Russian Journal of Nonlinear Dynamics, 2021, vol. 17, no. 1, pp. 23–37.
Full-texts are available at http://nd.ics.org.ru
DOI: 10.20537/nd210103

NONLINEAR PHYSICS AND MECHANICS

MSC 2010: 37D15

Stable Arcs Connecting Polar Cascades on a Torus

O. V. Pochinka, E. V. Nozdrinova

The problem of the existence of an arc with at most countable (finite) number of bifurcations connecting structurally stable systems (Morse – Smale systems) on manifolds was included in the list of fifty Palis – Pugh problems at number 33.

In 1976 S. Newhouse, J. Palis, F. Takens introduced the concept of a stable arc connecting two structurally stable systems on a manifold. Such an arc does not change its quality properties with small changes. In the same year, S. Newhouse and M. Peixoto proved the existence of a simple arc (containing only elementary bifurcations) between any two Morse – Smale flows. From the result of the work of J. Fliteas it follows that the simple arc constructed by Newhouse and Peixoto can always be replaced by a stable one. For Morse – Smale diffeomorphisms defined on manifolds of any dimension, there are examples of systems that cannot be connected by a stable arc. In this connection, the question naturally arises of finding an invariant that uniquely determines the equivalence class of a Morse – Smale diffeomorphism with respect to the relation of connection by a stable arc (a component of a stable isotopic connection).

In the article, the components of the stable isotopic connection of polar gradient-like diffeomorphisms on a two-dimensional torus are found under the assumption that all non-wandering points are fixed and have a positive orientation type.

Keywords: stable arc, saddle-node, gradient-like diffeomorphism, two-dimensional torus

Received February 28, 2021
Accepted March 21, 2021

This work is supported by the Russian Science Foundation under grant 17-11-01041, except of study of the dynamics of diffeomorphisms of the class under consideration supported by Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS” (project 19-7-1-15-1).

Olga V. Pochinka
olga-pochinka@yandex.ru
Elena V. Nozdrinova
maati@mail.ru
Higher School of Economics — Nizhny Novgorod
ul. B. Pecherskaya 25/12, Nizhny Novgorod, 603150 Russia

1. Introduction and formulation of results

The problem of the existence of an arc with no more than a countable (finite) number of bifurcations connecting structurally stable systems (Morse – Smale systems) on manifolds is on the list of fifty Palis – Pugh problems [21] under number 33.

In 1976, S. Newhouse, J. Palis and F. Takens [15] introduced the concept of a stable arc connecting two structurally stable systems on a manifold. Such an arc does not change its quality properties with a small perturbation. In the same year, S. Newhouse and M. Peixoto [17] proved the existence of a simple arc (containing only elementary bifurcations) between any two Morse – Smale flows. It follows from the result of G. Fleitas [8] that a simple arc constructed by Newhouse and Peixoto can always be replaced by a stable one [16]. For Morse – Smale diffeomorphisms given on manifolds of any dimension, examples of systems that cannot be connected by a stable arc are known.

Obstructions appear already for orientation-preserving diffeomorphisms of the circle S^1, which are connected by a stable arc only if the rotation numbers coincide [18].

Beginning with dimension two, additional obstructions appear to the existence of stable arcs between isotopic diffeomorphisms. They are associated with the existence of periodic points [6, 20], heteroclinic intersections [13], wild embeddings of separatrices [10], etc.

On the 6-dimensional sphere, examples of source-sink diffeomorphisms are known that are not connected by any smooth arc [7], which, in fact, became the source for constructing different smooth structures on a sphere of dimension 7. For $n = 2, 3$ the nontrivial fact of the existence of an arc without bifurcations between two source-sink diffeomorphisms was established in [7, 19].

Polar diffeomorphisms, i.e., gradient-like diffeomorphisms with a unique source and a unique sink, are a natural generalization of source-sink systems. It follows from Morse theory that such diffeomorphisms exist on any manifolds.

In this paper, we consider the class G of polar gradient-like diffeomorphisms on the two-dimensional torus $\mathbb{T}^2$ under the assumption that all nonwandering points are fixed and of positive orientation type. In Chapter 2 it is established that any diffeomorphism $f \in G$ has exactly two saddle points and is isotopic to the identity. Moreover, all diffeomorphisms of the class under consideration are pairwise topologically conjugate (see, for example, [5, 9]). Moreover, the closures of stable (unstable) manifolds of saddle points of different diffeomorphisms can belong to different homotopy classes of closed curves on the torus. Therefore, in the general case there is no arc without bifurcations connecting two diffeomorphisms of the class under consideration.

The main result of this work is the proof of the following theorem.

Theorem 1. *Any diffeomorphisms $f, f' \in G$ can be connected by a stable arc with a finite number of saddle-node bifurcations.*

2. Diffeomorphisms of class G

2.1. General properties

In this section, we establish the basic dynamical properties of diffeomorphisms $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ from the class G.

Recall that a diffeomorphism f is *gradient-like* if its nonwandering set Ω_f consists of a finite number of hyperbolic points and the invariant manifolds of different saddle points do not intersect.

A gradient-like diffeomorphism f is called *polar* if the set Ω_f contains exactly two nodal points, namely, one sink and one source.

Fix a system of generators of the fundamental group of torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$:

$$a = \mathbb{S}^1 \times \{0\} = \langle 1, 0\rangle, \qquad b = \{0\} \times \mathbb{S}^1 = \langle 0, 1\rangle.$$

Recall that the *algebraic torus automorphism* $\widehat{L}\colon \mathbb{T}^2 \to \mathbb{T}^2$, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is called the diffeomorphism defined by the matrix $L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, belonging to the set $GL(2, \mathbb{Z})$ *unimodular matrices* — integer matrices with determinant ± 1. That is,

$$\widehat{L}(x, y) = (\alpha x + \beta y, \gamma x + \delta y) \pmod 1.$$

The following statement follows directly from the relationship of gradient-like dynamics with the topology of the ambient surface and the homotopy properties of the torus.

Statement 2.1. *Any diffeomorphism $f \in G$ has the following properties:*

1. *The nonwandering set Ω_f of the diffeomorphism f consists of exactly four fixed hyperbolic points: the sink ω_f, the source α_f, and the saddles σ_f^1, σ_f^2, the closures of invariant manifolds of which are closed curves:*

$$c_f^{s1} = cl\, W^s_{\sigma_f^1} = W^s_{\sigma_f^1} \cup \alpha_f, \qquad c_f^{u1} = cl\, W^u_{\sigma_f^1} = W^u_{\sigma_f^1} \cup \omega_f,$$
$$c_f^{s2} = cl\, W^s_{\sigma_f^2} = W^s_{\sigma_f^2} \cup \alpha_f, \qquad c_f^{u2} = cl\, W^u_{\sigma_f^2} = W^u_{\sigma_f^2} \cup \omega_f.$$

2. *There is only one choice of saddle points numbering σ_f^1, σ_f^2 and the orientation of the closures of their invariant manifolds such that the curves c_f^{s1}, c_f^{u2} are of homotopy type $\langle \mu_f^1, \nu_f^1\rangle$ and the curves c_f^{s2}, c_f^{u1} are of homotopy type $\langle \mu_f^2, \nu_f^2\rangle$ in the basis a, b; also, $J_f = \begin{pmatrix} \mu_f^1 & \mu_f^2 \\ \nu_f^1 & \nu_f^2 \end{pmatrix}$ is a unimodular matrix with the following properties:*

   a) $\mu_f^1 \geqslant \mu_f^2 \geqslant 0$,

   b) $\nu_f^1 > \nu_f^2$, *if* $\mu_f^1 = \mu_f^2$,

   c) $\nu_f^2 = 1$, *if* $\mu_f^2 = 0$.

