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SOLUTION OF THE 33RD PALIS-PUGH PROBLEM FOR GRADIENT-LIKE DIFFEOMORPHISMS OF A TWO-DIMENSIONAL SPHERE

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Abstract. In the present paper, a solution to the 33rd Palis-Pugh problem for gradient-like diffeomorphisms of a two-dimensional sphere is obtained. It is precisely shown that with respect to the stable isotopic connectedness relation there exists countable many of equivalence classes of such systems. **43** words.

1. **Introduction.** The problem of the existence of an arc with no more than a countable (finite) number of bifurcations connecting structurally stable systems (Morse-Smale systems) on manifolds is on the list of fifty Palis-Pugh problems [28] under number 33. In this paper, this problem is solved for gradient-like diffeomorphisms of a two-dimensional sphere.

First the notion of rough (or structural stable) system was introduced in tha classical paper by A. Andronov and L. Pontryagin [3]. In 1976, S. Newhouse, J. Palis, F. Takens [24] introduced the concept of a stable arc connecting two structurally stable systems on a manifold. Such an arc does not change its quality properties with a little perturbation. In the same year, S. Newhouse and M. Peixoto [25] proved the existence of a simple arc (containing only elementary bifurcations) between any two Morse-Smale flows. It follows from the result of G. Fleitas [9] that a simple arc constructed by Newhouse and Peixoto can always be replaced by a stable one. For Morse-Smale diffeomorphisms given on manifolds of any dimension, examples of systems that cannot be connected by a stable arc are known. In this direction, the question naturally arises of finding an invariant that uniquely determines the equivalence class of the Morse-Smale diffeomorphism with respect to the relation of the connection by a stable arc.

According to [23], for diffeomorphisms of a closed manifold M^n with a finite limit set, the stability of the arc $\{f_t \in Diff(M^n),\, t \in [0,1]\}$ is characterized by a finite number of bifurcation values $0 < b_1 < \cdots < b_q < 1$, while the bifurcation diffeomorphism $\varphi_{b_i}, i \in \{1, \dots, q\}$ has the following properties:

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1) diffeomorphism φ_{b_i} has exactly one non-hyperbolic periodic orbit, namely a flip or a non-critical saddle-node, while the arc unfolds generically through a bifurcation value;

2) all invariant manifolds of the periodic points of the diffeomorphism φ_{b_i} intersect transversally and it has no cycles.

We say that diffeomorphisms $f_0, f_1 : M^n \to M^n$ belong to the same *component of stable connectedness* if in the space of diffeomorphisms they can be connected by an arc with the properties described above.

Classification from the point of view of the introduced equivalence relation is already non-trivial for orientation-preserving diffeomorphisms of the circle S^1. Here a countable set of such classes appears, each of which is uniquely determined by the rotation number of the rough transformation of the circle [26], which is $\frac{k}{m}$, where $k \in (\mathbb{N} \cup 0), m \in \mathbb{N}, k < m, (k, m) = 1$.

In the present paper we also prove that here is a countable set of the components of the stable connectedness for orientation-preserving gradient-like diffeomorphisms of 2-sphere.

Namely, consider S^1 as the equator of the sphere S^2. Then the structurally stable diffeomorphism of the circle with exactly two periodic orbits of the period $m \in \mathbb{N}$ and the rotation number $\frac{k}{m}$ can be extended to the diffeomorphism $\phi_{k,m} : S^2 \to S^2$, which has two fixed sources at the north and south poles. Denote by $C_{k,m}$ the component of stable connectedness of the diffeomorphism $\phi_{k,m}$ and by $C^-_{k,m}$ the component of stable connectedness of the diffeomorphism $\phi^{-1}_{k,m}$. Also denote by C_0 the component of stable connectedness of the source-sink diffeomorphism ϕ_0.

Theorem 1.1. *Every orientation-preserving gradient-like diffeomorphism of 2-sphere belongs to one of the components C_0, $C_{k,m}$, $C^-_{k,m}$, k, $m \in \mathbb{N}$, $k < m/2$, $(k, m) = 1$. Herewith:*

- *the components C_0, $C_{k,m}$, $C^-_{k,m}$, k, $m \in \mathbb{N}$, $k < m/2$, $(k, m) = 1$ are pairwise disjoint;*
- $C_{k,m} = C_{m-k,m}$, $C^-_{k,m} = C^-_{m-k,m}$, $C_{1,2} = C^-_{1,2} = C_{0,1} = C^-_{0,1} = C_0$.

Notice that it follows from a result by P. Blanchard [6] that $\phi_{k,m}, \phi_{k',m'} : S^2 \to S^2$ belong to different components for $m = 2^r \cdot q, m' = 2^{r'} \cdot q'$, where integers $r, r' \geq 0$ and positive integers $q \neq q'$. He obtained some necessary conditions for the connection of Morse-Smale diffeomorphisms on the surface by a stable arc. However, the question on sufficient conditions was not considered in [6].

The obtained result is closely related with the classification of periodic homeomorphisms of a two-dimensional sphere obtained by B. von Kerekjarto [18]. The topological conjugacy class of the periodic transformation of the period m on a 2-sphere is also completely determined by the rotation number $\frac{k}{m}$ around the north pole-south pole axis. Since any orientation-preserving gradient-like diffeomorphism is topologically conjugate to a composition of a periodic homeomorphism with a one-time shift of a gradient-like flow [5], [14], the natural question is about an interrelation between these rotation numbers.

The proof of the theorem 1.1 shows that they are not coincide in general. In any case the construction of a stable arc between diffeomorphisms is an independent problem that does not directly follow from the topological classification of diffeomorphisms. In support of this, it suffices to note that all orientation-preserving source-sink systems on the n-sphere are pairwise topologically conjugate for a fixed n. However, they are not connected by a stable arc in general, for example, for

$n = 7$ [8]. For $n = 2, 3$ the non-trivial fact of the existence of an arc without bifurcations between two source-sink diffeomorphisms was established in [27], [8].

2. Backgrounds.

2.1. Morse-Smale diffeomorphisms. Let a diffeomorphism $f : M^n \to M^n$ be given on a smooth closed (compact without boundary) n-manifold ($n \geq 1$) M^n with a metric d.

A point $x \in M^n$ is called *wandering* for f if there is an open neighborhood U_x of the point x, such that $f^n(U_x) \cap U_x = \emptyset$ for all $n \in \mathbb{N}$. Otherwise, the point x is called *non-wandering*. The set of non-wandering points f is called *non-wandering set* and is denoted by Ω_f.

For example, all the limit points of a diffeomorphism are non-wandering. Recall that a point $y \in M^n$ is called a ω-*limit* point for a point $x \in M^n$ if there exists a sequence $t_k \to +\infty$, $t_k \in \mathbb{Z}$ such that $\lim\limits_{t_k \to +\infty} d(f^{t_k}(x), y) = 0$. The set $\omega(x)$ of all ω-limit points for the point x is called ω-*limit set*. Replacing $+\infty$ with $-\infty$ determines the α-*limit set* $\alpha(x)$ for the point x. The set $L_f = cl\,(\bigcup\limits_{x \in M^n} \omega(x) \cup \alpha(x))$ is called the *limit set* of the diffeomorphism f.

If the set Ω_f is finite, then every point $p \in \Omega_f$ is periodic, we denote by $m_p \in \mathbb{N}$ the period of the periodic point p. Any periodic point p is associated with *stable* and *unstable* manifolds defined as follows

$W^s_p = \{x \in M^n : \lim\limits_{k \to +\infty} d(f^{km_p}(x), p) = 0\}$,

$W^u_p = \{x \in M^n : \lim\limits_{k \to +\infty} d(f^{-km_p}(x), p) = 0\}$.

Stable and unstable manifolds are called *invariant manifolds*. It is said that the periodic orbits $\mathcal{O}_1, \dots, \mathcal{O}_k$ form a *cycle* if $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_{i+1}} \neq \emptyset$ for $i \in \{1, \dots, k\}$ and $\mathcal{O}_{k+1} = \mathcal{O}_1$.

A periodic point $p \in \Omega_f$ is called *hyperbolic* if the absolute values of the eigenvalues of the Jacobi matrix $\left(\frac{\partial f^{m_p}}{\partial x}\right)|_p$ are not equal one. If all of them are less (greater) than one, then p is called the *sink* (*source*) *point*. Sink or source points are called *nodal*. If a hyperbolic periodic point is not *nodal*, then it is called *saddle point*.

The stable W^s_p and the unstable W^u_p manifolds of the periodic point p are injective immersions of the spaces $\mathbb{R}^{q_p}$ and $\mathbb{R}^{n-q_p}$, where q_p is the number of eigenvalues of the Jacobi matrix, modulo large ones (see, for example, [31]). The number ν_p, equal to $+1\,(-1)$ if the map $f^{m_p}|_{W^u_p}$ preserves (changes) the orientation of W^u_p, is called an *orientation type* of p. The path-connected component of the set $W^u_p \setminus p$ ($W^s_p \setminus p$) is called an *unstable* (*stable*) *separatrix* of the point p.

A closed f-invariant set $A \subset M^n$ is called an *attractor* of a discrete dynamical system generated by f f if it has a compact neighborhood U_A such that $f(U_A) \subset int\, U_A$ and $A = \bigcap\limits_{k \geq 0} f^k(U_A)$. The neighborhood U_A is called *trapping*. *A repeller* is defined as an attractor for f^{-1}. If the trapping neighborhood of an attractor A is the complement of a trapping neighborhood of a repeller R then pair A, R is called *dual*.

A diffeomorphism $f : M^n \to M^n$ is called *Morse-Smale*, if

1) the non-wandering set Ω_f consists of a finite number of hyperbolic orbits;

2) manifolds W^s_p, W^u_q intersect transversely for any non-wandering points p, q.

A Morse-Smale diffeomorphism is called *gradient-like* if the condition $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$ for different $\sigma_1, \sigma_2 \in \Omega_f$ implies that $dim\ W^u_{\sigma_1} < dim\ W^u_{\sigma_2}$. The *Morse-Smale flow* is similar defined and is called *gradient-like* in the absence of periodic trajectories.

Notice that the gradient-like diffeomorphism on a surface has no *heteroclinic points* – intersection points of the invariant manifolds of different saddle points.

Proposition 1. [14, Lemma 3.3] *All saddle separatrices of an orientation-preserving gradient-like diffeomorphism f on a surface has the same period $\mu_f \in \mathbb{N}$.*

A homeomorphism $\phi : M^2 \to M^2$ is called *periodic of order* $\mu \in \mathbb{N}$ if $\phi^\mu = id$ and $\phi^j \neq id$ for any positive integer $j < \mu$.

Proposition 2. [14, Theorems 3.1, 3.3] *Every orientation-preserving gradient-like diffeomorphism f on a surface is topologically conjugate to a composition of a periodic homeomorphism ϕ_f of the period μ_f with the one-time shift of a gradient-like flow. Moreover, $f|_{\Omega_f} = \phi_f|_{\Omega_f}$.*

We denote by G the class of orientation-preserving gradient-like diffeomorphisms on the two-dimensional sphere S^2. According to the classification given by B. Kerekyarto [18], an orientation-preserving periodic homeomorphism of the period μ on 2-sphere has periodic points of only two periods 1 and μ, while the set of its fixed points is not empty. This gives us the following corollary of the propositions 1 and 2 for the class G.

Proposition 3. *Any $f \in G$ has periodic points of only two periods 1 and μ_f (possibly $\mu_f = 1$). Moreover,*

1) if f has saddle points with the negative orientation type then all such points are fixed and $\mu_f = 2$;

2) any saddle point with a positive orientation type has the period μ_f.

2.2. **Stable arcs in the space of diffeomorphisms.** Consider a one-parameter family of diffeomorphisms (*arc*) $\varphi_t : M^n \to M^n, t \in [0,1]$. Denote by $\mathcal{Q}$ the set of arcs $\{\varphi_t\}$, that begin and end in Morse-Smale diffeomorphisms and every diffeomorphism φ_t has a finite limit set.

According to [23], an arc $\{\varphi_t\} \in \mathcal{Q}$ is called *stable* if it is an internal point of the equivalence class with respect to the following relation: arcs $\{\varphi_t\}$, $\{\tilde{\varphi}_t\} \in \mathcal{Q}$ are called *conjugate*, if there are homeomorphisms $h : [0,1] \to [0,1]$, $H_t : M^n \to M^n$ such that $h(b)$ is a bifurcation value for every bifurcation value b, $H_t\varphi_t = \tilde{\varphi}_{h(t)}H_t, t \in [0,1]$ and H_t continuously depends on t (see figure 2.2).

In [23] also established that the arc $\{\varphi_t\} \in \mathcal{Q}$ is stable if and only if all its points are structurally stable diffeomorphisms with the exception of a finite number of bifurcation points φ_{b_i}, $i = 1, \dots, q$ such that φ_{b_i}:

1) has a unique non-hyperbolic periodic orbit, which is a non-critical saddle-node or flip;

2) has no cycles;

3) the invariant manifolds of all periodic points intersect transversally;

4) the arc φ_t unfolds generically through φ_{b_i}.

Recall the definition of *unfolding generically arc φ_t through the saddle-node or flip.* We give the definition for a fixed non-hyperbolic point, in the case when it has a period $k > 1$, a similar definition is given for the arc φ^k_t.

