The maximum length of paths in H_x^i is $k - 2$.

The induction base of the proof is $H_x^0 = H_x + x \backslash r_x$. It was described above. Assume that both statements hold for the graph H_x^{i-1}.

Let $v, w \in V(H_x^{i-1})$, such that path $v\text{--}x\text{--}w$ exists, and u be an arbitrary vertex of a terminal subgraph, added by the procedure $AddTF(v, h(v))$ on the i-th iteration of the step 6.

By the inductive assumption, we have $h(v) \leq k - d(v\text{--}x\text{--}w) - 1$. By Lemma 2, we have $d_c(v, u) \leq h(v) - 1$, i.e. $d_c(u, v) \leq k - d(v\text{--}x\text{--}w) - 2$. Then, $d_c(u, v) + d(v\text{--}x\text{--}w) \leq k - 2$. But, every path $u\text{--}x$ contains v, so the maximum length of paths $u\text{--}x\text{--}w$ is $k - 2$. Since it holds for arbitrary u and w, we proved the statement 2.

Now, we evaluate the value of $h(w)$ in H_x^i. We denote it by $h_i(w)$. Let $h(w)$ be the corresponding value in H_x^{i-1}. By the description of the step 6, we have

$$h_i(w) = min\{h(w), k - d(v\text{--}x\text{--}w) - h(v)\}.$$

By the inductive assumption, we have $h(w) \leq k - d(v\text{--}x\text{--}w) - 1$. So,

$$h_i(w) \leq h(w) \leq k - d(v\text{--}x\text{--}w) - 1, d(u\text{--}x\text{--}w) = d_c(u, v) + d(v\text{--}x\text{--}w),$$

for each path $u\text{--}x\text{--}w$. But $d_c(v, u) \leq h(v) - 1$. So,

$$d(u\text{--}x\text{--}w) \leq d(v\text{--}x\text{--}w) + h(v) - 1 \text{ and } d(v\text{--}x\text{--}w) \geq d(u\text{--}x\text{--}w) - h(v) + 1.$$

Finally,

$$\begin{aligned} h_i(w) &\leq k - d(v\text{--}x\text{--}w) - h(v) \leq k - d(u\text{--}x\text{--}w) + h(v) - 1 - h(v) \\ &= k - d(u\text{--}x\text{--}w) - 1. \end{aligned}$$

Since it holds for arbitrary u, v, and w, we proved the statement 1. □

Definition 4 Let G be a graph and xy be an edge of G and $i \geq 3$ be a natural. Let us define the procedure of the (B, F_i)-*extension of the edge* xy.

1. Delete xy from G.
2. Add to G an arbitrary T_i^*-graph H_x, in such a way that x is the root of H_x. Denote by V_x the set of leaves of H_x.
3. Add the edge $u_x y$, for each $u_x \in V_x$.
4. For each vertex v of H_x, put

$$h(v) := \min\{d(v, x), \min_{\substack{v\text{--}y\text{--}w,\\ x \notin v\text{--}y\text{--}w}} \{k - d(v\text{--}y\text{--}w) - 1\}\}.$$

5. Perform all the steps of the following procedure:
   {
   while(true)
   (a) Chose an arbitrary vertex $v \in \{v : \ h(v) \geq 2\}$
   (b) Apply the procedure $AddTF(v, h(v))$.
   (c) For each vertex w, where $x \notin v\text{--}y\text{--}w$, change
   $$h(w) := \min\{h(w), i - d(v\text{--}y\text{--}w) - h(v)\}.$$
   }
   Hereafter, we denote by H_x^n the subgraph, obtained from H_x after applying all the procedures $AddTF$.

Lemma 5 *The graph H_x^n is a F_{2i+1}-graph.*

Proof Since $H_x \in T_i^*$, its diameter cannot be more than $2(i-1)$. It means that any maximum path of H_x consists of $2(i-1)+1 = 2i-1$ vertices.

Let v be a vertex of H_x and u be a vertex from V_x, which is nearest to v. It is easy to see that $h(v) \leq d(v, u) + 1$ at every iteration of the step 5. Let $H^{''}$ be a terminal subgraph, added by the procedure $AddTF(v, h(v))$. By Lemma 2, we have $d_c(v, w) \leq h(v) - 1$, for each vertex w of $H^{''}$. Thus, $d_c(v, w) \leq d(v, u)$. But, every maximum path, beginning at w, contains v. So, it length cannot be more than $2i - 2$. □

Lemma 6 *The graph $H_x^n + y \backslash x$ is a F_{2i+1}-graph.*

The proof of this Lemma is the same as in the proof of Lemma 4.

Definition 5 Let M be a pseudograph. Let us define the procedure of *RT_k-extention of the pseudograph M* as follows:

1. Each cyclic edge of M must be extended by the (A, F_k)-extention.
2. Some non-cyclic edges of M can be extended by the (B, F_i)-extention, where $i \leq s$.
3. For some vertices $v \in V(M)$, the procedure $AddTF(v, k)$ can be applied.