3. *The diffeomorphism f is isotopic to the identity map.*

2.2. Construction of model diffeomorphisms in the class G

In this section, for any unimodular matrix $J = \begin{pmatrix} \mu^1 & \mu^2 \\ \nu^1 & \nu^2 \end{pmatrix}$ such that $\mu^1 \geqslant \mu^2 \geqslant 0$ and $\nu^1 > \nu^2$ if $\mu^1 = \mu^2$, we construct a model diffeomorphism $f_J \in G$ for which $J_{f_J} = J$.

The simplest example of a diffeomorphism from the class G is the direct product of two copies of a source-sink diffeomorphism on the circle $\mathbb{S}^1$, which we denote by f_0. First, we construct a source-sink diffeomorphism on the circle. In order to do this, consider the map $\overline{F}_0\colon \mathbb{R} \to \mathbb{R}$ given by the formula

$$\overline{F}_0(x) = x - \frac{1}{4\pi}\sin\left(2\pi\left(x - \frac{1}{4}\right)\right).$$

By construction, $x = \frac{1}{4}$ and $x = \frac{3}{4}$ are fixed points of the map $\overline{F}_0$ on the segment $[0, 1]$ (Fig. 1).

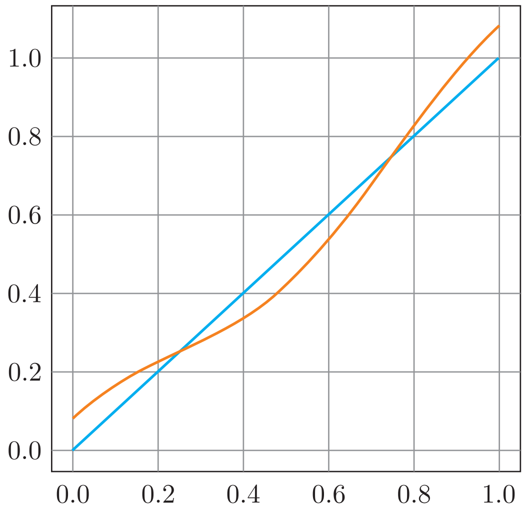


Fig. 1. Graph of the map $\overline{F}_0$.

Consider the projection $\pi\colon \mathbb{R} \to \mathbb{S}^1$ given by the formula $\pi(x) = e^{2\pi i x}$. As $\overline{F}_0$ is strictly increasing and satisfies the condition $\overline{F}_0(x+1) = \overline{F}_0(x)+1$, there is a diffeomorphism projecting it to the circle

$$F_0 = \pi\overline{F}_0\pi^{-1}\colon \mathbb{S}^1 \to \mathbb{S}^1.$$

By construction, the diffeomorphism F_0 has a fixed hyperbolic sink at the point $N = \pi\left(\frac{1}{4}\right)$ and a fixed hyperbolic source at the point $S = \pi\left(\frac{3}{4}\right)$.

Define the diffeomorphism $f_0\colon \mathbb{T}^2 \to \mathbb{T}^2$ by the formula (Fig. 2)

$$f_0(z, w) = (F_0(z), F_0(w)), \quad z, w \in \mathbb{S}^1.$$

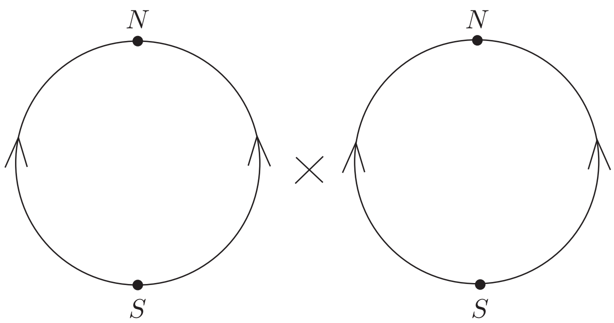


Fig. 2. Cartesian square of the diffeomorphism F_0.

By construction, the diffeomorphism f_0 contains a fixed hyperbolic sink at the point $\omega = (N, N)$, a hyperbolic source $\alpha = (S, S)$ and has two saddle points $\sigma_1 = (N, S)$, $\sigma_2 = (S, N)$ (Fig. 3). Moreover, the closures of their invariant manifolds lie in the classes of generators a and b, namely,

$$c^{s1}_{f_0} = cl\, W^s_{\sigma_1} = \mathbb{S}^1 \times \{S\}, \qquad c^{u1}_{f_0} = cl\, W^u_{\sigma_1} = \{N\} \times \mathbb{S}^1,$$
$$c^{s2}_{f_0} = cl\, W^s_{\sigma_2} = \{S\} \times \mathbb{S}^1, \qquad c^{u2}_{f_0} = cl\, W^u_{\sigma_2} = \mathbb{S}^1 \times \{N\}.$$

Let $f_J = \widehat{J} f_0 \widehat{J}^{-1}$. We will call the diffeomorphism f_J a *model diffeomorphism*. By construction, $f_E = f_0$.

Using the methods of [20], one can construct an arc without bifurcations from the diffeomorphism $f \in G$ to the model one, namely, to prove the following statement:

Statement 2.2. *Every diffeomorphism $f \in G$ is connected by an arc without bifurcations $H_{f,t}$ with the diffeomorphism f_{J_f}.*

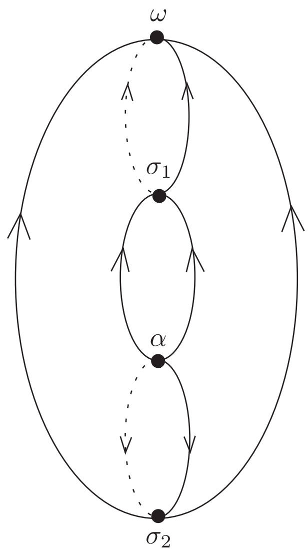


Fig. 3. Diffeomorphism f_0.

3. On stable arcs of diffeomorphisms

Consider a 1-parametric family of diffeomorphisms (*an arc*) $\varphi_t\colon M \to M$, $t \in [0, 1]$. An arc φ_t is called *smooth* if the map $F\colon M \times [0, 1] \to M$ defined by the formula $F(x, t) = \varphi_t(x)$ is smooth.