One says that an arc $\{\varphi_t\} \in \mathcal{Q}$ *unfolds generically through a saddle-node bifurcation* φ_{b_i} (see figure 2), if in some neighborhood of the non-hyperbolic point (p, b_i)

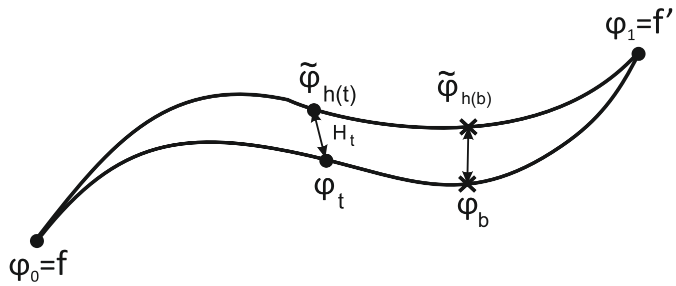

FIGURE 1. Conjugate arcs

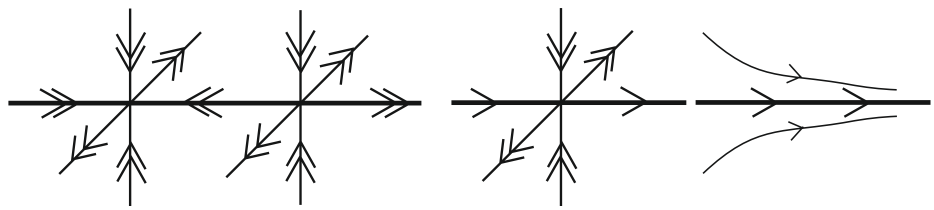

FIGURE 2. Saddle-node bifurcation

the arc φ_t is conjugate to the arc

$$\tilde{\varphi}_{\tilde{t}}(x_1, x_2, \dots, x_{1+n_u}, x_{2+n_u}, \dots, x_n) =$$
$$\left(x_1 + 2x_1^2 + \tilde{t}, \pm 2x_2, \dots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \dots, \frac{\pm x_n}{2}\right),$$

where $(x_1, \dots, x_n) \in \mathbb{R}^n$, $|x_i| < 1/2$, $|\tilde{t}| < 1/10$.

In the local coordinates $(x_1, \dots, x_n, \tilde{t})$ the bifurcation occurs at time $\tilde{t} = 0$ and the origin $O \in \mathbb{R}^n$ is a saddle-node point. The axis Ox_1 is called a *central manifold*, the half-space $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0,\ x_{2+n_u} = \dots = x_n = 0\}$ is the unstable manifold, half-space $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 \leq 0,\ x_2 = \dots = x_{1+n_u} = 0\}$ is the stable manifold of the point O.

If p is a saddle-nodal point of the diffeomorphism φ_{b_i}, then there exists a unique φ_{b_i}-invariant foliation F_p^{ss} with smooth leaves such that W_p^s is a leave of this foliation [17]. F_p^{ss} is called a *strongly stable foliation* (see figure 3). A similar *strongly unstable foliation* is denoted by F_p^{uu}. A point p is called *s-critical*, if there exists some hyperbolic periodic point q such that W_q^u non-transversally intersect some leaf of the foliation F_p^{ss}; *u-criticality* is defined similarly. Point p is called

- *semi-critical* if it is either s- or u-critical;
- *bi-critical* if it is s- and u-critical;
- *non-critical* if it is not semi-critical[1].

Remark 1. For surface diffeomorphisms, the non-criticality of the saddle-node point p means the absence of intersection of the central manifold of the point p with the separatrices of saddle points.

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[1]For the first time, the effect of arc instability in a neighborhood of a non-critical saddle-node was discovered in 1974 by V. Afraimovich and L. Shilnikov [1], [2]. The existence of invariant foliations F_p^{ss}, F_p^{uu} was also proved earlier in the works of V. Lukyanov and L. Shilnikov [19].

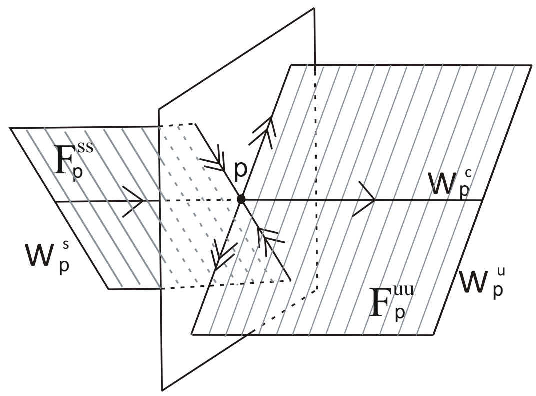

Figure 3. Strongly stable and unstable foliations

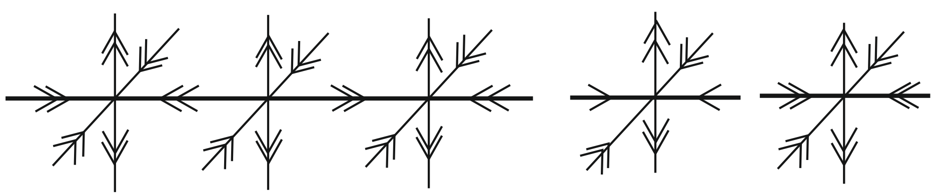

Figure 4. Flip bifurcation

One says that an arc $\{\varphi_t\} \in \mathcal{Q}$ *unfolds generically through a flip bifurcation* φ_{b_i} (see figure 4), if in some neighborhood of the non-hyperbolic point (p, b_i) the arc φ_t is conjugate to the arc

$$\tilde{\varphi}_{\tilde{t}}(x_1, x_2, \dots, x_{1+n_u}, x_{2+n_u}, \dots, x_n) =$$
$$\left(-x_1(1 \pm \tilde{t}) + x_1^3, \pm 2x_2, \dots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \dots, \frac{\pm x_n}{2}\right),$$

where $(x_1, \dots, x_n) \in \mathbb{R}^n$, $|x_i| < 1/2$, $|\tilde{t}| < 1/10$.

We say that Morse-Smale diffeomorphisms $f_0, f_1 : M^n \to M^n$ belong to the same *component of stable connectedness*, if in the space of diffeomorphisms they can be connected by a stable arc.

A diffeomorphism *source-sink* on a closed manifold is a gradient-like diffeomorphism having exactly two fixed points: a source and a sink. The ambient manifold for such diffeomorphisms is always a n-sphere.

Proposition 4. [27, Theorem 1] *Every source-sink diffeomorphisms $f_0, f_1 : S^2 \to S^2$ are connected by an arc without bifurcations.*

2.3. **Reduction of confluence objects to a canonical form.** To construct an arc that unfolds generically through a saddle-node or flip bifurcation, it is necessary to reduce the confluence objects to a canonical form. In this section, we give necessary facts that make it possible, without loss of generality, to consider any diffeomorphism by linear in a neighborhood of a hyperbolic point, and the closure of any saddle separatrix of a 2-diffeomorphism lying on a smooth arc.

Proposition 5. [22, Theorem 5.8 (Tom's theorem on the continuation of isotopy)] *Let Y be a smooth manifold without boundary, X be a smooth compact submanifold of Y and $\{f_t : X \to Y, t \in [0,1]\}$ is a smooth isotopy such that f_0 is the inclusion of X in Y. Then for any compact set $A \subset Y$, containing the isotopy support*[2] *$supp\{f_t\}$ there exists a smooth isotopy $\{g_t \in Diff(Y), t \in [0,1]\}$ such that $g_0 = id$, $g_t|_X = f_t|_X$ for any $t \in [0,1]$ and $supp\{g_t\}$ belongs to A.*

Proposition 6. [11, Theorem 1 (A "pathwise" Franks' lemma)] *Let a diffeomorphism $\varphi_0 : M^n \to M^n$ has an isolated hyperbolic point r_0 of the period m_0 and let (U_0, h) is a local map of the manifold M^n such that $r_0 \in U_0$, $h(r_0) = O$. Then for any hyperbolic automorphism Q, having the same index as the automorphism $(D\varphi_0^{m_0})_{r_0}$, there exist neighborhoods U_1, U_2 of the point r_0, $U_2 \subset U_1 \subset U_0$, and the arc $\varphi_t : M^n \to M^n, t \in [0,1]$ without bifurcations such that:*

1) the diffeomorphism φ_t, $t \in [0,1]$, coincides with the diffeomorphism φ_0 utside the set $\bigcup\limits_{k=0}^{m-1} \varphi_0^k(U_1)$;

2) $\mathcal{O}_{r_0} = \bigcup\limits_{k=0}^{m-1} \varphi_0^k(r_0)$ is an isolated hyperbolic orbit of period m_0 of the same index as the automorphism $(D\varphi_0^{m_0})_{r_0}$, for every φ_t;

3) $W^s_{\mathcal{O}_{r_0}}(\varphi_t) = W^s_{\mathcal{O}_{r_0}}(\varphi_0)$ and $W^u_{\mathcal{O}_{r_0}}(\varphi_t) = W^u_{\mathcal{O}_{r_0}}(\varphi_0)$ outside the set $\bigcup\limits_{k=0}^{m-1} \varphi_0^k(U_1)$;

4) the diffeomorphism $h\varphi_1^m h^{-1}$ coincides with the diffeomorphism Q on the set $h(U_2)$.

Proposition 7. [21, Lemma 2] *Let a diffeomorphism $\varphi_0 : M^2 \to M^2$ has an isolated hyperbolic sink ω_0 and an isolated hyperbolic saddle σ_0 such that the unstable separatrix γ_{φ_0} of the saddle σ_0 lies in the sink basin $W^s_{\omega_0}$ and has the same period m, as the sink ω_0. Let (V_0, ψ_0) be a local chart in ω_0 such that the diffeomorphism $\psi_0\varphi_0^m\psi_0^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear contraction $g(x_1, x_2) = (x_1/2, x_2/2)$. Then there exist neighborhoods V_1, V_2 of the point ω_0, $V_2 \subset V_1 \subset V_0$ and the arc $\varphi_t : M^2 \to M^2, t \in [0,1]$ without bifurcations with the following properties:*

1) the diffeomorphism φ_t, $t \in [0,1]$ coincides with the diffeomorphism φ_0 outside the set $\bigcup\limits_{k=0}^{m-1} \varphi_0^k(V_1)$ and $\bigcup\limits_{k=0}^{m-1} \varphi_0^k(\omega_0)$ is the hyperbolic sink orbit of the period m for all φ_t;

2) the diffeomorphism φ_1 coincides with the diffeomorphism φ_0 on the set $\bigcup\limits_{k=0}^{m-1} \varphi_0^k$ (V_2) and $\psi_0(\gamma_{\varphi_1} \cap V_2) \subset OX_1$, where γ_{φ_1} is an unstable separatrix of the saddle σ_0 with respect to the diffeomorphism φ_1.

2.4. **Necessary information from the graph theory.** Recall some definitions from the graph theory (see, for example, [16]).

Graph Γ is a pair (V_Γ, E_Γ), where V_Γ is a set *of vertices*, and E_Γ is a set of *pairs of vertices*, called *edges*.

Two vertices are called *adjacent*, if they are connected by an edge (that is, they form an edge), and the edge in this case is called *incidental* to each of the vertices. The number of edges incident to a vertex is called the *degree* of the vertex.

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[2] *The support $supp\{f_t\}$ of isotopy $\{f_t\}$* s the closure of the set $\{x \in X : f_t(x) \neq f_0(x) \text{ for some } t \in [0,1]\}$.

The set of vertices $\{v_1, (v_1, v_2), v_2, \dots, v_{k-1}, (v_{k-1}, v_k), v_k\}$ is called *path* of length k. A path is called a *cycle*, if $v_1 = v_k$. A graph without cycles is called *acyclic*. A graph is called *connected*, if every two of its vertices are connected by a path. *Tree* is a connected acyclic graph, that is, any two of its vertices are connected in exactly one way.

Everywhere below Γ is a tree.

Every tree with at least 2 vertices has at least two *hanging vertices*, that is, vertices of degree 1. Then every such tree Γ is uniquely associated with the sequence

$$\Gamma_0, \Gamma_1, \dots, \Gamma_s$$

trees such that $\Gamma_0 = \Gamma$, Γ_s contain one or two vertices and for any $i \in \{1, ., , , s\}$, the tree Γ_i, is obtained from Γ_{i-1} y removing all its hanging vertices. All vertices of the tree Γ_s are called *central vertices* of the tree Γ and if Γ_s has an edge, then it is called *central edge* of the tree Γ.

A tree Γ is called *central* if it has exactly one central vertex, and *bi-central*, otherwise.

Vertex rank $x \in V_\Gamma$, denoted by $rank(x)$, is defined by the formula

$$rank(x) = max\{i : x \in V_{\Gamma_i}\}.$$

It follows from the definition that if the vertices v, w are incident to an off-center edge, then $|rank(v) - rank(w)| = 1$, and the central vertices of the bi-central tree have the same rank.

Automorphism P_Γ of the tree Γ is a bijective map of the set V_Γ onto itself, preserving the adjacency, i.e.

$$(u, v) \in E_\Gamma \Leftrightarrow (P_\Gamma(u), P_\Gamma(v)) \in E_\Gamma.$$

The automorphism P_Γ can be represented as a superposition of cyclic permutations. Then the set V_Γ can be decomposed into P_Γ-*orbits* – subsets invariant under the permutations. It is clear that every P_Γ-orbit consists of vertices of the same rank and if the tree is central (bi-central), then its central vertex (central edge) remains fixed for any automorphism.

The automorphism P_Γ naturally induces a map of the set of edges E_Γ, which we will also denote by P_Γ.

Proposition 8. [12, Corollary 2.2] *Let $(v, w) \in E_\Gamma$ be an off-center edge and $rank(v) < rank(w)$. Then the period of the edge (v, w) is equal to the period of the vertex v.*

3. Dynamics of gradient-like surface diffeomorphisms.

3.1. Dynamics on an arbitrary surface.

Consider an orientation-preserving gradient-like diffeomorphism f, defined on a smooth orientable closed surface M^2. In this section, we describe the general dynamic properties of such diffeomorphisms.