Denote by RT_k the set of all RT_k-extentions of all pseudographs.

Theorem 3 *Every subgraph of each RT_k-graph is a RT_k-graph.*

Proof Let G be a RT_k-graph, which obtained by the RT_k-extention from a pseudograph M. Let $G^{'} = G \backslash e$ be the graph, which obtained from G by deleting an edge e.

We may consider, without loss of generality, that if one of the graphs H_x or H_y is empty after the (B, F_i)-extension of an edge, then the root of another one has at least two children. Otherwise, we can consider an root edge as the edge of M.

If e is a bridge of G, then it is a non-cyclic edge of M or an edge of some terminal subgraph added by the procedure $AddTF$, or connect such terminal subgraph with its contact vertex. In this cases, $G^{'}$ can be obtained by the RT_k-extention from $H \backslash e$ or $M \cup K_1$, respectively.

If e is not a bridge of G, then the following cases are possible:

1. The edge e is in a terminal subgraph T, added by $AddTF$, or connect such a terminal subgraph with its contact vertex. Then, $T\backslash e$ can be added by $AddTF$ to the same contact vertex. Hence, $G^{'}$ can be obtained by the RT_k-extention from M.
2. The edge e is in a subgraph, added by the (B, F_i)-extension of the edge xy, but not in a terminal subgraph. Suppose that e is in H_x. Its deletion divides H_x into two connected components. Denote by $H_x^{'}$ those of them, which contains x, and by $H_x^{''}$ the another one. It is easy to see that $H_x^{''}$ is a terminal F_i-subgraph of G, so it can be added to y by the procedure $AddTF$. In its turn, either $H_x^{'}\backslash x$ is a terminal F_i-subgraph of G too or a T_i^*-graph, each leaves of each are adjacent to y. Hence, $G^{'}$ can be obtained by the RT_k-extention from M.
3. The edge e is in a subgraph, added by the (A, F_k)-extension of the edge xy, but not in a terminal subgraph. If deletion of e breaks all the paths $x \notin v$--y in the corresponding H^c, then consider the pseudograph $M^{'} = M\backslash xy$.

Denote by $E^{'}$ the set of cyclic edges of M, which are non-cyclic in $M^{'}$. Subdivide each edge $ab \in E^{'}$ with two vertices r_a and r_b in such a manner as ar_a, br_b are the new edges. Add the vertices r_x, r_y and the edges xr_x if $e \notin H_x^c$, yr_y if $e \notin H_y^c$, r_xr_y if $e \neq r_xr_y$ in H^c (it is easy to see, that exactly 2 edges are added). Denote the obtained graph by $M^{''}$.

We can see now that each (A, F_k)-extension of the edge $ab \in E^{'}$ in M corresponds to some (B, F_s)-extension of ar_a, br_b in $M^{''}$ and, maybe, after applying some procedures $AddTF(r_a, k)$ and $AddTF(r_b, k)$. In addition, $H^c\backslash e$ can be obtained by some (B, F_s)-extension of xr_x and yr_y (if the corresponding edge exists) and, maybe, after applying the procedures $AddTF(r_x, k)$, $AddTF(r_y, k)$, $AddTF(x, k)$, $AddTF(y, k)$. This follows from the definition of the procedures and Lemma 4.

Hence, $G^{'}$ can be obtained by the RT_k-extention from $M^{''}$.

If there are some paths $x \notin v$--y in the corresponding H^c after deleting of e, then the connected component of H^c, containing r_x and r_y can still be added by the (A, F_k)-extension of the edge xy, and, by Lemma 4 the other components are terminal F_k-graphs, so, it can be added after applying the procedures $AddTF(x, k)$ or $AddTF(y, k)$.

Hence, $G^{'}$ can be obtained by the RT_k-extention from M. □

Theorem 4 *Each RT_k-graph is König for $\langle P_k\rangle$.*

Proof Let G be a RT_k-graph, which obtained by the RT_k-extention from a pseudograph M. Considering Lemma 3, we must only prove that $\mu_{\langle P_k\rangle}(G) = \beta_{\langle P_k\rangle}(G)$.

The proof is by induction on the edge number of G. If every connected component of G is a F_k-graph, then, obviously, $\mu_{\langle P_k\rangle}(G) = \beta_{\langle P_k\rangle}(G) = 0$.