The smooth arc φ_t is called a *smooth* product of the smooth arcs φ^1_t and φ^2_t such that $\varphi^1_1 = \varphi^2_0$, if $\varphi_t = \begin{cases} \varphi^1_{2\tau(t)}, & 0 \leqslant t \leqslant \dfrac{1}{2}, \\ \varphi^2_{2\tau(t)-1}, & \dfrac{1}{2} \leqslant t \leqslant 1, \end{cases}$ where $\tau\colon [0, 1] \to [0, 1]$ is a smooth monotone map such that $\tau(t) = 0$ for $0 \leqslant t \leqslant \frac{1}{3}$ and $\tau(t) = 1$ for $\frac{2}{3} \leqslant t \leqslant 1$. We will write $\varphi_t = \varphi^1_t * \varphi^2_t$.

Following [16], an arc φ_t is called *stable* if it is an inner point of the equivalence class with respect to the following relation: two arcs φ_t, φ'_t are called *conjugate* if there are homeomorphisms $h\colon [0, 1] \to [0, 1]$, $H_t\colon M \to M$ such that $H_t\varphi_t = \varphi'_{h(t)}H_t$, $t \in [0, 1]$ and H_t continuously depend on t.

In [16] it is also established that the arc $\{\varphi_t\}$, consisting of diffeomorphisms with a finite limit set, is stable iff all its points are structurally stable diffeomorphisms with the exception of a finite number of bifurcation points, φ_{b_i}, $i = 1, \ldots, q$ such that φ_{b_i}:

1) has no cycles;

2) has a unique nonhyperbolic periodic orbit which is a noncritical saddle-node or flip;

3) the invariant manifolds of all periodic points of the diffeomorphismφ_{b_i} intersect transversally;

4) the transition through φ_{b_i} is a generically unfolded saddle-node or period-doubling bifurcation, with the saddle-node point being noncritical.

Recall the definition of the generically unfolding arc φ_t through the saddle-node or flip. We give the definition for a fixed nonhyperbolic point in the case where it has a period $k > 1$. A similar definition is given for the arc φ^k_t.

An arc $\{\varphi_t\} \in \mathcal{Q}$ *unfolds generically* through a saddle-node bifurcation φ_{b_i} (Fig. 4) if in some neighborhood of the nonhyperbolic point (p, b_i) the arc φ_t is conjugate to

$$\widetilde{\varphi}_{\widetilde{t}}(x_1, x_2, \ldots, x_{1+n_u}, x_{2+n_u}, \ldots, x_n) = \left(x_1 + \frac{x_1^2}{2} + \widetilde{t}, \pm 2x_2, \ldots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \ldots, \frac{\pm x_n}{2}\right),$$

where $(x_1, \ldots, x_n) \in \mathbb{R}^n$, $|x_i| < 1/2$, $|\widetilde{t}| < 1/10$.

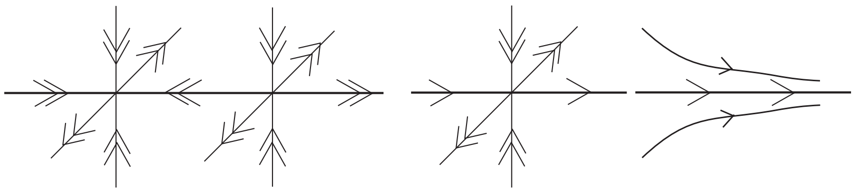


Fig. 4. Saddle-node bifurcation.

In the local coordinates $(x_1, \ldots, x_n, \widetilde{t})$ the bifurcation occurs at time $\widetilde{t} = 0$ and the origin $O \in \mathbb{R}^n$ is a saddle-node point. The axis Ox_1 is called a *central manifold* W^c_O, the half-space $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \colon x_1 \geqslant 0,\, x_{2+n_u} = \ldots = x_n = 0\}$ is the *unstable manifold* W^u_O, and the half-space $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \colon x_1 \leqslant 0,\, x_2 = \ldots = x_{1+n_u} = 0\}$ is the *stable manifold* W^s_O of the point O.

If p is a saddle-node point of the diffeomorphism φ_{b_i}, then there exists a unique φ_{b_i}- invariant foliation F^{ss}_p with smooth leaves such that ∂W^s_p is a leave of this foliation [11]. F^{ss}_p is called a *strongly stable foliation* (Fig. 5). A similar *strongly unstable foliation* is denoted by F^{uu}_p. A point p is called *s-critical* if there exists some hyperbolic periodic point q such that W^u_q nontransversally intersect some leaf of the foliation F^{ss}_p; *u-criticality* is defined similarly. Point p is called

– *semicritical* if it is either s- or u-critical;

– *bicritical* if it is s- and u-critical;

– *noncritical* if it is not semicritical[1].

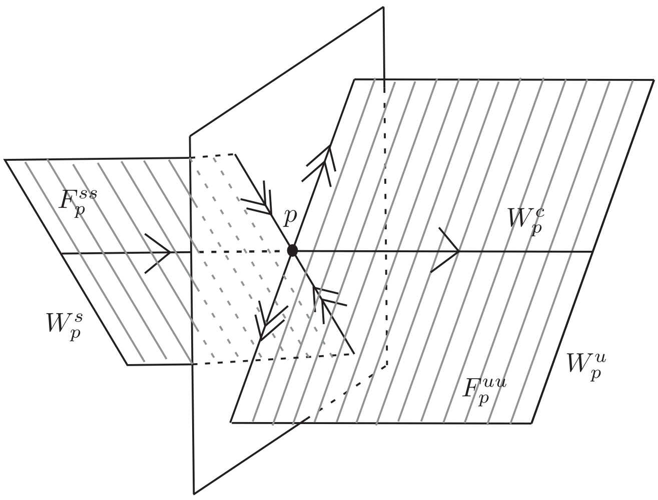


Fig. 5. Strongly stable and unstable foliations.

[1]For the first time, the effect of arc instability in a neighborhood of a critical saddle was discovered in 1974 by V. Afraimovich and L. Shilnikov [1, 2]. The existence of invariant foliations F^{ss}_p, F^{uu}_p was also proved earlier in the work of V. Lukyanov and L. Shilnikov[12].

In particular, for M^2, the noncriticality of the saddle-node point means that the saddle-node separatrix does not intersect with the one-dimensional manifold of the saddle-node point. The two-dimensional manifold of a saddle-node point must intersect the transversally invariant layer (Fig. 6).

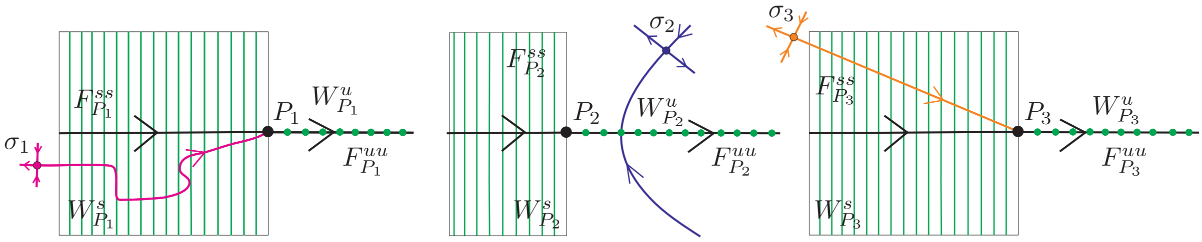


Fig. 6. p_1 — s-critical saddle-node point, p_2 — u-critical saddle-node point, p_3 — noncritical saddle-node point.