Denote by Ω^0_f, Ω^1_f, Ω^2_f the set of sinks, saddles and sources of f. For any (possibly empty) f-invariant set $\Sigma \subset \Omega^1_f$ we set

$$A_\Sigma = \Omega^0_f \cup W^u_\Sigma,\ R_\Sigma = \Omega^2_f \cup W^s_{\Omega^1_f \setminus \Sigma}.$$

It follows from [13] that A_Σ, R_Σ are dual attractor and repeller. The set

$$V_\Sigma = M^2 \setminus (A_\Sigma \cup R_\Sigma)$$

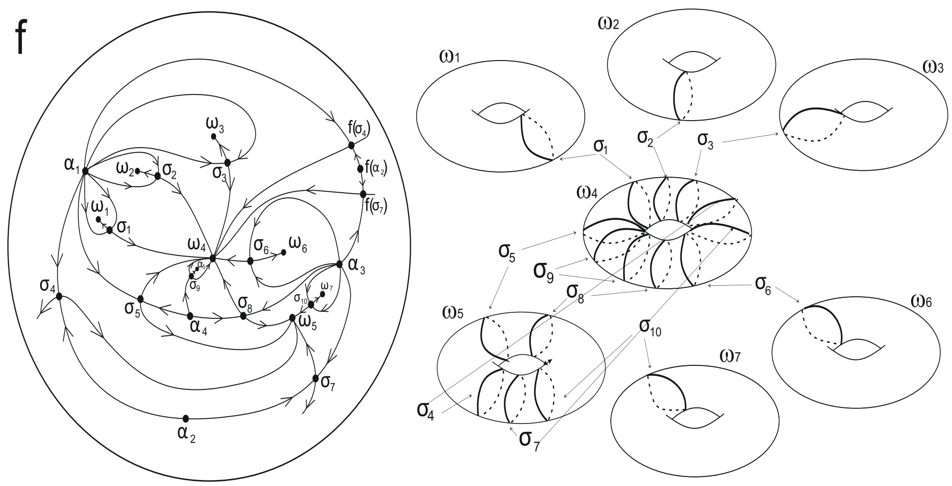

FIGURE 5. Illustration to the proof of lemma 3.1

is called *characteristic space*. We denote by $\hat{V}_\Sigma$ the orbit space of the action of the diffeomorphism f on V_Σ and by $p_{\Sigma} : V_\Sigma \to \hat{V}_\Sigma$ the natural projection. According to [15], each connected component of the manifold $\hat{V}_\Sigma$ is diffeomorphic to a two-dimensional torus.

Lemma 3.1. *For every orientation-preserving gradient-like diffeomorphism $f : M^2 \to M^2$ there exists a set Σ, such that the orbit space $\hat{V}_\Sigma$ is connected.*

Proof. Let $\Sigma_0 = \emptyset$ and consider the corresponding dual attractor and repeller $A_{\Sigma_0} = \Omega^0_f$ and $R_{\Sigma_0} = \Omega^2_f \cup W^s_{\Omega^1_f}$. If the orbit space of $\hat{V}_{\Sigma_0}$ is connected, then $\Sigma = \Sigma_0$.

Otherwise, denote by $\hat{V}_1, \dots, \hat{V}_l$ the connected components of the space $\hat{V}_{\Sigma_0}$. For any saddle point $\sigma \in \Omega^1_f$ we set $\hat{L}^u_\sigma = p_{\Sigma_0}(W^u_\sigma \setminus \sigma)$. Due to [14], the set $\hat{L}^u_\sigma$ consists of two closed curves if $\nu_\sigma = +1$ and one closed curve if $\nu_\sigma = -1$.

Consider the dual attractor and repeller for the set Σ^1_f, that is, $A_{\Sigma^1_f} = \Omega^0_f \cup W^u_{\Omega^1_f}$ and $R_{\Sigma^1_f} = \Omega^2_f$. In this case, the repeller has dimension zero and, by [13], the attractor $A_{\Sigma^1_f}$ is connected. Then, up to the renumbering of the components $\hat{V}_1, \dots, \hat{V}_l$, there is a sequence of saddle points $\sigma_1, \dots, \sigma_{l-1}$ such that the set $\hat{L}^u_{\sigma_i}$ consists of two closed curves $\hat{\ell}^{1u}_{\sigma_i} \subset \hat{V}_i$, $\hat{\ell}^{2u}_{\sigma_i} \subset \hat{V}_{i+1}$ (see figure 3.1). Let $\mathcal{O}_{\sigma_i}$ denote the orbit of the point σ_i. Let

$$\Sigma_i = \Sigma_0 \cup \bigcup_{j=1}^{i} W^u_{\mathcal{O}_{\sigma_j}},\ i \in \{1, \dots, l-1\}.$$

According to [15], the orbit space of $\hat{V}_{\Sigma_i}$ consists of $l - i$ connected components. Thus, the space $\hat{V}_{\Sigma_{l-1}}$ is connected, and $\Sigma = \Sigma_{l-1}$ is the desired set. □

For any gradient-like diffeomorphism $f : M^2 \to M^2$ and a set Σ, satisfying the conditions of the lemma 3.1, let

$$A_f = A_\Sigma,\ R_f = R_\Sigma,\ V_f = V_\Sigma.$$

Figure 6 shows a phase portrait of a gradient-like diffeomorphism f with a pair A_f, R_f.

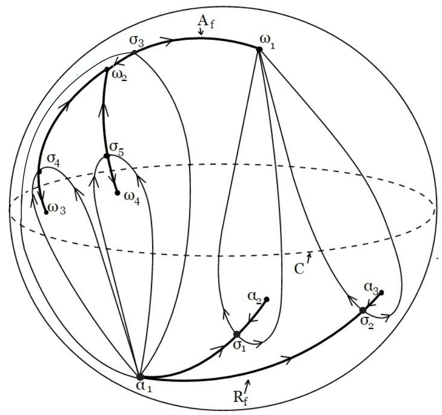

FIGURE 6. An example of an attractor A_f and repeller R_f for a gradient-like diffeomorphism f

Note that the set Σ, satisfying the conditions of Lemma 3.1, is not unique. Thus in Figure 7 depicted diffeomorphism $f \in G$ with two choices of the pair A_f, R_f.

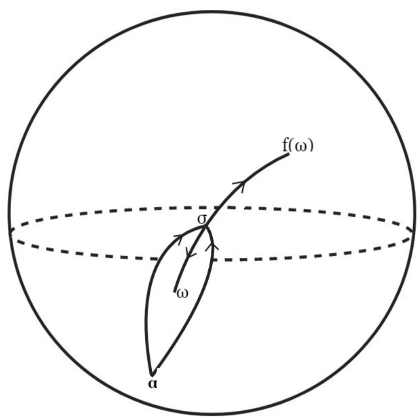

FIGURE 7. For the shown diffeomorphism, there are two ways to choose the pair A_f, R_f: 1) $A_f = cl\, W^u_\sigma$, $R_f = \alpha$ and 2) $A_f = \omega \cup f(\omega)$, $R_f = cl\, W^s_\sigma$

For any chosen pair A_f, R_f let

$$\hat{V}_f = \hat{V}_\Sigma,\ p_f = p_\Sigma.$$

Then the set $\hat{V}_f$ is connected and homeomorphic to the torus, while the set V_f is not connected in general. Denote by m_f the number of connected components of the set V_f. Note that the number m_f depends on the choice of the pair A_f, R_f. For example, for a diffeomorphism in the figure 7 in case 1) $m_f = 1$, and in case 2) $m_f = 2$.

In any case the set V_f is diffeomorphic to $(\mathbb{R}^2 \setminus O) \times \mathbb{Z}_{m_f}$ and the restriction of the diffeomorphism f to V_f is topologically conjugate by a homeomorphism $h_f : V_f \to (\mathbb{R}^2 \setminus O) \times \mathbb{Z}_{m_f}$ to periodic contraction $a_{m_f} : (\mathbb{R}^2 \setminus O) \times \mathbb{Z}_{m_f} \to (\mathbb{R}^2 \setminus O) \times \mathbb{Z}_{m_f}$, given by the formula

$$a_{m_f}(x, y, i) = \begin{cases} (x, y, i+1),\ i = 0, \dots, m_f - 2, \\ (x/2, y/2, 0),\ i = m_f - 1. \end{cases}$$

Let $c_i = h_f^{-1}(\mathbb{S}^1 \times \{i\}), i = 0, \dots, m_f - 1$, $c = h_f^{-1}(\mathbb{S}^1 \times \mathbb{Z}_{m_f})$ and $\hat{c} = p_f(c)$ (see figure 6, where the curve c is shown). Curve $\hat{c}$ is called *equator*, it is a simple closed curve on the torus $\hat{V}_f$ and its homotopy type is uniquely defined by a diffeomorphism f, that is, does not depend on the choice of the conjugating homeomorphism h_f.

3.2. **Dynamics on the two-dimensional sphere.** Let us recall that G is the class of orientation-preserving gradient-like diffeomorphisms on the two-dimensional sphere S^2. Consider $f \in G$. In this case, the attractor and repeller A_f, R_f can be described in more detail. To do this, note that $\bigcup\limits_{i=0}^{m_f-1} f^i(c)$ divides the sphere S^2 into two disjoint parts U and V such that

$$f(U) \subset U,\ A_f = \bigcap_{j \in \mathbb{N}} f^j(U);\ f^{-1}(V) \subset V,\ R_f = \bigcap_{j \in \mathbb{N}} f^{-j}(V).$$

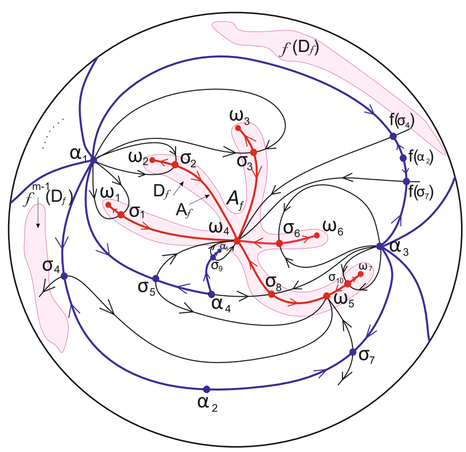

FIGURE 8. Illustration to the lemma 3.2

Lemma 3.2. *For any diffeomorphism $f \in G$ (up to a consideration of the diffeomorphism f^{-1}) the following is true (see figure 8):*

1) the set U consists of m_f pairwise disjoint disks $D_f, f(D_f), \dots, f^{m_f-1}(D_f)$ such that $f^{m_f}(cl\, D_f) \subset int\, D_f$;

2) the attractor A_f consists of m_f connected components $A, f(A), \dots, f^{m_f-1}(A)$ such that $A = \bigcap\limits_{j \in \mathbb{N}} f^{jm_f}(D_f)$ and $f^{m_f}(A) = A$;

3) repeller R_f is connected.

Proof. It follows from the definition of the equator $\hat{c}$ hat the set c consists of m_f simple closed curves $c_0, \dots, c_{m_f-1}$ on the sphere S^2 such that $c_{i+1} = f(c_i), i = 0, \dots, m_f - 2$. Since there are a finite number of such curves, among them there necessarily exists a curve $f^k(c)$, $k \in \{0, \dots, m_f - 1\}$, bounding the disk $D_f : int\, D_f \cap (\bigcup\limits_{i=0}^{m_f-1} f^i(c)) = \emptyset$. For definiteness, we assume that the disk D_f is a connected component of the set U (otherwise, this holds for the diffeomorphism f^{-1}).

Since the restriction of the diffeomorphism f^{m_f} to $D_f \cap V_f$ is associated with linear contraction, $f^{m_f}(cl\, D_f) \subset int\, D_f$. Thus, the set $A = \bigcap\limits_{j \in \mathbb{N}} f^{jm_f}(D_f)$ is connected. Since $A_f = \bigcap\limits_{j \in \mathbb{N}} f^{jm_f}(U)$, A is a connected component of the attractor A_f and $D_f = (D_f \cap V_f) \cup A$. Further, we consider separately two cases: (1) $m_f = 1$, (2) $m_f > 1$.

(1) If $m_f = 1$, then $A_f = A$, $R_f = \bigcap\limits_{j \in \mathbb{N}} f^{-j}(S^2 \setminus D_f)$ are connected attractor and repeller and the lemma is proved.

(2) If $m_f > 1$, then $f(c) \cap (D_f \cap V_f) = \emptyset$ due to conjugation to periodic contraction, also $f(c) \cap A = \emptyset$, since $f(c) \subset V_f$. So $f(D_f) \cap D_f = \emptyset$ since $f(c) \cap D_f = \emptyset$. Therefore, the disk $f(D_f)$ contains the connected component $f(A)$ of the attractor A_f, which does not intersect with A. Reasoning in the same way, we get m_f of pairwise disjoint connected components $A, f(A), \dots, f^{m_f-1}(A)$ of the attractor A_f,this means that the attractor A_f consists of one orbits of the period m_f. Thus, the set U is the union of pairwise disjoint disks $D_f, f(D_f), \dots, f^{m_f-1}(D_f)$. This implies that the set $V = S^2 \setminus U$ is connected, that implies the connectedness of the repeller R_f. □

4. **Proof of the theorem 1.1.** In this section, we give a scheme of the proof of theorem 1.1 with links to statements that will be proven in the following sections.

Recall that by G we denote the class of orientation-preserving gradient-like diffeomorphisms on the two-dimensional sphere S^2 and in the section 5 we constructed a family of model diffeomorphisms $\phi_{k,m}, \phi_0 \in G$. We denote by $C_{k,m}$ the component of stable connectedness of the diffeomorphism $\phi_{k,m}$ and we denote by $C^-_{k,m}$ the component of stable connectedness of the diffeomorphism $\phi^{-1}_{k,m}$. We denote by C_0 the component of the stable isotopic connection of the source-drain diffeomorphism ϕ_0.

We show that any diffeomorphism $f \in G$ belongs to one of the components C_0, $C_{k,m}$, $C^-_{k,m}$, k, $m \in \mathbb{N}$, $k < m/2$, $(k, m) = 1$.