Now, suppose that $G \notin F_k$ and $\mu_{\langle P_k\rangle}(G^{'}) = \beta_{\langle P_k\rangle}(G^{'})$ holds, for any subgraph $G^{'}$ of G with any fewer number of edges. We can assume for G to be connected. Since G contains at least one q-tuple, one of the following conditions holds for it:

1. The graph G contains a terminal F_k-subgraph T with a contact vertex y, such that $T + y$ contains a q-tuple. Or G contains a pair of terminal F_k-subgraphs T_1 and T_2 with a common contact vertex y, such that $T_1 + T_2 + y$ contains a q-tuple. Note that each q-tuple of $T + y$ and $T_1 + T_2 + y$ contains the vertex y. Let Q

be one of such q-tuples. Consider the graph $G^{'} = G\backslash Q$. Let P be a maximum $\langle P_k\rangle$-packing and C be a minimum $\langle P_k\rangle$-cover of the graph $G^{'}$. By the inductive assumption, we have $|P| = |C|$. But, $P \cup \{Q\}$ is a $\langle P_k\rangle$-packing of the size $|P|+1$ and $C \cup \{y\}$ is a $\langle P_k\rangle$-cover of the same cardinality in the graph G.

2. The graph G does not contain terminal F_k-subgraphs from the case 1, but it contains a cycle of length k or more. Denote by D the set of all such vertices of all such cycles. Denote by C_0 the set of all cyclic vertices of the pseudograph M. Obviously, $C_0 \subseteq D$. Each vertex x of C_0 corresponds to at least one q-tuple r_1--x--r_2, where r_1 and r_2 are roots of two H_x graphs, added in the type A extention of two cyclic edges, incident to x. Denote by P_0 the set of such q-tuples. All q-tuples in P_0 pairwise have not common vertices, and they are all contained in D. Obviously, $|C_0| = |P_0|$.
   Consider the graph $G^{'} = G\backslash D$. Let P be a maximum $\langle P_k\rangle$-packing and C be a minimum $\langle P_k\rangle$-cover of the graph $G^{'}$. By the inductive assumption, we have $|P| = |C|$. The set $P \cup P_0$ is a $\langle P_k\rangle$-packing of the graph G.
   We show that $C \cup C_0$ is a $\langle P_k\rangle$-cover of graph G. Consider two vertices $x, y \in C_0$ adjacent in the pseudograph M (the case $x = y$ is possible, if xx is a loop in M). By Lemma 3, the RT_k-extention converts the edge xy of the pseudograph M into a F_k-subgraph of the graph G. Thus, $G\backslash C_0$ is the union of the graph $G^{'}$ and a number of F_k-graphs, so it contains the same set of q-tuples as $G^{'}$. In other words, C_0 covers all q-tuples of the graph G, which are not covered by C, i.e. $C \cup C_0$ is a $\langle P_k\rangle$-cover of graph G.
   Since $|C_0| = |P_0|$, we have $\mu_{\langle P_k\rangle}(G) = \beta_{\langle P_k\rangle}(G)$.
3. The graph G does not contain cycles of length k or more and terminal F_k-subgraphs from the case 1. The pseudograph M is a tree in this case.
   Without loss of generality, we can consider that all maximal terminal F_k-subgraphs of the graph G added by some $AddTF$ procedures. In this case, either $M = O_1$ (then, it is easy to see, that $\mu_{\langle P_k\rangle}(G) = \beta_{\langle P_k\rangle}(G) = 0$), or at least one $AddTF$ procedure was applied to each its leaf. In other words, now, we assume that each terminal subgraph, a contact vertex of which is a leaf, is maximal

Let y be a leaf of M, T be the union of all terminal F_k-subgraphs of the graph G with the contact vertex y, and x be a neighbour of y in the tree M. Suppose that xy is an edge of G. Then, the graph $T + y$ is a terminal F_k-subgraph, because it satisfies the case 1, otherwise. Then, connected components of T are not maximal terminal F_k-subgraphs, which contradicts the statement above. Thus, the edge xy is extended by the type B extention.

Note that the type B extention can convert xy into $H_y^n + x$ or $H_x^n + y$. In both cases, $H_y^n + T$ or $H_x^c + T + y\backslash x$ is the terminal subgraph in G and, therefore, it contains s q-tuple. Each such a q-tuple contains y and at least one more vertex from H_y^n or H_x^n. Consider these two cases separately.

If xy is converted into $H_x^n + y$, then H_x^n has at least 2 leaves. Without loss of generality, we can assume that x has degree 2 or more in H_x. Otherwise, we can use the lowest descendant of x with degree 3 or more as x. Let b, d be any vertices of H_x adjacent to x. Since $H_x \in T_i^*$, it has two non-intersected paths a--b and c--d, where a, c are leaves of H_x and, therefore, adjacent to y. Note that $d(a$--$b) = d(c$--$d)$. We can assume that $a\text{--}a^{'}$ is the longest path of $H_x^n\backslash x$, for some vertex $a^{'}$. Since each vertex of $a\text{--}a^{'}$ is a descendant of b (or equals to b), $a\text{--}a^{'}$ does not intersect $c\text{--}d$.