4. Construction of a stable arc between model torus diffeomorphisms

4.1. Construction of auxiliary functions

In this section, we construct model functions that will later be used to construct a stable arc. The construction is based on the principle of gluing infinitely smooth functions by means of the following *sigmoid function*.

Let $a < b$ and $\delta_{a;b}\colon \mathbb{R} \to [0, 1]$ be a sigmoid function defined by the formula (Fig. 7)

$$\delta_{a;b}(x) = \begin{cases} 0, & x \leqslant a, \\ \dfrac{1}{1 + \exp\left(\dfrac{(a+b)/2 - x}{(x-a)^2(x-b)^2}\right)}, & a < x < b, \\ 1, & x \geqslant b. \end{cases}$$

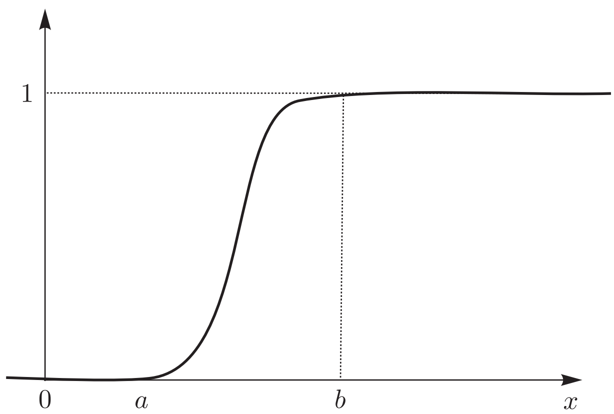


Fig. 7. Graph of the sigmoid function.

Define the function $\overline{\phi}_1 : \mathbb{R} \to \mathbb{R}$ by the formula (Fig. 8)

$$\overline{\phi}_1(x) = x - \frac{1}{12\pi} \sin\left(6\pi\left(x - \frac{1}{4}\right)\right).$$

Define the function $\overline{g}_1 : \mathbb{R} \to \mathbb{R}$ by the formula (Fig. 9)

$$\overline{g}_1(x) = \begin{cases} \overline{\phi}_0(x), & 0 \leqslant x \leqslant 0.26, \\ (1 - \delta_{0.26;0.27}(x))\overline{\phi}_0(x) + \delta_{0.26;0.27}(x)\overline{\phi}_1(x), & 0.26 < x < 0.27, \\ \overline{\phi}_1(x), & 0.27 \leqslant x \leqslant 0.76, \\ (1 - \delta_{0.76;0.77}(x))\overline{\phi}_1(x) + \delta_{0.76;0.77}(x)\overline{\phi}_0(x), & 0.76 < x < 0.77, \\ \overline{\phi}_0(x), & 0.77 \leqslant x \leqslant 1. \end{cases}$$

Define the function $\overline{\phi}_2 : \mathbb{R} \to \mathbb{R}$ by the formula (Fig. 10)

$$\overline{\phi}_2(x) = x + \frac{1}{4\pi} \sin\left(\frac{5}{6}\pi\left(x - \frac{5}{12}\right)\right).$$

Define the function $\overline{g}_2 : \mathbb{R} \to \mathbb{R}$ by the formula (Fig. 11)

$$\overline{g}_2(x) = \begin{cases} \overline{g}_1(x), & 0 \leqslant x \leqslant 0.42, \\ (1 - \delta_{0.42;0.43}(x))\overline{g}_1(x) + \delta_{0.42;0.43}(x)\overline{\phi}_2(x), & 0.42 < x < 0.43, \\ \overline{\phi}_2(x), & 0.43 \leqslant x \leqslant 0.98, \\ (1 - \delta_{0.98;0.99}(x))\overline{\phi}_2(x) + \delta_{0.98;0.99}(x)\overline{g}_1(x), & 0.98 < x < 0.99, \\ \overline{g}_1(x), & 0.99 \leqslant x \leqslant 1. \end{cases}$$

4.2. Construction of the model arcs

In this section, we will construct arcs which are the main components making up an arc $H_{J,t}$.

For $n \in \mathbb{Z}$ let $J_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$.

Lemma 1. *The diffeomorphism f_0 is connected with the diffeomorphism f_{J_1} by a stable arc $H_{0,1,t}$ with two of generically unfolding noncritical saddle-node bifurcations.*

Proof. In this proof, the bar-free mappings are projections on $\mathbb{S}^1$ by π of the bar-mappings given on the line $\mathbb{R}$. The stable arc $H_{0,1,t}$, connecting the diffeomorphism f_0 with the diffeomorphism f_{J_1} is the product of the arcs Γ_t^1, Γ_t^2, constructed in step 1 and step 2 below, and the arc $H_{\Gamma_1^2,t}$.

Step 1. First saddle-node bifurcation.

1. The birth of a saddle-node point. We start with the diffeomorphism $f_0 : T^2 \to T^2$ defined by the formula

$$f_0(z, w) = (\phi_0(z), \phi_0(w)), \quad z, w \in \mathbb{S}^1.$$

Let

$$\overline{\eta}_t^1(x) = (1 - t)\overline{\phi}_0(x) + t\overline{g}_1(x), \quad x \in \mathbb{R}, \quad t \in [0, 1]$$

and

$$\overline{\eta}_{t,\tau}^1(x) = (1 - \tau)\overline{\eta}_t^1(x) + \tau\overline{\phi}_0(x), \quad x \in \mathbb{R}, \quad t \in [0, 1], \quad \tau \in [0, 1].$$

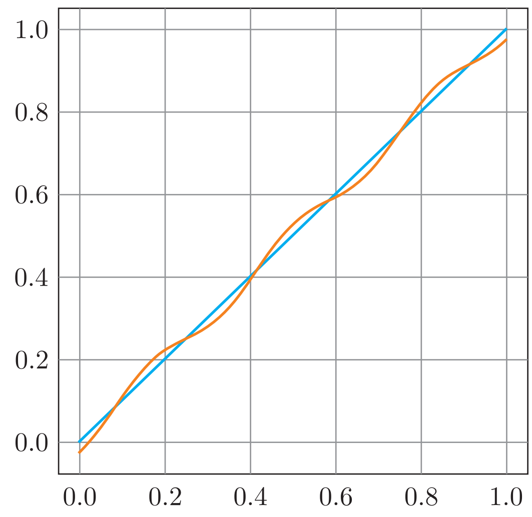


Fig. 8. Graph of the map $\overline{\phi}_1(x)$.

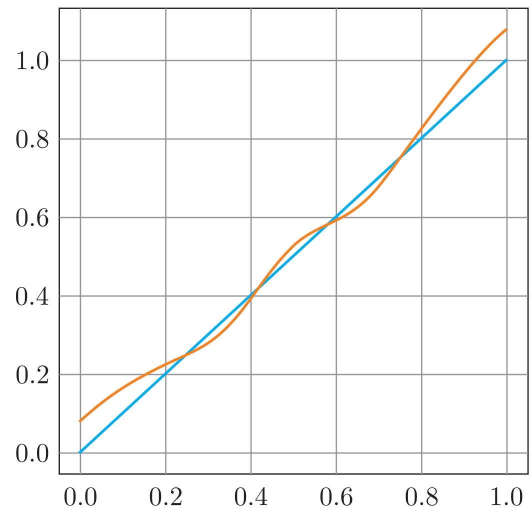


Fig. 9. Graph of the map $\overline{g}_1(x)$.