Proof. Let $f \in G$. By lemma 3.2, diffeomorphism f (with respect to f^{-1}) has a (not unique) dual pair A_f, R_f, in which the repeller R_f is connected and the attractor consists of the m_f connected components of the period m_f. Let us show that f is connected by a stable arc either with the diffeomorphism ϕ_0, or with the diffeomorphism $\phi_{k,m}$, $(k, m) = 1$.

It follows from the results of sections 6, 7 that if the non-wandering set of f contains a fixed sink or a saddle of negative orientation type, then there exists a fixed pair A_f, R_f $(m_f = 1)$. Otherwise, $m_f = \mu_f$ for any pairs A_f, R_f, where μ_f is a period of non-wandering points of the diffeomorphism f that is different from 1.

Denote by G_1 the subset of G consisting of diffeomorphisms of f for which there exists a fixed pair A_f, R_f $(m_f = 1)$ and by $G_m, m > 1$ the subset of G, consisting of f diffeomorphisms for which $m_f = m$ for any pair A_f, R_f.

By the lemmas 6.3, 6.4, any diffeomorphism $f \in G_1$ connected by a stable arc with the diffeomorphism ϕ_0. By the lemmas 8.1, 8.2, 8.3, 8.5, any diffeomorphism $f \in G_m$ is connected by a stable arc with some diffeomorphism $\phi_{k,m}$.

Finitely, in section 9 we give a complete classification of the model diffeomorphisms $\phi_{k,m}$ with respect to the stable connectedness. □

5. **Model diffeomorphisms.** In this section, we give an exact description of the model diffeomorphisms $\phi_{k,m}, \phi_0 : S^2 \to S^2$.

For $m \in \mathbb{N}$ we define a vector field on the plane $\mathbb{R}^2$ using the following system of differential equations in polar coordinates (r, φ)

$$\begin{cases} \dot{r} = -r(r-1), \\ \dot{\varphi} = -\varphi\left(\varphi - \frac{\pi}{m}\right)\dots\left(\varphi - \frac{(2m-1)\pi}{m}\right) \end{cases}.$$

Denote by χ_m^t the flow induced by this vector field and denote by χ_m its one-time shift. The resulting diffeomorphism has a hyperbolic source at the origin O, hyperbolic saddles at $A_{2i}\left(1, \frac{2\pi i}{m}\right)$ and hyperbolic sinks at $A_{2i+1}\left(1, \frac{2\pi i+1}{m}\right)$, $i \in \{0, \dots, m-1\}$ (see figure 9).

For any integer $k \geq 0$ such that $k < m$, $(k, m) = 1$ we define a diffeomorphism $\theta_{k,m} : \mathbb{R}^2 \to \mathbb{R}^2$ as follows $\theta_{k,m}(r, \varphi) = (r, \varphi + \frac{2\pi k}{m})$. We define the diffeomorphism $\bar{\phi}_{k,m} : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$\bar{\phi}_{k,m} = \theta_{k,m} \circ \chi.$$

By construction, the non-wandering set of the diffeomorphism $\bar{\phi}_{k,m}$ coincides with the non-wandering set of the diffeomorphism χ_m, and all sink points form a unique orbit of the period m and all saddle points form a unique orbit of the period m of the diffeomorphism $\bar{\phi}_{k,m}$.

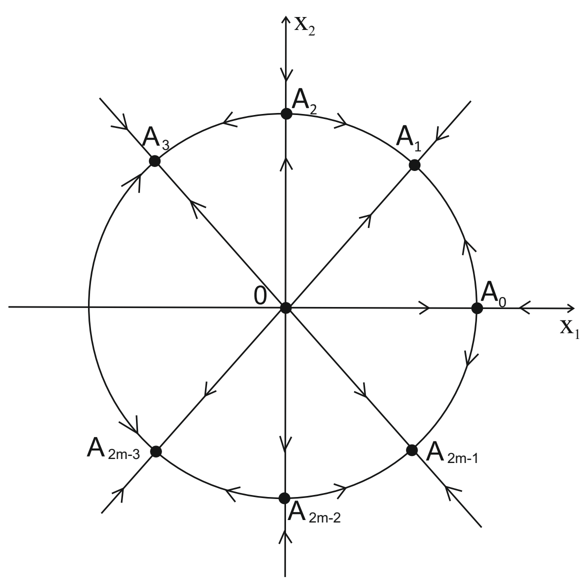

FIGURE 9. Phase portrait of diffeomorphism χ_m

Consider the standard two-dimensional sphere

$$\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Denote by $S(0,0,-1)$ south pole and define a stereographic projection $\vartheta : \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$ by the formula

$$\vartheta(x_1, x_2, x_3) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3}\right).$$

Then the inverse map $\vartheta^{-1} : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{S\}$ is given by the formula

$$\vartheta^{-1}(x_1, x_2) = \left(\frac{2x_1}{x_1^2 + x_2^2 + 1}, \frac{2x_2}{x_1^2 + x_2^2 + 1}, \frac{1-(x_1^2 + x_2^2)}{x_1^2 + x_2^2 + 1}\right).$$

Define a diffeomorphism $\phi_{k,m} : \mathbb{S}^2 \to \mathbb{S}^2$ by the formula

$$\phi_{k,m}(s) = \begin{cases} \vartheta^{-1} \circ \bar{\phi}_{k,m} \circ \vartheta(s),\ s \in \mathbb{S}^2 \setminus \{S\}, \\ S,\ s = S. \end{cases}$$

By construction, the diffeomorphism $\phi_{k,m}$ is a gradient-like diffeomorphism of a 2-sphere with the following non-wandering set (see figure 10):

- fixed source points:
  at the north pole $\alpha_1 = \vartheta^{-1}(O)$, at the south pole $\alpha_2 = S$;
- saddle and sink orbits of the period m at the equator:
  saddle orbit $\mathcal{O}_\sigma = \{\vartheta^{-1}(A_0), \vartheta^{-1}(A_2), \dots, \vartheta^{-1}(A_{2m-2})\}$,
  sink orbit $\mathcal{O}_\omega = \{\vartheta^{-1}(A_1), \vartheta^{-1}(A_3), \dots, \vartheta^{-1}(A_{2m-1})\}$.

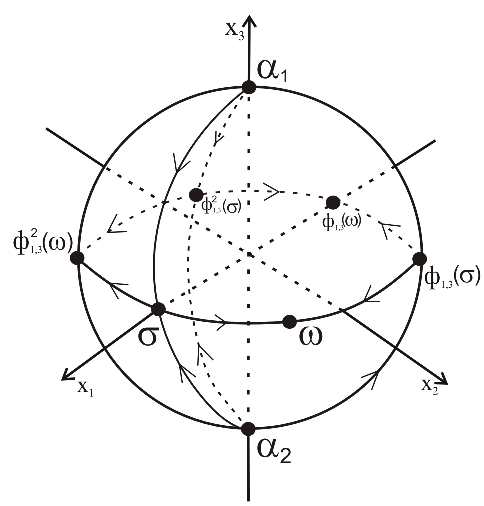

FIGURE 10. Diffeomorphism $\phi_{1,3}$

Let us define χ_0 as one-time shift of the flow $\chi_0^t : \mathbb{R}^2 \to \mathbb{R}^2$ given by the vector field $\dot{r} = -r$. Define a diffeomorphism $\phi_0 : \mathbb{S}^2 \to \mathbb{S}^2$ by the formula

$$\phi_0(s) = \begin{cases} \vartheta^{-1} \circ \chi_0 \circ \vartheta(s),\ s \in \mathbb{S}^2 \setminus \{S\}, \\ S,\ s = S. \end{cases}$$

By construction, the diffeomorphism ϕ_0 is a source-sink diffeomorphism with the source $\alpha = \vartheta^{-1}(O)$ and the sink $\omega = S$.

6. **Diffeomorphisms of class G_1.** Recall that by G_1 we denote the subset G, consisting of diffeomorphisms f, for which there exists a fixed pair A_f, R_f ($m_f = 1$).

6.1. **Attractor structure.** Let $f \in G_1$. We associate the attractor A_f with the graph Γ_f so that its vertices V_{Γ_f} are in one-to-one correspondence with periodic points, and the edges E_{Γ_f} – with saddle separatrices of the diffeomorphism f, belonging to the attractor A_f. Moreover, the diffeomorphism f naturally induces the automorphism P_{Γ_f} of the graph Γ_f.

Lemma 6.1. *If $f \in G_1$, then the graph Γ_f is a tree.*

Proof. We show that if the attractor A_f is different from the sink, then it does not contain cycles.

Suppose the opposite: A_f contains a cycle formed by the closures of the unstable manifolds of saddle points $\sigma_1, \dots, \sigma_r$. Then the closed curve $\bigcup\limits_{i=1}^{r} cl\, W^u_i$ bounds a disk $d \subset D_f$. It implies that one of the stable separatrices of every saddle σ_i lies in the disk d. Consequently, the closure of this separatrix lies in the disk d. Thus, $R_f \cap D_f \neq \emptyset$, which contradicts lemma 3.2. □

The following lemma follows directly from the proposition 8.

Lemma 6.2. *If $f \in G_1$ and the attractor A_f of the diffeomorphism f is different from the sink, then exactly one of the following statements is true:*

1) $A_f = cl\, W^u_\sigma$, where σ is a saddle point with a negative orientation type;

2) there is a saddle point $\sigma \in A_f$ with a positive orientation type and a sink point $\omega \in A_f$ such that $m_\sigma = m_\omega$, $\omega \in cl\, W^u_\sigma$ and the intersection $W^s_\omega \cap A_f$ onsists of exactly one unstable separatrix of the saddle σ and the sink ω.

6.2. **Trivialization of the attractor for $f \in G_1$.** Denote by H_1 a subset of G_1, consisting of diffeomorphisms for which the attractor A_f consists of one sink orbit.

Lemma 6.3. *Any diffeomorphism $f \in G_1$ is connected by a stable arc with some diffeomorphism $g \in H_1$, coinciding with f on $S^2 \setminus D_f$.*

Proof. We divide the set G_1 into subsets $G_1 = G_{1,1} \cup G_{1,2} \cup \dots \cup G_{1,\lambda} \cup \dots$, where $\lambda \in \mathbb{N}$ is the number of sink orbits in the attractor A_{f_λ} for a diffeomorphism $f_\lambda \in G_{1,\lambda}$. Note that $G_{1,1} = H_1$, then to prove the statement it is enough to construct a stable arc connecting a diffeomorphism $f_\lambda \in G_{1,\lambda}$, $\lambda > 1$ with a diffeomorphism $f_{\lambda-1} \in G_{1,\lambda-1}$.

Let $f = f_\lambda$. By lemma 6.2 there exist points $\sigma, \omega \in A_f$ such that $q_\omega = 0, q_\sigma = 1$, $\omega \in cl\, W^u_\sigma$ and the intersection $W^s_\omega \cap A_f$ consists of exactly one unstable separatrix γ of the saddle σ and the sink ω, while the period of ω, we denote it by m. By lemma 7, without loss of generality, we can assume that there exists a local map (U, ψ) of S^2 such that $\omega \in U$, $\psi(\omega) = O$, $f^m(U) \subset U \subset D_f$ and $\psi(\gamma \cap U) \subset OX_1$. According to lemma 6.2 for the diffeomorphism f two cases are possible: 1) $\nu_\sigma = -1$; 2) $\nu_\sigma = 1$. We construct the desired arc separately for each case.

1) In this case $A_f = W^u_\sigma \cup \omega \cup f(\omega)$ and $m = 2$. Let $l = W^u_\sigma \cup \psi^{-1}(OX_1) \cup f(\psi^{-1}(OX_1))$. Then l is a smooth curve containing A_f, the points $\omega, f(\omega)$ are internal and $f(l) \subset l$. Let $\Pi_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| \le \frac{1}{2}\}$. Define the diffeomorphism $\tilde{\varphi} : \Pi_1 \to \mathbb{R}^2$ by the formula

$$\tilde{\varphi}(x_1, x_2) = \left(-\frac{11}{10}x_1 + x_1^3, -\frac{x_2}{2}\right).$$

By construction $\tilde{\varphi}(\Pi_1) \subset int\,\Pi_1$, the diffeomorphism $\tilde{\varphi}$ has a saddle point O and a sink periodic orbit $\{P_0, \tilde{\varphi}(P_0)\}$, where $P_0(-x_0, 0)$, $\tilde{\varphi}(P_0) = (x_0, 0)$, $x_0 \in (0, 1/2)$. Let $\Pi_2 = \tilde{\varphi}(\Pi_1)$. We choose a closed neighborhood V of the attractor A_f and a diffeomorphism $\beta : V \to \Pi_1$ so that $f(V) \subset int\,V$, $\beta(l \cap V) = Ox_1 \cap \Pi_1$, $\beta(f(V)) = \Pi_2$, $\beta(\omega) = P_0$ and $\beta(f(\omega)) = \tilde{\varphi}(P_0)$ (see figure 11). Let $\tilde{f} = \beta f \beta^{-1} : \Pi_1 \to \Pi_2$. Then on the set Π_2 the family of maps $\chi_t : \Pi_2 \to \mathbb{R}^2$ is correctly defined by the formula

$$\chi_t = (1-t)\tilde{f} + t\tilde{\varphi}.$$

By construction $\chi_t(\Pi_2) \subset int\,\Pi_2$ for all $t \in [0, 1]$. Note that the origin is a