Let Q be a q-tuple in $H_x^n + T + y\backslash x$. If $a \notin Q$, then exchange $Q \cap V(H_x^n)$ into a part of the same length beginning from a. Since $a\text{--}a^{'}$ is the longest path of $H_x^n\backslash x$, we can find such a part in this path. In other words, we can assume that $Q = u\text{--}ya\text{--}v$, where $u \in V(T)$ and $a\text{--}v \subseteq a\text{--}a^{'}$.

Consider the graph $G^{'}$, obtained from the graph G by deleting vertices of the part $u\text{--}y$ of the path Q. Let P be a maximum $\langle P_k\rangle$-packing and C be a minimum $\langle P_k\rangle$-cover of the graph $G^{'}$. By the inductive assumption, $|P| = |C|$. By Lemma 5, since $i \leq s$, $H_x^n \in F_k$. Therefore and since x is the cut vertex of $G^{'}$, only one q-tuple of P can contain some vertices from H_x^n.

If a q-tuple $Q^{'} \in P$ contains b, then exchange $Q^{'} \cap V(H_x^c)$ into part of the same length beginning from d. Since $d(a\text{--}b) = d(c\text{--}d)$, we can find such a part in $c\text{--}d$. In other words, we can assume that not a q-tuple of P contain a vertex from $a\text{--}a^{'}$.

Then, $P \cup \{Q\}$ is a $\langle P_k\rangle$-packing of the size $|P| + 1$ and $C \cup \{y\}$ is a $\langle P_k\rangle$-cover of the same cardinality in the graph G.

If xy is converted into $H_y^n + x$, then y has degree 2 or more in H_y. Repeating the foregoing argument, we find that H_y has two non-intersected paths $a\text{--}b$ and $c\text{--}d$, where $a, c \in N(y)$, b, d are leaves of H_y, adjacent to x, $d(a\text{--}b) = d(c\text{--}d)$. We can assume that $b\text{--}b^{'}$ is the longest path of $H_y^n\backslash y$, for some vertex $b^{'}$, and $b\text{--}b^{'}$ does not intersect $c\text{--}d$.

Let Q be a q-tuple in $H_y^n + T$. If $a \in Q$, then exchange $Q \cap V(H_y^n)$ into a part of the same length beginning from c. Since $d(a\text{--}b) = d(c\text{--}d)$, we can find such a part in $c\text{--}d$. In other words, we can assume that $Q = u\text{--}yc\text{--}v$, where $u \in V(T)$ and $v \in H_y^n\backslash y\backslash b\text{--}b^{'}$.

Consider the graph $G^{'}$, obtained from the graph G by deleting vertices of the part $u\text{--}y$ of the path Q. Let P be a maximum $\langle P_k\rangle$-packing and C be a minimum $\langle P_k\rangle$-cover of graph $G^{'}$. By the inductive assumption, we have $|P| = |C|$. By Lemma 6, since $i \leq s$, $H_y^n + x\backslash y \in F_k$. Therefore and since x is the cut-vertex of $G^{'}$, only one q-tuple of P can contain some vertices from $H_y^n + x\backslash y$.

If a q-tuple $Q^{'} \in P$ does not contain b, then exchange $Q^{'} \cap V(H_x^c)$ into a part of the same length beginning from b. Since $b\text{--}b^{'}$ is the longest path of $H_y^n\backslash y$, we can find such a part in this path. In other words, we can assume that if P contain a q-tuple intersected to $H_y^n\backslash y$, then each its vertex of this subgraph is in $b\text{--}b^{'}$.

Then, $P \cup \{Q\}$ is a $\langle P_k\rangle$-packing of the size $|P| + 1$ and $C \cup \{y\}$ is a $\langle P_k\rangle$-cover of the same cardinality in the graph G.

Thus, $\mu_{\langle P_k\rangle}(G) = \beta_{\langle P_k\rangle}(G)$. By Lemma 3, each subgraph of the graph G is a RT_k-graph. Hence, each RT_k-graph is a König graph for $\langle P_k\rangle$. □

5 Conclusions and future work

We have considered the family of König graphs for odd paths and all their spanning supergraphs. Some infinite sets of minimal forbidden subgraphs for them have been revealed. A procedure for constructing some of the considered König graphs has been presented. Both results are not finalized, i.e. they both do not give complete descriptions. Describing the classes of König graphs for odd paths and all their spanning supergraphs in the languages of minimal forbidden subgraphs and/or constructive procedures are interesting research problems for future work, which are open at the moment.

Recall that, according to the König theorem, the papers [2] and [18], the classes of $\langle P_k \rangle$-König graphs are monotone for $k \in \{2, 3, 5\}$. We conjecture that the class of $\langle P_k \rangle$-König graphs is monotone for any $k \geq 2$. Proving or disproving it is also a challenging research problem for future work.

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