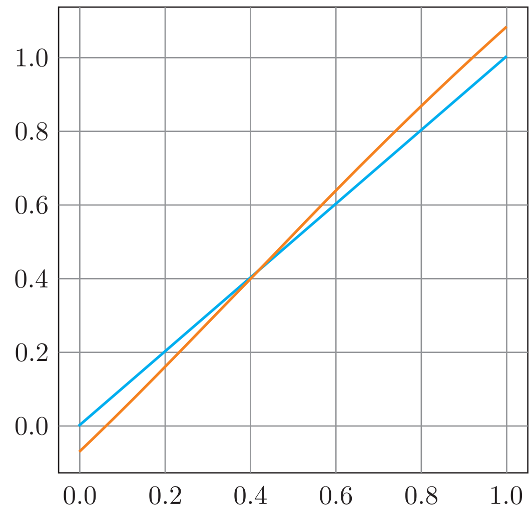


Fig. 10. Graph of the map $\overline{\phi}_2(x)$.

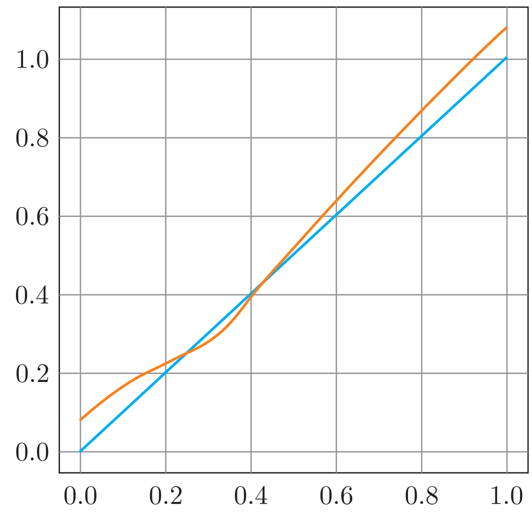


Fig. 11. Graph of the map $\overline{g}_2(x)$.

Define a smooth arc $H_t^1 \colon \mathbb{T}^2 \to \mathbb{T}^2$, $t \in [0, 1]$ by the formula

$$H_t^1(z, w) = \begin{cases} (\phi_0(z), \eta^1_{t,|8x-2|}(w)), \quad z = \pi(x), \quad x \in \left(\frac{1}{8}, \frac{3}{8}\right), \quad w \in \mathbb{S}^1, \\ f_0(z, w), \quad z = \pi(x), \quad x \in \left(-\frac{5}{8}, \frac{1}{8}\right), \quad w \in \mathbb{S}^1. \end{cases}$$

For $t = \frac{3}{4}$, the diffeomorphism $H^1_{\frac{3}{4}}$ has a saddle-node point $p = (N, \pi(0))$ whose stable manifold is diffeomorphic to a half-plane whose boundary is arc γ_p (Fig. 12).

2. Rotation of the separatrix of the saddle $\boldsymbol{\sigma_2}$.

Consider the fundamental domain $K = \left[\pi(0), \pi\left(\frac{1}{4\pi}\right)\right] \times \mathbb{S}^1$, a restriction of the diffeomorphism f_0 to $V = \left[\pi\left(-\frac{1}{4}\right), \pi\left(\frac{1}{4}\right)\right] \times \mathbb{S}^1$. Let $\widehat{V} = V/f_0$. Then $\widehat{V}$ is a torus obtained from K

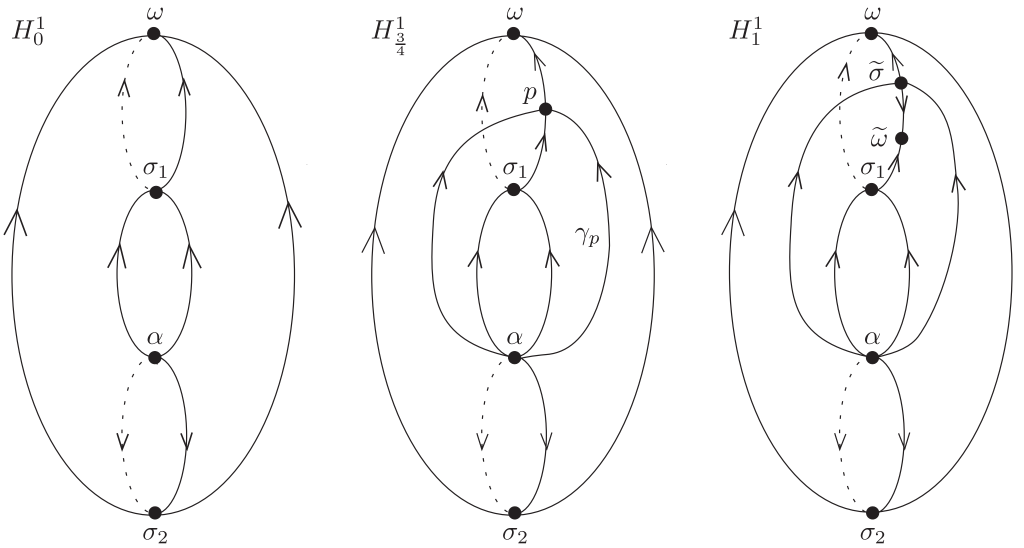


Fig. 12. Isotopy H_t^1 on the torus.

by identifying the boundaries with the map f_0. Denote by $q\colon V \to \widehat{V}$ the natural projection. Let $\widehat{\gamma}_2 = q(W^u_{\sigma_2} \cap V)$ and $\widehat{\gamma}_1 = q(W^s_{\sigma_1} \cap V)$. Since for all $t \in [0, 1]$ the diffeomorphism H_t coincides with f_0 on the annulus $\left[\pi\left(-\frac{1}{4}\right), \pi\left(\frac{1}{8}\right)\right] \times \mathbb{S}^1$, it follows that the circle $\widehat{\gamma}_p = q(\gamma_p \cap K)$ is defined correctly.

Let $W = \left[\pi\left(-\frac{1}{4}\right); \pi\left(\frac{1}{4}\right)\right] \times \left[\pi\left(-\frac{1}{4}\right); \pi\left(\frac{1}{4}\right)\right]$ and $\widehat{W} = p(W)$. By construction, the circle $\widehat{\gamma}_p$ divides the annulus $\widehat{W}$ into two annuli, the closures of which are denoted by$\widehat{W}_1$, $\widehat{W}_2$, assuming that $\widehat{\gamma}_1 \subset \widehat{W}_1$ and $\widehat{\gamma}_2 \subset \widehat{W}_2$ (Fig. 13).

Choose a circle $\widehat{\gamma} \subset \operatorname{int} \widehat{W}_1$ that is not homotopic to zero, transversal to the projection of a strongly stable foliation of a saddle-node point. Such a curve always exists, since the projection of each layer of this foliation is a curve wrapped around a knot $\widehat{\gamma}_1$ (Fig. 13). According to [3, 14], there exists a diffeomorphism $\widehat{h}_1\colon \widehat{V} \to \widehat{V}$ smoothly isotopic to the identity such that $\widehat{h}_1(\widehat{\gamma}_2) = \widehat{\gamma}$.