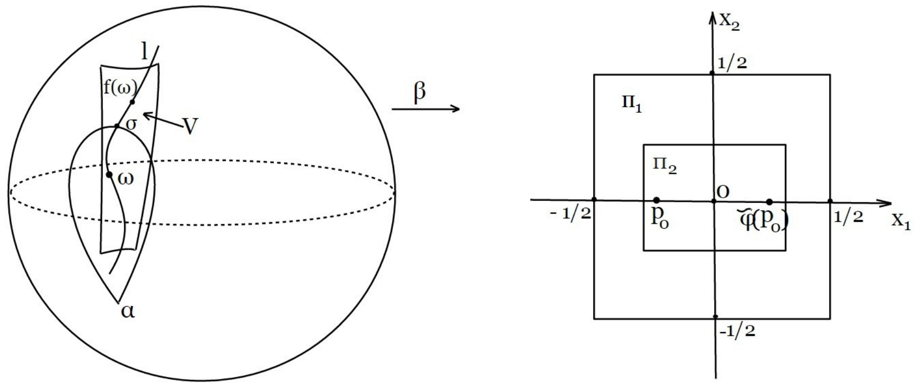

FIGURE 11. Illustration to the lemma 6.3, case 1)

fixed saddle point for the diffeomorphism χ_t and the points P_0, $\tilde{\varphi}(P_0)$ form a sink orbit. In addition, the isotopy $\xi_t = \tilde{f}^{-1}\chi_t|_{\Pi_2}$ connects the identity map with the diffeomorphism $\tilde{f}^{-1}\tilde{\varphi}$ and $\xi_t(\Pi_2) \subset int\,\Pi_1$. By proposition 5, there exists an isotopy $\Xi_t : \Pi_1 \to \Pi_1$, coinciding with ξ_t on Π_2 and identical on $\partial\Pi_1$. Let

$$\tilde{f}_t = \tilde{f}\Xi_t.$$

Note that $\tilde{f}_1 = \tilde{\varphi}$ on Π_2. Let $\Pi_3 = \tilde{\varphi}(\Pi_2)$. Define the arc $\eta_t : \Pi_3 \to \mathbb{R}^2$ by the formula

$$\eta_t(x_1, x_2) = \left(-x_1\left(1 + \frac{1}{10}(1-2t)\right) + x_1^3, -\frac{x_2}{2}\right).$$

By construction, $\eta_t(\Pi_3) \subset int\,\Pi_3$ for all $t \in [0, 1]$, in addition, the isotopy $\zeta_t = \tilde{\varphi}^{-1}\eta_t$ connects the identity map with the diffeomorphism $\tilde{\varphi}^{-1}\eta_1$ and $\zeta_t(\Pi_3) \subset int\,\Pi_2$. By the proposition 5, there exists an isotopy $\theta_t : \mathbb{R}^2 \to \mathbb{R}^2$, which coincides with ζ_t on Π_3 and is identical outside Π_2. Let

$$\tilde{\Theta}_t = \tilde{\varphi}\theta_t.$$

Then the desired arc is the product of the arcs $f_t, \Theta_t : S^2 \to S^2$, where f_t coincides with f outside V, $f_t(z) = \beta^{-1}(\tilde{f}_t(\beta(z)))$ for $z \in V$ and Θ_t coincides with f_1 outside $f_1(V)$, $\Theta_t(z) = \beta^{-1}(\tilde{\Theta}_t(\beta(z)))$ for $z \in f_1(V)$.

2) In this case the saddle σ and the sink ω have the same period m. Let $l = W^u_\sigma \cup \psi^{-1}(OX_1)$. Then l is a smooth curve containing γ and for which the points ω, σ are internal. Let $\Pi_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| \leq \frac{1}{2}\}$, $\tilde{U}_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| < \frac{3}{4}\}$,

$\tilde{U}_2 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| < \frac{2}{3}\}$. Define the diffeomorphism $\tilde{\varphi} : \tilde{U}_1 \to \mathbb{R}^2$ by the formula

$$\tilde{\varphi}(x_1, x_2) = \left(x_1 + 2x_1^2 - \frac{1}{10}, \frac{x_2}{2}\right).$$

By construction, the diffeomorphism $\tilde{\varphi}$ has a saddle point $P_1(0, x_1), x_1 \in (0, 1/2)$ and a sink point $P_2(-x_2, 0), x_2 \in (0, 1/2)$. Let $\Pi_2 = \tilde{\varphi}(\Pi_1)$.

We choose a closed neighborhood V of the arc γ, an open neighborhood of $U_1 \supset V$ of the arc γ and a diffeomorphism $\beta : U_1 \to \tilde{U}_1$ so that $\beta(\sigma) = P_1$, $\beta(\omega) = P_2$, $\beta(l \cap U_1) = Ox_1 \cap \tilde{U}_1$, $\beta(V) = \Pi_1$, $\beta(f^m(V)) = \Pi_2$ (see figure 12). Let $\tilde{f} = \beta f^m \beta^{-1} : \tilde{U}_1 \to \tilde{\varphi}(\tilde{U}_1)$. Then on the set Π_1 the family of maps $\chi_t : \Pi_1 \to \Pi_2$ is correctly defined by the formula

$$\chi_t = (1 - t)\tilde{f} + t\tilde{\varphi}.$$

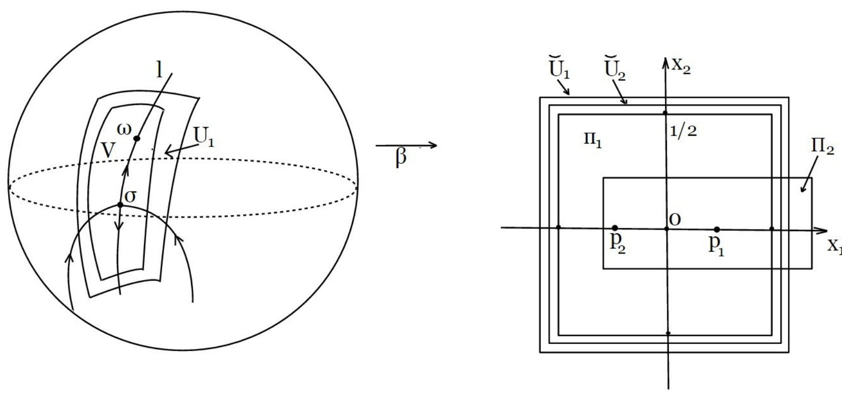

FIGURE 12. Illustration to the lemma 6.3, case 2)

Note that the point P_1 is a fixed saddle point and the point P_2 is a fixed sink point for the diffeomorphism χ_t. In addition, the isotopy $\xi_t = \tilde{f}^{-1}\chi_t|_{\Pi_1} : \Pi_1 \to \Pi_1$ connects the identity map with the diffeomorphism $\tilde{f}^{-1}\tilde{\varphi}$. By proposition 5, there exists an isotopy $\Xi_t : \tilde{U}_1 \to \tilde{U}_1$, coinciding with ξ_t on Π_1 and identical on $\partial\tilde{U}_1$. Let

$$\tilde{f}_t = \tilde{f}\Xi_t.$$

Note that $\tilde{f}_1 = \tilde{\varphi}$ on Π_1. Define the arc $\eta_t : \Pi_1 \to \mathbb{R}^2$ by the formula

$$\eta_t(x_1, x_2) = \left(x_1 + 2x_1^2 + \frac{1}{10}(2t - 1), \frac{x_2}{2}\right).$$

By construction $\eta_t(\Pi_1) \subset \Pi_2$ for all $t \in [0, 1]$, in addition, the isotopy $\zeta_t = \tilde{\varphi}^{-1}\eta_t$ connects the identity map with the diffeomorphism $\tilde{\varphi}^{-1}\eta_1$ and $\zeta_t(\Pi_1) \subset \Pi_1$. By proposition 5, there exists an isotopy $\theta_t : \tilde{U}_2 \to \tilde{U}_2$, coinciding with ζ_t on Π_1 and identical on $\partial\tilde{U}_2$. Let

$$\tilde{\Theta}_t = \tilde{\varphi}\theta_t.$$

Let $U_2 = \beta^{-1}(\tilde{U}_2)$. Then the desired arc is the product of the arcs $f_t, \Theta_t : S^2 \to S^2$, where f_t coincides with f outside $\bigcup\limits_{k=0}^{m-1} f^k(U_1)$, $f_t(z) = f(z)$ for $z \in f^k(U_1)$, $k \in \{0, \ldots, m-2\}$ and $f_t(z) = \beta^{-1}(\tilde{f}_t(\beta(f^{1-m}(z))))$ for $z \in f^{m-1}(U_1)$; Θ_t coincides with f_1 outside $\bigcup\limits_{k=0}^{m-1} f^k(U_2)$, $\Theta_t(z)=f_1(z)$ for $z\in f_1^k(U_2)$, $k\in\{0,\dots,m-2\}$ and $\Theta_t(z)=\beta^{-1}(\tilde\Theta_t(\beta(f_1^{1-m}(z))))$ for $z\in f^{m-1}(U_2)$. □

6.3. **Trivialization of the repeller for $f\in H_1$.**

Lemma 6.4. *Any diffeomorphism $f\in H_1$ is connected by a stable arc with diffeomorphism ϕ_0.*

Proof. Let $f\in H_1$. Then the diffeomorphism f^{-1} belongs to the class G_1 and has a connected attractor $A_{f^{-1}}=R_f$ in the disk $D_{f^{-1}}=S^2\setminus int\,D_f$.

By lemma 6.3 there exists a stable arc $\Gamma_{f^{-1},h,t}$ connecting the diffeomorphism f^{-1} with some diffeomorphism $h\in H_1$ and such that $\Gamma_{f^{-1},h,t}=f^{-1}$ on D_f. By construction, the diffeomorphism h is a source-sink diffeomorphism, as well as h^{-1}. Thus, the arc $\Gamma^{-1}_{f^{-1},h,t}$ connects the diffeomorphism f^{-1} with the source-sink diffeomorphism.

Then the arc $\Gamma^{-1}_{f^{-1},h,t}$ connects the diffeomorphism f with a source-sink diffeomorphism. By the proposition 4 it can be connected by an arc without bifurcations with the diffeomorphism ϕ_0. □

7. **Properties of the number m_f.** Denote by G^+ the subset of G, consisting of diffeomorphisms all of whose saddle points have a positive orientation type. Let $G^-=G\setminus G^+$.

7.1. **Diffeomorphisms $f\in G^-$.**

Lemma 7.1. *$G^-\subset G_1$.*

Proof. Let $f\in G^-$. Choose a pair A_f,R_f, satisfying lemma 3.1. By proposition 3, $\mu_f=2$ and, therefore, $m_f\le 2$.

If $m_f=1$, then the lemma is proved. Consider the case $m_f=2$. Let σ be a saddle with a negative orientation type. By the lemma 3.2, all periodic points belonging to the attractor A_f have a period at least two. According to proposition 3, σ is a fixed point, therefore σ does not belong to the attractor A_f. Adding W^u_σ to A_f,we get a new attractor $\tilde A_f$ and a dual repeller $\tilde R_f$ to it. By construction, $\tilde A_f$ is connected and lies in the disk, just like the $\tilde R_f$ repeller. Thus $\tilde m_f=1$ for the pair $\tilde A_f$, $\tilde R_f$. □

7.2. **Diffeomorphisms $f\in G^+$.** Recall that by proposition 3, for any diffeomorphism $f\in G^+$ there exists a natural number μ_f such that all periodic (non-fixed) points of the diffeomorphism f have period μ_f.

Lemma 7.2. *For any diffeomorphism $f\in G^+$ the number m_f is uniquely determined, that is, it does not depend on the choice of the pair A_f,R_f. Moreover, $m_f=1$, if the diffeomorphism f has at least one fixed sink and $m_f=\mu_f$ otherwise.*

Proof. It follows from lemma 3.2 that all periodic points of the attractor A_f of the diffeomorphism $f\in G^+$ have a period at least m_f and there is at least one sink point of the period m_f. If $m_f>1$, then, by proposition 3, all periodic points of the attractor A_f, and therefore all sinks of f have period $\mu_f=m_f$. Thus, the number m_f is uniquely determined, that is, it does not depend on the choice of the pair A_f,R_f. Moreover, $m_f=1$, if f has at least one fixed sink, and $m_f=\mu_f$ otherwise. □

Remark 2. If $m_f = 1$ for some diffeomorphism $f \in G^+$, then μ_f may be different from 1 (see the picture 13).

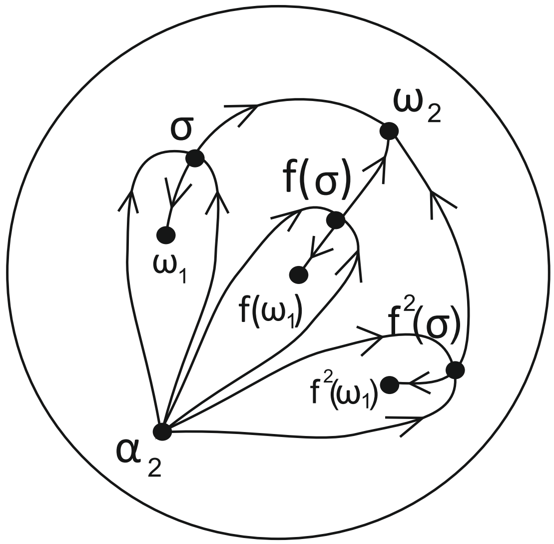

FIGURE 13. A diffeomorphism $f \in G^+$, for which $m_f = 1$, $\mu_f = 3$

Thus, the set $G^+ \setminus G_1$ is represented as pairwise disjoint subsets of

$$G^+ \setminus G_1 = G_2 \dots \cup G_m \cup \dots$$

such that $m_f = \mu_f = m$ for any diffeomorphism $f \in G_m$, $m > 1$.

8. Diffeomorphisms of the class $G_m, m > 1$.

8.1. **Trivialization of the attractor for $f \in G_m$.** For $m > 1$ denote by H_m the subset of G_m, consisting of diffeomorphisms for which the attractor A_f consists of one sink orbit $\mathcal{O}_\omega$ (of the period m by lemma 3.2).

Lemma 8.1. *Any diffeomorphism $f \in G_m$ is connected by a stable arc with some diffeomorphism $g \in H_m$, coinciding with f on $S^2 \setminus (D_f \cup \dots \cup f^{m_f-1}(D_f))$ (see figure 14).*

Proof. By the lemma 3.2, the attractor A_f of $f \in G_m$, $m > 1$ lies on the disjoint union of disks $D_f, \dots, f^{m-1}(D_f)$ and the diffeomorphism $f^m|_{D_f}$ is conjugate to linear contraction. Let g_0 be a 2-sphere diffeomorphism coinciding with f^m on D_f and having a unique hyperbolic source in $S^2 \setminus D_f$. By construction, $g_0 \in G_1$. By lemma 6.3 there exists a stable arc $g_t : S^2 \to S^2$, $t \in [0, 1]$, connecting the diffeomorphism g_0 with the diffeomorphism $g_1 \in H_1$ and such that $g_t = g_0$ on $S^2 \setminus D_f$. Then the desired arc f_t has the form

$$f_t(x) = \begin{cases} f(x),\ x \in f^i(D_f),\ i = 0, \dots, m-2, \\ g_t(f^{1-m}(x)),\ x \in f^{m-1}(D_f), \\ f(x),\ x \in S^2 \setminus (D_f \cup f(D_f) \cup \dots \cup f^{m-1}(D_f)). \end{cases}$$

□

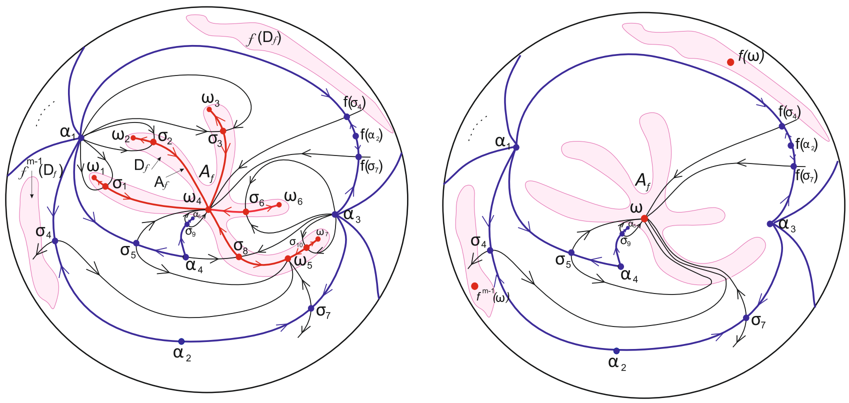

FIGURE 14. Transition from the diffeomorphism $f \in G_m$ to the diffeomorphism $g \in H_m$

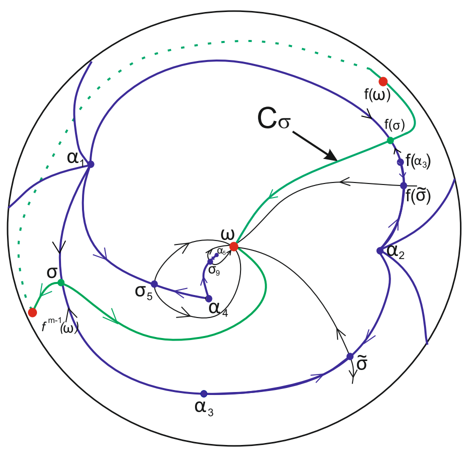

FIGURE 15. Curve C_σ

Lemma 8.2. *For any diffeomorphism $f \in H_m$ there exists a saddle orbit $\mathcal{O}_\sigma$ of period m such that $cl\, W^u_{\mathcal{O}_\sigma}$ is a f-invariant closed curve C_σ (see figure 15).*

Proof. Let $f \in H_m$. Then A_f consists of m sink points $\omega, ..., f^{m-1}(\omega)$. Since the set $cl\, W^u_{\Sigma^1_f}$ is connected, there exists a saddle point $\sigma \in \Sigma^1_f$ such that the connected components of the set $W^u_\sigma \setminus \sigma$ are in the different sink basins. Since the period of the point σ coincides with the period of its separatrices and is m, the set $C_\sigma = cl \bigcup\limits_{i=0}^{m-1} W^u_{f^i(\sigma)}$ is a closed curve. $\square$

Lemma 8.3. *For any saddle orbit $\mathcal{O}_\sigma$, satisfying the conclusion of lemma 8.2, there exists a unique number $k < m/2$, $(k, m) = 1$ such that the map $f|_{C_\sigma}$ is topologically conjugate to a rough transformation of the circle with the rotation number $\frac{k}{m}$.*

Proof. Let C_σ be an f-invariant closed curve constructed in lemma 8.3. Then the homeomorphism $f|_{C_\sigma} : C_\sigma \to C_\sigma$ is topologically conjugate to an orientation-preserving rough circle transformation. By [20] there is a unique number $k < m/2$, $(k, m) = 1$ such that $\frac{k}{m}$ is the rotation number of this transformation. Suppose that there exists another closed curve $C_{\tilde\sigma}$, that satisfies lemma 8.3. We show that the homeomorphism $f|_{C_{\tilde\sigma}} : C_{\tilde\sigma} \to C_{\tilde\sigma}$ is topologically conjugate to a rough transformation of a circle with the same rotation number $\frac{k}{m}$.

For this we note that the set $\tilde A_f = C_\sigma$ is a connected attractor of the diffeomorphism f and divides the sphere S^2 into two disks D_1 and D_2. The dual repeller $\tilde R_f = R_f \setminus W^s_{\mathcal{O}_\sigma}$ consists of two f-invariant connected components $R_1 \subset D_1$, $R_2 \subset D_2$. Since the curve $C_{\tilde\sigma}$ is f-invariant, it lies in the closure of one of the disks, suppose D_1, for definiteness (see figure 16).

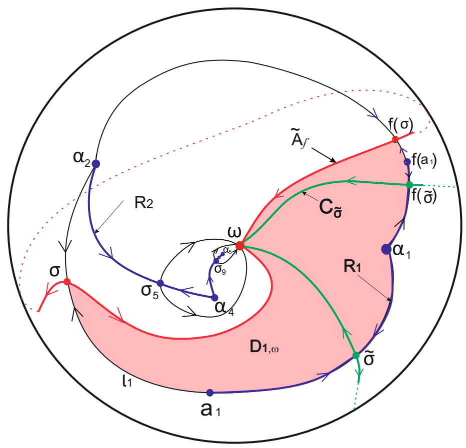

FIGURE 16. Curve $C_{\tilde\sigma}$

Similarly to the lemma 6.1 it can be shown that the repeller R_1 is a tree, we denote it by Γ_1. Moreover, the diffeomorphism f induces an automorphism P_{Γ_1}, for which all edges have a period m. This means (see section 2.4), that the graph is central, that is, the repeller R_1 contains a single fixed point, which is the source, we denote it by α_1. Denote by l_1 the connected component $W^u_\sigma \setminus \sigma$, belonging to the disk D_1. Let $a_1 = cl(l_1) \setminus (l_1 \cup \sigma)$ (see the picture 16). Then in the tree R_1 there is a unique path L_{a_1,α_1}, connecting the source a_1 with the source α_1. It follows from the properties of the tree that the path L_{a_1,α_1} consists of vertices of pairwise different ranks, that is, $L_{a_1,\alpha_1} \cap f(L_{a_1,\alpha_1}) = \alpha_1$.

Let $L_1 = l_1 \cup L_{a_1,\alpha_1}$. Then the set $\mathcal{L}_1 = L_1 \cup f(L_1) \cup \cdots \cup f^{m-1}(L_1)$ divides the disk D_1 into m of pairwise disjoint parts $D_{1,\omega}, \dots, D_{1,f^{m-1}(\omega)}$, each of which contains a single sink $\omega, \dots, f^{m-1}(\omega)$, accordingly, in its closure. Moreover, by continuity, the diffeomorphism f induces on the components $D_{1,\omega}, \dots, D_{1,f^{m-1}(\omega)}$ a rotation with the same rotation number as on the circle C_σ. Since any saddle point lying inside such a part has unstable separatrices going to the same sink, $\tilde\sigma \in \mathcal{L}_1$. Thus, the homeomorphism $f|_{C_{\tilde\sigma}}$ is topologically conjugate to the rough transformation of the circle with the rotation number $\frac{k}{m}$. $\square$

Lemma 8.4. *For any diffeomorphism $f \in G_m$, $m > 1$ the following properties hold:*

- *there exists a simple closed f-invariant curve (may be not unique) C_f composed by the closures of the unstable manifolds of saddle points;*
- *for all such curves C_f the homeomorphisms $f|_{C_f}$ have the same rotation number $\frac{k}{m}$, $k < m/2$, $(k, m) = 1$.*

Proof. According lemmas 6.1 and 8.1, every connected component of A_f is a tree for $f \in G_m$, $m > 1$. Moreover, by lemma 8.1, there is a diffeomorphism $g \in H_m$, coinciding with f on $S^2 \setminus (D_f \cup \cdots \cup f^{m_f-1}(D_f))$. Let A be a connected component of A_f belonging to D_f. Then for the saddle orbit $\mathcal{O}_\sigma$, satisfying the conclusion of lemma 8.2, denote by A_σ the intersection $cl\, W^u_{\mathcal{O}_\sigma} \cap A$. If A_σ consists of a unique point then $C_f = cl\, W^u_{\mathcal{O}_\sigma}$. In the opposite case A_σ consists of two vertex of the tree A_f. Let c_σ be a simple path connected them. Then $C_f = cl\, W^u_{\mathcal{O}_\sigma} \cup c_\sigma$. By lemma 8.3, for all such curves C_f the homeomorphisms $f|_{C_f}$ have the same rotation number $\frac{k}{m}$, $k < m/2$, $(k, m) = 1$. □

For $k \in (\mathbb{N} \cup 0)$, $m \in \mathbb{N}$, $k < m/2, (k, m) = 1$ we denote by $G_{k,m}$ the subset of G_m such that $f|_{C_f}$ is topologically conjugate to a rough transformation of the circle with the rotation number $\frac{k}{m}$ for any $f \in G_{k,m}$. We denote by $H_{k,m}$ the subset of $G_{k,m}$ consisting from diffeomorphisms with unique sink orbit.

8.2. **Trivialization of the repeller for $f \in H_{k,m}$.** Denote by $F_{k,m}$ the subset of $H_{k,m}$, consisting of diffeomorphisms having a repeller R_f, containing a unique saddle orbit (of the period m by lemma 8.2).

Lemma 8.5. *Any diffeomorphism $f \in H_{k,m}$ is connected by a stable arc with some diffeomorphism $g \in F_{k,m}$.*

Proof. The set $A_f = cl\, W^u_{\mathcal{O}_\sigma}$ is a connected attractor, homeomorphic to a circle. Then there exists a neighborhood K of this attractor diffeomorphic to an annulus and such that $f(K) \subset int\, K$. Then the set $S^2 \setminus K$ consists of two disjoint disks D_1, D_2. Denote by g_i a 2-sphere diffeomorphism coinciding with f on D_i and having a unique hyperbolic sink in $S^2 \setminus D_i$. By construction, $g_i \in G_1$. By lemma 6.4 there exists a stable arc $g_{i,t} : S^2 \to S^2$, $t \in [0, 1]$, connecting the diffeomorphism g_i with the source-sink diffeomorphism, while $g_{i,t} = g_i$ on $S^2 \setminus f^{-1}(D_i)$. Define the arc f_t by the formula

$$f_t(x) = \begin{cases} f(x),\ x \in f^{-1}(K), \\ g_{i,t}(x),\ x \in f^{-1}(D_i). \end{cases}$$

□

Lemma 8.6. *Any diffeomorphism $f \in F_{k,m}$ is connected by an arc without bifurcations with diffeomorphism $\phi_{k,m}$.*

Proof. Let $f \in F_{k,m}$. Then the non-wandering set of diffeomorphism f consists of one saddle orbit $\mathcal{O}_\sigma = \{\sigma, f(\sigma), \dots, f^{m-1}(\sigma)\}$, one sink orbits $\mathcal{O}_\omega = \{\omega, f(\omega), \dots, f^{m-1}(\omega)\}$ and fixed sources α_1, α_2. Moreover, the closures of unstable saddle separatrices form a circle

$$C_\sigma = W^u_{\mathcal{O}_\sigma} \cup \mathcal{O}_\omega.$$

Also a similar non-wandering set has a diffeomorphism $\phi_{k,m}$, which we will denote by ϕ for brevity (see figure 20). By proposition 7 the circle C_σ can be considered smooth. Since all orientation-preserving diffeomorphisms of a 2-sphere are smoothly

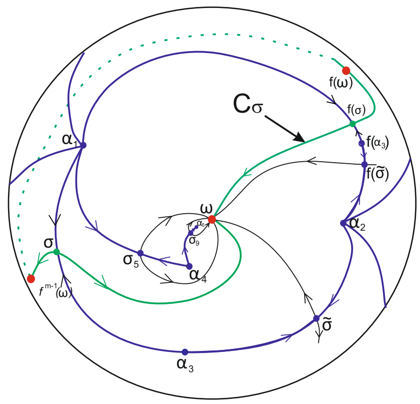

FIGURE 17. $f \in H_{k,m}$

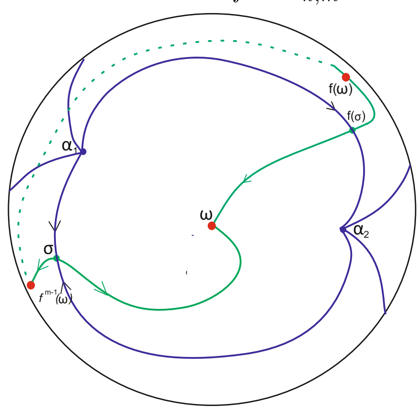

FIGURE 18. $g \in F_{k,m}$

isotopic to the identity map (see, for example, [30]), the circle C_σ can be considered to coincide with the similar circle of the diffeomorphism ϕ, we can also consider the same non-wandering sets of two diffeomorphisms. By proposition 6, we can assume that the diffeomorphisms f and ϕ coincide in some neighborhoods of the periodic points.