For $x_i \in \left[-\frac{1}{4}; 0\right]$ let $K_i = \left[\pi(x_i); (\pi(\overline{\phi}_0^{-1}(x_i))\right] \times \mathbb{S}^1$. Choose an open cover $D = \{D_1, \ldots, D_{k_1}\}$ of the torus $\mathbb{T}^2$ such that the connected component $\overline{D}_i$ of the set $q^{-1}(D_i)$ is a subset of K_i for some $x_i < \overline{\phi}_0^{-1}(x_{i-1})$. According to the fragmentation lemma [4], there exist diffeomorphisms $\widehat{w}_1, \ldots, \widehat{w}_{k_1}\colon \mathbb{T}^2 \to \mathbb{T}^2$ smoothly isotopic to the identity, with the following properties:

i) for each $i \in \{1, \ldots, k_1\}$ there exists a smooth isotopy $\{\widehat{w}_{i,t}\}$ which is the identity outside D_i and which joins the identity and $\widehat{w}_i$;

ii) $\widehat{h}_1 = \widehat{w}_1 \ldots \widehat{w}_{k_1}$.

Let $w_{i,t}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism that coincides with $(q|_{K_i})^{-1}\widehat{w}_{i,t}q$ on K_i and coincides with the identity map outside K_i. Let

$$\zeta_t = w_{1,t} \ldots w_{k_1,t} f_0, \qquad G_t^1 = \begin{cases} \zeta_{2t}, & 0 \leqslant t < \dfrac{1}{2}, \\[2mm] \zeta_1, & \dfrac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

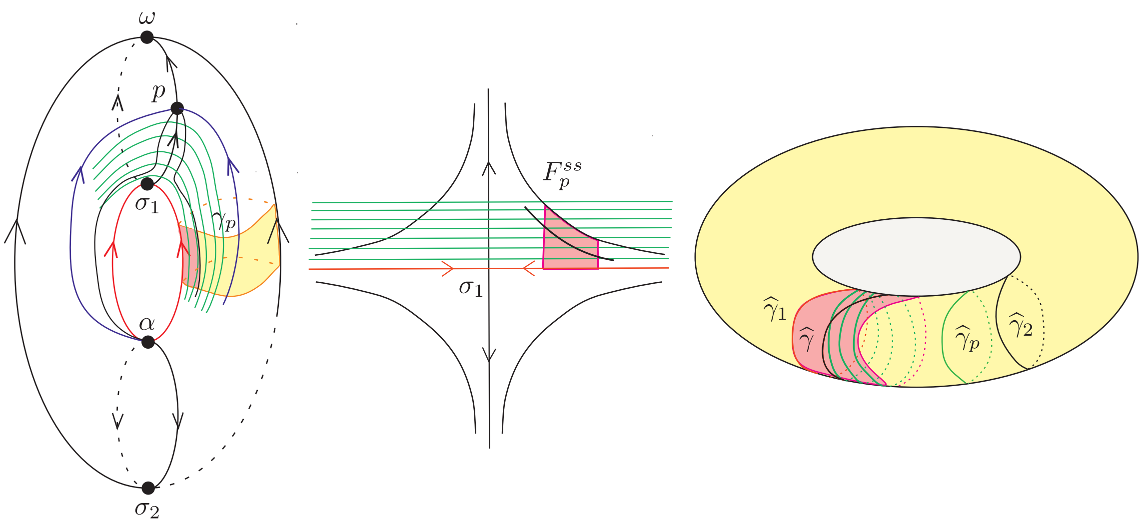


Fig. 13. Curve $\widehat{\gamma}$.

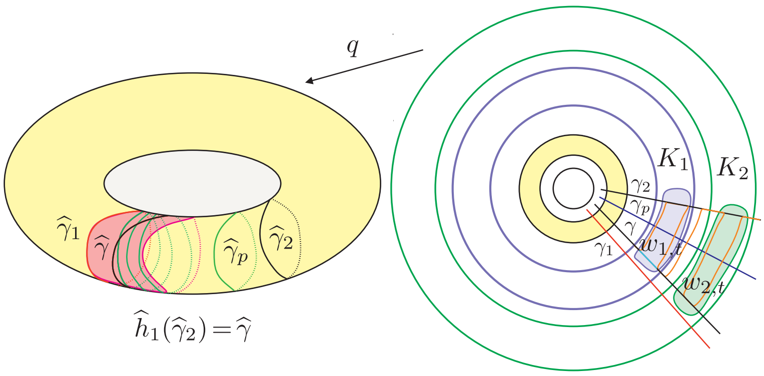


Fig. 14. Application of the fragmentation lemma.

3. Combining isotopies H_t^1 and G_t^1.

Define a smooth arc $\Gamma_t^1 \colon \mathbb{T}^2 \to \mathbb{T}^2$, $t \in [0,1]$ by the formula (Fig. 16)

$$\Gamma_t^1(z,w) = \begin{cases} H_t^1(z,w), & z = \pi(x), \quad x \in \left(\dfrac{1}{8}, \dfrac{3}{8}\right), \quad w \in \mathbb{S}^1, \\ G_t^1(z,w), & z = \pi(x), \quad x \in \left(-\dfrac{1}{4}, 0\right), \quad w \in \mathbb{S}^1, \\ f_0(z,w), & z = \pi(x), \quad x \in \left[-\dfrac{5}{8}, -\dfrac{1}{4}\right] \cup \left[0; \dfrac{1}{8}\right], \quad w \in \mathbb{S}^1. \end{cases}$$

Step 2. Second saddle-node bifurcation.

1. Merging saddle and node points

For all $t \in [0;1]$ let $\overline{\eta}_t^2(x) = t\overline{g_2}(x) + (1-t)\overline{g_1}(x)$, $x \in \mathbb{R}$ and

$$\overline{\eta}_{t,\tau}^2(x) = (1-\tau)\overline{\eta}_t^2(x) + \tau\overline{\phi}_0(x), \quad x \in \mathbb{R}, \quad t \in [0,1], \quad \tau \in [0,1].$$

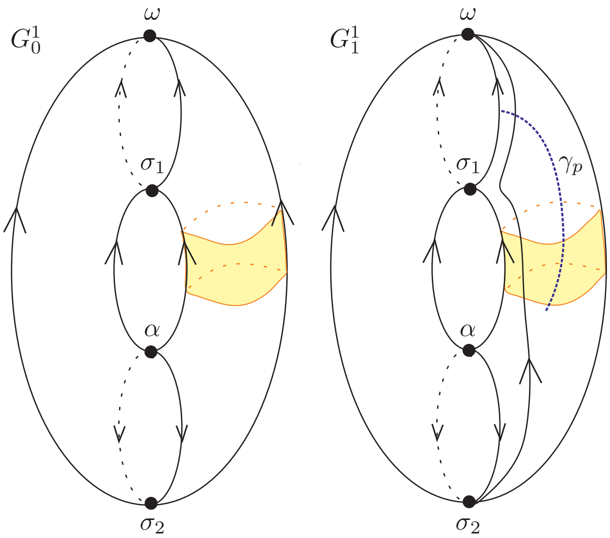


Fig. 15. Isotopy G^1_t on the torus.