Since the circle C_σ is an attractor of both diffeomorphisms, there exist smooth annulus K_f, K_ϕ, containing C_σ and such that $f(K_f) \subset int\, K_f$, $\phi(K_\phi) \subset int\, K_\phi$. We choose a diffeomorphism $\beta : K_f \to K_\phi$ so that $\beta|_{C_\sigma} = id$ and $\beta(f(K_f)) = \phi(K_\phi)$. Let $\tilde{f} = \beta f \beta^{-1} : K_\phi \to \phi(K_\phi)$. Then on the set K_ϕ the family of maps $\chi_t : \phi(K_\phi) \to \mathbb{S}^2$ is correctly defined by the formula

$$\chi_t = (1-t)\tilde{f} + t\phi.$$

By construction, $\chi_t(\phi(K_\phi)) \subset int\, \phi(K_\phi)$ for all $t \in [0,1]$. Note that the circle C_σ is χ_t-invariant and $C_\sigma = W^u_{\mathcal{O}_\sigma} \cup \mathcal{O}_\omega$ for any diffeomorphism χ_t. In addition, the isotopy $\xi_t = \tilde{f}^{-1}\chi_t|_{\phi(K_\phi)}$ connects the identity map with the diffeomorphism $\tilde{f}^{-1}\phi$ and $\xi_t(\phi(K_\phi)) \subset int\, K_\phi$. By proposition 5, there is an isotopy $\Xi_t : K_\phi \to K_\phi$,

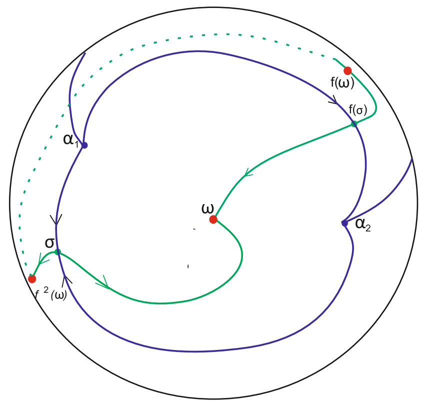

FIGURE 19. $f \in F_{1,3}$

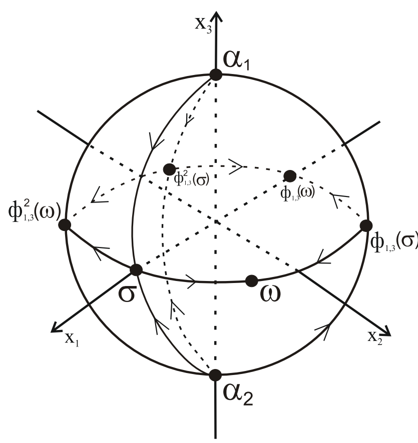

FIGURE 20. $\phi_{1,3}$

coinciding with ξ_t on $\phi(K_\phi)$ and identical on ∂K_ϕ. Let

$$\tilde{f}_t = \tilde{f}\Xi_t.$$

Note that $\tilde{f}_1 = \phi$ on $\phi(K_\phi)$. Define the arc $f_t : \mathbb{S}^2 \to \mathbb{S}^2$ so that f_t coincides with f outside K_f, $f_t(z) = \beta^{-1}(\tilde{f}_t(\beta(z)))$ for $z \in K_f$.

Let $\gamma = f_1$ and $D_i, i = 1, 2$ denotes the connected component of the set $\mathbb{S}^2 \setminus C_\sigma$, containing α_i. Let $\gamma_i = \gamma|_{D_i}$. By construction, there is a neighborhood V_{α_i} of the point α_i, where $\gamma_i|_{V_{\alpha_i}} = \phi|_{V_{\alpha_i}}$. Define the diffeomorphism $\psi_{\gamma_i} : D_i \to D_i$ by the formula

$$\psi_{\gamma_i}(w) = \phi^k(\gamma_i^{-k}(w)),$$

where $k \in \mathbb{Z}$ such that $\gamma_i^{-k}(w) \in \phi(K_\phi)$ for $w \in D_i$. Then $\gamma_i = \psi_{\gamma_i}^{-1}\phi\psi_{\gamma_i}$. If ψ_{γ_i} can be smoothly extended to the point α_i by the condition $\psi_{\gamma_i}(\alpha_i) = \alpha_i$, then, by [30], there exists a smooth isotopy $\rho_{i,t} : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\rho_{i,0} = \psi_{\gamma_i}$, $\rho_{i,1} = id$. Let $\delta_{i,t} = \rho_{i,t}^{-1}\phi\rho_{i,t}$. Denote by $\delta_t : \mathbb{S}^2 \to \mathbb{S}^2$ the arc coinciding $\delta_{i,t}$ on D_i and from ϕ to $\phi(K_\phi)$. Then the product of the arcs f_t and δ_t is the desired arc.

In the case when at least one of the diffeomorphisms ψ_{γ_i}, $i \in \{1,2\}$ can not be smoothly continued to α_i we show that there is an arc $\zeta_{i,t} : D_i \to D_i$, connecting the diffeomorphism $\zeta_{i,0} = \gamma|_{D_i}$ with some diffeomorphism $\zeta_{i,1}$ such so that $\psi_{\zeta_{i,1}}$ can be smoothly extended to α_i by the condition $\psi_{\zeta_{i,1}}(\alpha_i) = \alpha_i$.

For definiteness, let $i = 1$. $\mathbb{B}_r(O) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ and $K_r = \mathbb{B}_r(O) \setminus \phi(int\,\mathbb{B}_r(O)))$ for $r > 0$. Let $\bar{\gamma}_1 = \vartheta\gamma_1\vartheta^{-1}$ and $\bar{\psi}_{\gamma_1} = \vartheta\psi_{\gamma_1}\vartheta^{-1}$, where $\vartheta : \mathbb{S}^2 \setminus \{\alpha_2\} \to \mathbb{R}^2$ is the stereographic projection and $\vartheta(\alpha_1) = O$. Then there exists $r_0 \in (0,1)$ such that $\bar{\gamma}_1 = \bar{\phi} = \vartheta\phi\vartheta^{-1}$ on $\mathbb{B}_{r_0}$ and the ring K_{r_0} is the fundamental domain of the diffeomorphism of $\bar{\phi}$ (and $\bar{\gamma}_1$) and $int\,\mathbb{B}_1(O) \setminus \{O\}$. Represent $\mathbb{T}^2$ as the orbit space of $(int\,\mathbb{B}_1(O) \setminus \{O\})/\bar{\phi}$. Let $p : \mathbb{B}_{r_0} \setminus \{O\} \to \mathbb{T}^2$ denote the natural projection. Then the curves $a = p(Ox_1), b = p(\partial\mathbb{B}_{r_0})$ are the generators of the fundamental group $\pi_1(\mathbb{T}^2)$. Since $\bar{\psi}_{\gamma_1}$ translates the orbits $\bar{\phi}$ into the orbits $\bar{\gamma}_1$ and K_{r_0} is a common fundamental domain for $\bar{\phi}, \bar{\gamma}_1$ on $int\,\mathbb{B}_1(O)\setminus\{O\}$, then $\bar{\psi}_{\gamma_1}$ is projected onto $\mathbb{T}^2$ by the formula $\hat{\psi}_{\gamma_1} = p\bar{\psi}_{\gamma_1}p^{-1}$. Then the induced isomorphism $\hat{\psi}_{\gamma_1 *} : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$ preserves the homotopy class of the generator a and, therefore, is given matrix

$$\begin{pmatrix} 1 & n_0 \\ 0 & 1 \end{pmatrix}$$

for some integer n_0.

The arc $\zeta_{1,t}$ will be the smooth product of the arcs ν_t and μ_t, where

1) ν_t is a smooth arc without bifurcations connecting the diffeomorphism $\nu_0 = \gamma_1$ with the diffeomorphism ν_1 such that $\hat{\psi}_{\nu_1}$ induces an identical isomorphism in $\pi_1(\mathbb{T}^2)$;

2) μ_t is a smooth arc without bifurcations connecting the diffeomorphism $\mu_0 = \nu_1$ with the diffeomorphism μ_1 such that $\hat{\psi}_{\mu_1} = id$, which means $\psi_{\mu_1} = \phi^k$ for some $k \in \mathbb{Z} \setminus \{0\}$, that is, the diffeomorphism ψ_{μ_1} is smoothly continued to α_i.

1) We introduce the polar coordinates r, φ on $\mathbb{R}^2$. Define the diffeomorphism $\bar{\theta}_t$

by the formula $\bar{\theta}_t(re^{i\varphi}) = \begin{cases} re^{i\varphi}, \rho > r_0, \\ re^{i\left(\varphi + 4n_0\pi t\left(1 - \frac{r}{r_0}\right)\right)}, \frac{r_0}{2} \leq r \leq r_0; \\ re^{i(\varphi + 2n_0\pi t)}, r < \frac{r_0}{2}. \end{cases}$

Let $\theta_t = \vartheta^{-1}\bar{\theta}_t\vartheta : D_1 \to D_1 \setminus \{\alpha_1\} \to D_1 \setminus \{\alpha_1\}$, then θ_t can be smoothly continued to α_1 by the condition $\theta_t(\alpha_1) = \alpha_1$. Moreover, by construction, $\hat{\psi}_{\theta_1\gamma_1}$ induces an identical isomorphism on $\pi_1(\mathbb{T}^2)$. Thus, $\nu_t = \theta_t\gamma_1 : D_1 \to D_1$ is the desired arc.

2) Here we are dealing with a diffeomorphism $\nu_1 : D_1 \to D_1$ such that a diffeomorphism $\hat{\psi}_{\nu_1}$ induces an identical isomorphism in $\pi_1(\mathbb{T}^2)$. Then, by [29, 7], the diffeomorphism $\hat{\psi}_{\nu_1}$ is smoothly isotopic to the identity map. We choose a cover $U = \{U_1, \ldots, U_q\}$ of the torus $\mathbb{T}^2$ consisting of disks such that a connected component of the set $p^{-1}(U_i)$ is a subset of K_{r_i} for some r_i such that $B_{r_i}(O) \subset \phi(B_{r_{i-1}}(O))$. By [4, Lemma de fragmentation] there exist smoothly isotopic to the identity diffeomorphisms $\hat{w}_1, \ldots, \hat{w}_q : \mathbb{T}^2 \to \mathbb{T}^2$ such that

i) for each $i = \overline{1, q}$ there exists $U_{j(i)} \in U$ such that for each $t \in [0,1]$ the map $\hat{w}_{i,t}$ is identical outside $U_{j(i)}$, where $\{\hat{w}_{i,t}\}$ is the smooth isotopy between the identity map and $\hat{w}_i$;

ii) $\hat{\psi}_{\nu_1} = \hat{w}_1 \ldots \hat{w}_q$.

Let $\bar{w}_{i,t} : D_1 \to D_1$ be a diffeomorphism that coincides with $(p|_{K_{r_i}})^{-1}\hat{w}_{i,t}p$ on K_{r_i} and is identical outside K_{r_i}. Let $\bar{\mu}_t = \bar{\nu}_1\bar{w}_{1,t}\dots\bar{w}_{q,t} : D_1 \setminus \{\alpha_1\} \to D_1 \setminus \{\alpha_1\}$. By construction, $\bar{\mu}_0 = \bar{\nu}_1$ and $\bar{\mu}_1$ has the property: $\hat{\psi}_{\mu_1} = \hat{w}_q^{-1}\dots\hat{w}_1^{-1}\hat{\psi}_{\nu_1} = \hat{w}_q^{-1}\dots\hat{w}_1^{-1}\hat{w}_1\dots\hat{w}_q = id$. □

9. Classification of the model diffeomorphisms with respect to the stable connectedness.

The classification directly follows from two lemmas below.

Lemma 9.1. *There is a stable arc connecting the diffeomorphism $\phi_{0,1}$ ($\phi_{1,2}$, $\phi_{1,2}^{-1}$, $\phi_{0,1}^{-1}$) with diffeomorphism ϕ_0.*

Proof. We show how to construct a stable arc connecting:

1) $\phi_{1,2}^{-1}$ with ϕ_0; 2) $\phi_{0,1}^{-1}$ with ϕ_0.

For diffeomorphisms $\phi_{1,2}$, $\phi_{0,1}$ the constructions are similar.

1) Let $f = \phi_{1,2}^{-1}$ (see picture 21). Consider a smooth curve $l = cl\, W^u_{\mathcal{O}_\sigma} \setminus \{\omega_2\}$, for which points $\sigma, f(\sigma)$ are internal, while $f(l) \subset l$.

Let $\Pi_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| \leq \frac{1}{2}\}$. Define a diffeomorphism $\tilde{\varphi} : \Pi_1 \to \mathbb{R}^2$ formula along the axis Ox_1

$$\tilde{\varphi}(x_1, x_2) = \left(-\frac{9}{10}x_1 - x_1^3, -\frac{x_2}{2}\right).$$

By construction $\tilde{\varphi}(\Pi_1) \subset int\,\Pi_1$, diffeomorphism $\tilde{\varphi}$ has a sink point O and saddle periodic orbit $\{P_0, \tilde{\varphi}(P_0)\}$, where $P_0(-x_0, 0)$, $\tilde{\varphi}(P_0) = (x_0, 0)$, $x_0 \in (0, 1/2)$. Let $\Pi_2 = \tilde{\varphi}(\Pi_1)$. We choose a closed neighborhood V of arc l and diffeomorphism $\beta : V \to \Pi_1$ in the following way $f(V) \subset int\,V$, $\beta(l \cap V) = Ox_1 \cap \Pi_1$, $\beta(f(V)) = \Pi_2$, $\beta(\omega) = P_0$ and $\beta(f(\omega)) = \tilde{\varphi}(P_0)$ (see picture 21). Set $\tilde{f} = \beta f \beta^{-1} : \Pi_1 \to \Pi_2$. Then on the set Π_2 correctly defined family of maps $\chi_t : \Pi_2 \to \mathbb{R}^2$ by the formula

$$\chi_t = (1-t)\tilde{f} + t\tilde{\varphi}.$$

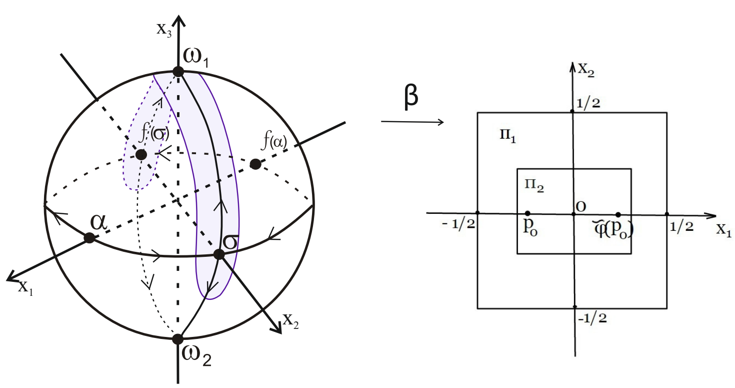

FIGURE 21. Illustration to the lemma 9.1, the case 1)

By construction $\chi_t(\Pi_2) \subset int\,\Pi_2$ for all $t \in [0, 1]$. Note that the origin is a fixed sink point for the diffeomorphism χ_t and the points P_0, $\tilde{\varphi}(P_0)$ form a saddle orbit. In addition, the isotopy $\xi_t = \tilde{f}^{-1}\chi_t|_{\Pi_2}$ connects the identity map with the diffeomorphism $\tilde{f}^{-1}\tilde{\varphi}$ and $\xi_t(\Pi_2) \subset int\,\Pi_1$. By proposition 5, there exists an isotopy $\Xi_t : \Pi_1 \to \Pi_1$, coinciding with ξ_t on Π_2 and identical on $\partial\Pi_1$. Let

$$\tilde{f}_t = \tilde{f}\Xi_t.$$

Note that $\tilde{f}_1 = \tilde{\varphi}$ on Π_2. Let $\Pi_3 = \tilde{\varphi}(\Pi_2)$. Define the arc $\eta_t : \Pi_3 \to \mathbb{R}^2$ by the formula

$$\eta_t(x_1, x_2) = \left(-x_1\left(1 + \frac{1}{10}(2t-1)\right) - x_1^3, -\frac{x_2}{2}\right).$$

By construction $\eta_t(\Pi_3) \subset int\,\Pi_3$ for all $t \in [0,1]$, in addition, isotopy $\zeta_t = \tilde{\varphi}^{-1}\eta_t$ connects the identity map with a diffeomorphism $\tilde{\varphi}^{-1}\eta_1$ and $\zeta_t(\Pi_3) \subset int\,\Pi_2$. By proposition 5, there exists isotopy $\theta_t : \mathbb{R}^2 \to \mathbb{R}^2$, coinciding with ζ_t on Π_3 and identical outside Π_2. Let

$$\tilde{\Theta}_t = \tilde{\varphi}\theta_t.$$

Let $\delta_t = f_t * \Theta_t : \mathbb{S}^2 \to \mathbb{S}^2$, where f_t coincides with f outside V, $f_t(z) = \beta^{-1}(\tilde{f}_t(\beta(z)))$ for $z \in V$ and Θ_t coincides with f_1 outside $f_1(V)$, $\Theta_t(z) = \beta^{-1}(\tilde{\Theta}_t(\beta(z)))$ for $z \in f_1(V)$. Phase portrait of diffeomorphism δ_1 depicted on the picture 22.

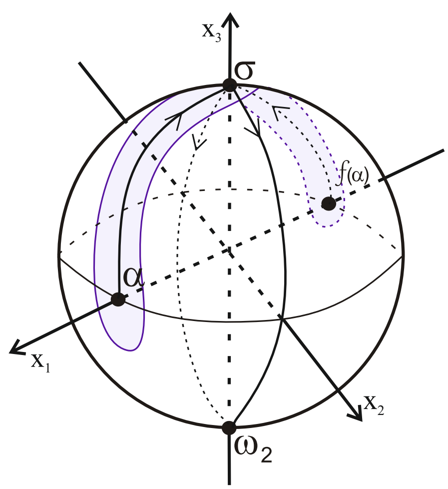

FIGURE 22. Illustration to the lemma 9.1, case 1)

Having done similar constructions in a neighborhood of the arc $cl\,W^s_\sigma$, we connect the diffeomorphism δ_1 with the source-sink diffeomorphism by a stable arc with one flip bifurcation. By the proposition4, any source-sink diffeomorphism is connected by an arc without bifurcations with the diffeomorphism ϕ_0.

2) Let $f = \phi_{0,1}^{-1}$ (see figure 23). For the diffeomorphism f the saddle σ and the drain ω_1 are fixed. Let $l = Ox_1x_3 \cap \mathbb{S}^2$ and denote by $\gamma \subset l$ an arc bounded by the points ω_1, σ and not containing α. Set $\Pi_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| \le \frac{1}{2}\}$, $\tilde{U}_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| < \frac{3}{4}\}$, $\tilde{U}_2 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| < \frac{2}{3}\}$. Define the diffeomorphism $\tilde{\varphi} : \tilde{U}_1 \to \mathbb{R}^2$ by the formula

$$\tilde{\varphi}(x_1, x_2) = \left(x_1 + 2x_1^2 - \frac{1}{10}, \frac{x_2}{2}\right).$$

By construction, a diffeomorphism $\tilde\varphi$ has a saddle point $P_1(0,x_1), x_1\in(0,1/2)$ and sink point $P_2(-x_2,0), x_2\in(0,1/2)$. Let $\Pi_2=\tilde\varphi(\Pi_1)$.

We choose a closed neighborhood V of arc γ, open neighborhood $U_1\supset V$ of arc γ and diffeomorphism $\beta: U_1\to\tilde U_1$ so that $\beta(\sigma)=P_1$, $\beta(\omega_1)=P_2$, $\beta(l\cap U_1)=Ox_1\cap\tilde U_1$, $\beta(V)=\Pi_1$, $\beta(f(V))=\Pi_2$ (see picture 23). Let $\tilde f=\beta f\beta^{-1}:\tilde U_1\to\tilde\varphi(\tilde U_1)$. Then on the set Π_1 correctly defined family of mappings $\chi_t:\Pi_1\to\Pi_2$ by the formula

$$\chi_t=(1-t)\tilde f+t\tilde\varphi.$$

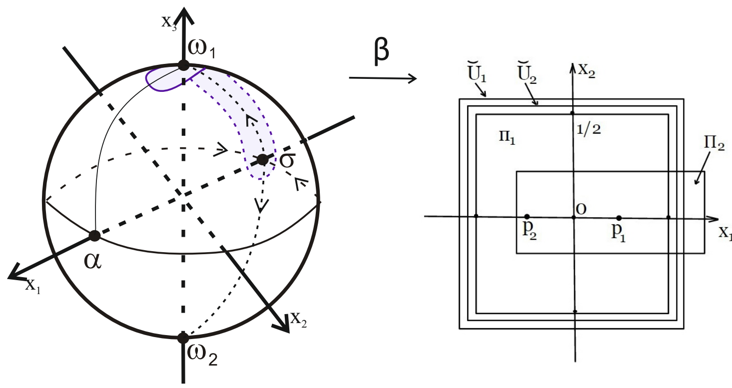

FIGURE 23. Illustration to the lemma 9.1, case 2)

Note that the point P_1 is a fixed saddle point and the point P_2 is a fixed sink for diffeomorphism χ_t. In addition, isotopy $\xi_t=\tilde f^{-1}\chi_t|_{\Pi_1}:\Pi_1\to\Pi_1$ connects the identity map with a diffeomorphism $\tilde f^{-1}\tilde\varphi$. By virtue of the proposition 5, there exists isotopy $\Xi_t:\tilde U_1\to\tilde U_1$, coinciding with ξ_t on Π_1 and identical on $\partial\tilde U_1$. Let

$$\tilde f_t=\tilde f\Xi_t.$$

Notice, that $\tilde f_1=\tilde\varphi$ on Π_1. Define an arc $\eta_t:\Pi_1\to\mathbb{R}^2$ by the formula

$$\eta_t(x_1,x_2)=\left(x_1+2x_1^2+\frac{1}{10}(2t-1),\frac{x_2}{2}\right).$$

By construction $\eta_t(\Pi_1)\subset\Pi_2$ for all $t\in[0,1]$, in addition, isotopy $\zeta_t=\tilde\varphi^{-1}\eta_t\ \tilde\varphi^{-1}\eta_1$ and $\zeta_t(\Pi_1)\subset\Pi_1$. By virtue of the proposition 5,there exists isotopy $\theta_t:\tilde U_2\to\tilde U_2$, coinciding with ζ_t on Π_1 and identical on $\partial\tilde U_2$. Let

$$\tilde\Theta_t=\tilde\varphi\theta_t.$$

Let $U_2=\beta^{-1}(\tilde U_2)$ and $\delta_t=f_t*\Theta_t:\mathbb{S}^2\to\mathbb{S}^2$, f_t coincides with f out of U_1 and $f_t(z)=\beta^{-1}(\tilde f_t(\beta(z)))$ for $z\in U_1$; Θ_t coincides with f_1 out of U_2 and $\Theta_t(z)=\beta^{-1}(\tilde\Theta_t(\beta(z)))$ for $z\in U_2$. By construction, a diffeomorphism δ_1 is a source-sink diffeomorphism. By virtue of the proposition 4, any source-sink diffeomorphism is connected by an arc without bifurcations with a diffeomorphism ϕ_0. □

Lemma 9.2. *Diffeomorphism $\phi_{k,m}$, $k<m/2$, $m>2$ is connected by a stable arc with a diffeomorphism $\phi_{k',m'}$ if and only if $m'=m$, $k'=m-k$; and is not connected with any diffeomorphism $\phi^{-1}_{k',m'}$.*

Proof. Assume that diffeomorphism $\phi_{k,m}$, $k < m/2$, $m > 2$ is connected by a stable arc φ_t with some diffeomorphism $\phi_{k',m'}$ such that $\frac{k}{m} \neq \frac{k'}{m'}$. By remark 1, all diffeomorphisms on φ_t, except bifurcations, belongs to class G. Firstly, let us show that all these φ_t belongs to the same subclass G_m.

Indeed, in the opposite case, as $\phi_{k,m} \in G_m$, there is a stable arc f_t with unique bifurcation value b such that $f_0 \in G_m$ and $f_1 \in G_{\tilde{m}}$, $\tilde{m} \neq m$. By proposition 3, f_0 has the periodic points of exactly two periods 1 and m, moreover, all saddle points have the positive type of the orientation and the period m. As a flip bifurcation is connected with an appearing or disappearing of points of two different periods k and $2k$, then for f_b is impossible to be disappearing because $m > 2$. If f_b is an appearing, then $k = 1$ or $k = m$ and, hence, f_1 necessary has periodic points of three different periods $1, 2, m$ or $1, m, 2m$, that is impossible according to proposition 3. Thus, f_b is a saddle-node bifurcation.

A saddle-node bifurcation is connected with an appearing or disappearing of saddle of the positive type of orientation and node points of the same period. If f_b is an appearing than, by proposition 3, this period equals m and, hence, by lemma 7.2, $\tilde{m} = m$, that is a contradiction. If f_b is a disappearing then there are two possibilities: 1) f_b has no saddle points; 2) f_b has a saddle point. In the case 1) f_b has a saddle-node cycle, that contradicts to definition of the stable arc. In the case 2), by lemma 7.2, $f_1 \in G_m$. Thus, $m = \tilde{m}$.

Let us assume now that $k \neq k'$. There are two possibilities: 1) φ_t has no bifurcation at all; 2) φ_t contains bifurcations. In the case 1) φ_0 is topologically conjugated with φ_1. Hence, they are conjugated on the equator, where φ_0 is a rough transformation of the circle with the rotation number $\frac{k}{m}$, $k < \frac{m}{2}$ and φ_1 – with $\frac{k'}{m}$. By [20], it implies $k' = m - k$. Let us show that there indeed exists a stable arc φ_t, connecting $\phi_{k,m}$ with $\phi_{m-k,m}$. To do this, denote by $\Theta_t : \mathbb{S}^2 \to \mathbb{S}^2$ a rotation $\mathbb{S}^2$ on the angle $2\pi t$ around an axis passing through points $(1, 0, 0)$ and $(-1, 0, 0)$. Then $\varphi_t = \Theta_t \phi_{k,m} \Theta_t^{-1}$.

In the case 2) let us show that all these φ_t belongs to the same subclass $G_{k,m}$.

Indeed, in the opposite case, as $\phi_{k,m} \in G_{k,m}$, there is a stable arc f_t with unique bifurcation value b such that $f_0 \in G_{k,m}$ and $f_1 \in G_{\tilde{k},m}$, $\tilde{k} \neq k$. Similar to the arguments above it is possible to show that for f_b is impossible to be a flip bifurcation. Thus, f_b is a saddle-node bifurcation connected with an appearing or disappearing of saddle of the positive type of orientation and node points of the same period m. By proposition 8.4, f_0 possesses a simple closed f-invariant curve C_f composed by the closures of the unstable manifolds of saddle points such that the homeomorphism $f|_{C_f}$ has the rotation number $\frac{k}{m}$. If f_b is an appearing than the curve is preserved for f_1 and, hence, $\tilde{k} = k$, that is a contradiction. If f_b is a disappearing then there are two possibilities: 1) the disappearing points do not belong to C_f; 2) the disappearing points belong to C_f. In the case 1) the curve C_f is preserved for f_1 and, hence, $\tilde{k} = k$, that is a contradiction. In the case 2) f_b has periodic points different from saddle-node on C_f (in the opposite case we have a saddle-node cycle) then, by [20], $f_b|_{C_f}$ and $f_1|_{C_f}$ has the rotation number $\frac{k}{m}$. Thus, $k = \tilde{k}$. □

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