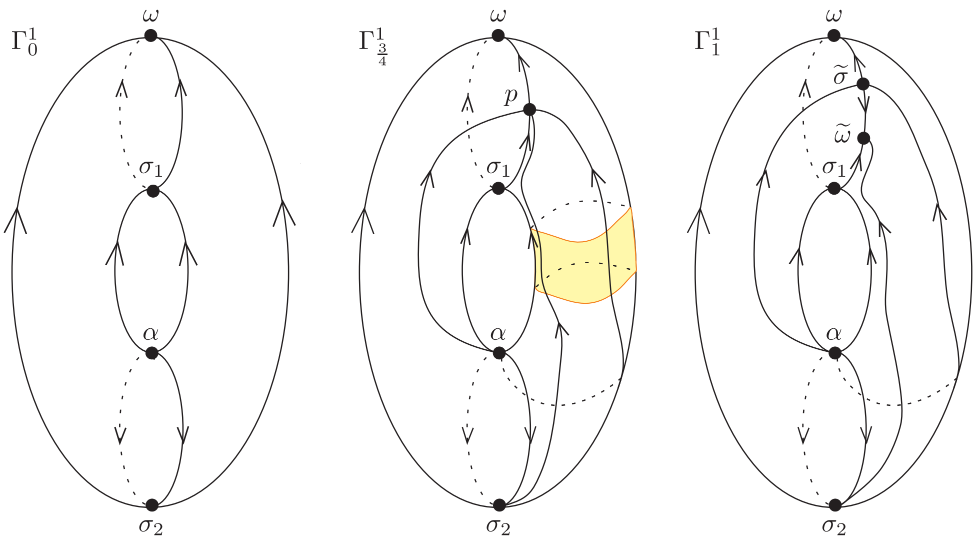


Fig. 16. Isotopy Γ^1_t on the torus

Define a smooth arc $H^2_t\colon \mathbb{T}^2 \to \mathbb{T}^2$, $t \in [0, 1]$ by the formula

$$H^2_t(z,w) = \begin{cases} (\phi_0(z), \eta^2_{t,|8x-2|}(z)), \quad z = \pi(x), \quad x \in \left(\dfrac{1}{8}, \dfrac{3}{8}\right), \quad w \in \mathbb{S}^1, \\ \Gamma_1(z,w), \quad z = \pi(x), \quad x \in \left(-\dfrac{5}{8}, \dfrac{1}{8}\right), \quad w \in \mathbb{S}^1. \end{cases}$$

The arc H^2_t realizes the merging of the sink $\widetilde{\omega}$ and the saddle σ_1 into the saddle-node point $\widetilde{p}$ and its further disappearance. Denote by $\beta_{\widetilde{p}}$ the boundary of the stable manifold of a saddle-node $\widetilde{p}$.

2. Rotation of the separatrix of the saddle σ_2.

Since for all $t \in [0, 1]$ the diffeomorphism H^2_t coincides with f_0 on the annulus K, it follows that the circles $\widehat{\beta}_2 = q(W^u_{\sigma_2} \cap K)$, $\widehat{\beta}_1 = q(W^s_{\widetilde{\sigma}} \cap K)$ and $\widehat{\beta}_{\widetilde{p}} = q(\beta_{\widetilde{p}} \cap K)$ are defined correctly.

Let $\widehat{W}_3$ be a neighborhood of the curve $\widehat{\beta}_1$, then choose a smooth nonzero homotopic curve $\widehat{\gamma} \subset \widehat{W}_3$, transversal to the projection of a strongly stable foliation of a saddle-node point. Such a curve always exists, since the projection of each layer of this foliation is a curve wrapped around the knot $\widehat{\beta}_1$ (we construct in the same way as in Step 1). According to [3] and [14], there exists a diffeomorphism $\widehat{h}_2 \colon \widehat{V} \to \widehat{V}$ smoothly isotopic to the identity such that $\widehat{h}_2(\widehat{\beta}_2) = \widehat{\beta}$ and $\widehat{h}_2(\widehat{\beta}_1) = \widehat{\beta}_1$.

Choose an open cover $U = \{U_1, \ldots, U_{k_2}\}$ of the torus $\mathbb{T}^2$ such that the connected component $\overline{U}_i$ of the set $q^{-1}(U_i)$ is a subset of K_i for some $x_i < \overline{\phi}_0^{\,-1}(x_{i-1})$. According to the fragmentation lemma [4], there exist diffeomorphisms $\widehat{v}_1, \ldots, \widehat{v}_{k_2} \colon \mathbb{T}^2 \to \mathbb{T}^2$ smoothly isotopic to the identity, with the following properties:

i) for each $i \in \{1, \ldots, k_2\}$ there exists a smooth isotopy $\{\widehat{v}_{i,t}\}$ which is the identity outside U_i and which joins the identity and $\widehat{v}_i$;

ii) $\widehat{h}_2 = \widehat{v}_1 \ldots \widehat{v}_{k_2}$.

Let $v_{i,t} \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism that coincides with $(q|_{K_i})^{-1}\widehat{v}_{i,t}q$ on K_i and coincides with the identity map outside K_i. Let

$$\xi_t = v_{1,t} \ldots v_{k_2,t}\Gamma_1, \quad G_t^2 = \begin{cases} \xi_{2t}, & 0 \leqslant t < \dfrac{1}{2}, \\ \xi_1, & \dfrac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

3. Combining isotopies H_t^2 and G_t^2.

Define a smooth arc $\Gamma_t^2 \colon \mathbb{T}^2 \to \mathbb{T}^2$, $t \in [0, 1]$ by the formula (Fig. 17)

$$\Gamma_t^2(z, w) = \begin{cases} H_t^2(z, w), & z = \pi(x), \quad x \in \left(\dfrac{1}{8}, \dfrac{3}{8}\right), \quad w \in \mathbb{S}^1, \\ G_t^2(z, w), & z = \pi(x), \quad x \in \left(-\dfrac{1}{4}, 0\right), \quad w \in \mathbb{S}^1, \\ f_0(z, w), & z = \pi(x), \quad x \in \left[-\dfrac{5}{8}, -\dfrac{1}{4}\right] \cup \left[0; \dfrac{1}{8}\right], \quad w \in \mathbb{S}^1. \end{cases}$$

According to statement 2.2, the diffeomorphism Γ_1^2 can be connected by an arc without bifurcations $H_{\Gamma_1^2,t}$ with the diffeomorphism f_{J_1}. $\square$

Denote by $H_{n,n+1,t}$ the arc with two saddle-node bifurcations connecting the diffeomorphisms f_{J_n}, $f_{J_{n+1}}$ and given by the formula

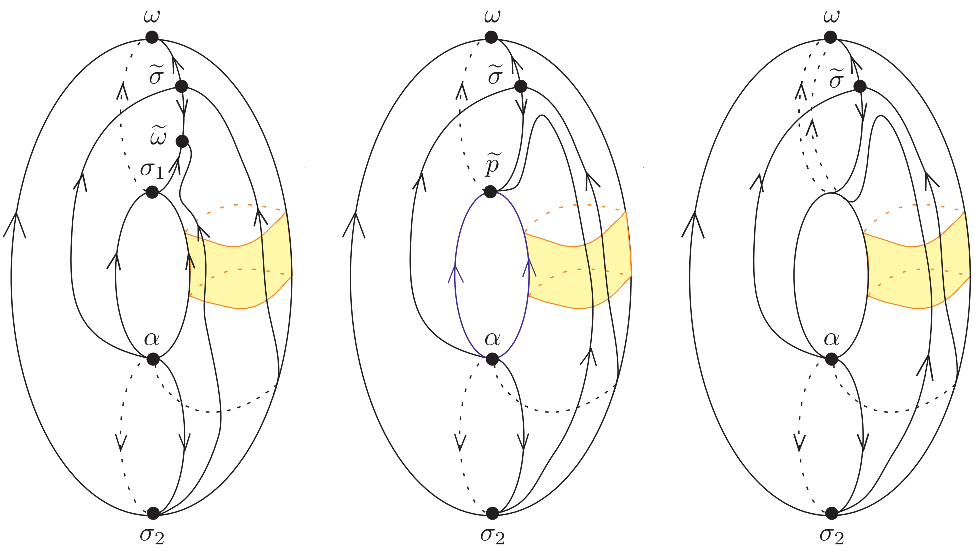
$$H_{n,n+1,t} = \widehat{J}_n H_{0,1,t} \widehat{J}_n^{-1}.$$

4.3. Arc construction algorithm $H_{J,t}$

In this section, using the model arcs constructed above, we will prove the following lemma.

Lemma 2. *The diffeomorphism f_J is joined by a stable arc $H_{J,t}$ with a finite number of generically unfolding noncritical saddle-node bifurcations with the diffeomorphism f_0.*

Proof. Let $J = \begin{pmatrix} \mu^1 & \mu^2 \\ \nu^1 & \nu^2 \end{pmatrix}$ be a unimodular matrix such that $\mu^1 \geqslant \mu^2 \geqslant 0$ and $\nu^1 > \nu^2$ if $\mu^1 = \mu^2$. Consider the following possibilities for the matrix J: 1) $\mu^2 = 0$; 2) $\mu^1 = \mu^2 = 1$; 3) $\mu^2 > \mu^1 > 0$. Construct the arc $H_{J,t}$ in each case separately.



(a) $t=0$ (b) $t=\frac{1}{2}$ (c) $t=1$

Fig. 17. Isotopy Γ_t^2 on the torus.

In case 1) $J = J_n$. If $n > 0$, then $H_{J_n,t} = H_{n-1,n,1-t} * \ldots * H_{0,1,1-t}$ is the required arc. If $n < 0$, then $H_{J_n,t} = \widehat{J}_n H_{J_{-n},1-t}\widehat{J}_n^{-1}$ is the required arc.

In case 2) $H_{J,t} = \widehat{J} H_{J_{-1},1-t}\widehat{J}^{-1} * H_{J_{\nu^2},t}$ is the required arc.

In case 3) applying Euclid's algorithm to the pair μ_1, μ_2 generates a sequence of natural numbers $n_1, \ldots, n_m$, $k_1, \ldots, k_m$ such that $\mu^1 = n_1\mu^2 + k_1$, $\mu^2 = n_2k_1 + k_2$, $k_1 = n_3k_2 + k_3, \ldots, k_{m-2} = n_mk_{m-1} + k_m$, where $k_{m-1} = 1$, $k_m = 0$. Let $k_{-1} = \mu^1$, $k_0 = \mu^2$. Then the sequence $k_{-1}, k_0, k_1, \ldots, k_m$ satisfies the recurrence relation

$$k_{i+1} = n_{i+1}k_i - k_{i-1},\ i = 0, \ldots, m-1.$$

Let $l_{-1} = \nu^1$, $l_0 = \nu^2$ and define the sequence $l_{-1}, l_0, l_1, \ldots, l_m$ by the recurrent relation

$$l_{i+1} = n_{i+1}l_i - l_{i-1},\ i = 0, \ldots, m-1.$$

Let $L_i = \begin{pmatrix} k_{i-1} & k_i \\ l_{i-1} & l_i \end{pmatrix}$, $i = 0, \ldots, m$. Then the arc $F_{i,t} = \widehat{L}_{i-1} H_{J_{-n_i},t}\widehat{L}_{i-1}^{-1}$, $i = 1, \ldots, m$ joins diffeomorphisms $f_{L_{i-1}}$ and f_{L_i} and contains $2n_i$ noncritical saddle-node bifurcations. Since $f_{L_m} = f_{J_{l_{m-1}}}$, it follows that $H_{J,t} = F_{1,t} * \ldots * F_{m,t} * H_{J_{l_{m-1}}}$ is the required arc. $\square$

References

[1] Afraimovich, V. S. and Shilnikov, L. P., Certain Global Bifurcations Connected with the Disappearance of a Fixed Point of Saddle-Node Type, *Dokl. Akad. Nauk SSSR*, 1974, vol. 219, pp. 1281–1284 (Russian).

[2] Afraimovich, V. S. and Shilnikov, L. P., Small Periodic Perturbations of Autonomous Systems, *Soviet Math. Dokl.*, 1974, vol. 15, pp. 734–742; see also: *Dokl. Akad. Nauk SSSR*, 1974, vol. 214, no. 4, pp. 739–742.

[3] Baer, R., Isotopie von Kurven auf orientierbaren, geschlossenen Flächen und ihr Zusammenhang mit der topologischen Deformation der Flächen, *J. Reine Angew. Math.,* 1928, vol. 159, pp. 101–116.

[4] Banyaga, A., On the Structure of the Group of Equivariant Diffeomorphisms, *Topology,* 1977, vol. 16, no. 3, pp. 279–283.

[5] Bezdenezhnykh, A. N. and Grines, V. Z., Dynamical Properties and Topological Classification of Gradient-Like Diffeomorphisms on Two-Dimensional Manifolds: 1, *Selecta Math. Soviet.,* 1992, vol. 11, no. 1, pp. 1–11.

[6] Blanchard, P. R., Invariants of the NPT Isotopy Classes of Morse – Smale Diffeomorphisms of Surfaces, *Duke Math. J.,* 1980, vol. 47, no. 1, pp. 33–46.

[7] Bonatti, C., Grines, V. Z., Medvedev, V. S., and Pochinka, O. V., Bifurcations of Morse – Smale Diffeomorphisms with Wildly Embedded Separatrices, *Proc. Steklov Inst. Math.,* 2007, vol. 256, no. 1, pp. 47–61; see also: *Tr. Mat. Inst. Steklova,* 2007, vol. 256, pp. 54–69.

[8] Fleitas, G., Replacing Tangencies by Saddle-Nodes, *Bol. Soc. Brasil. Mat.,* 1977, vol. 8, no. 1, pp. 47–51.

[9] Grines, V. Z., Kapkaeva, S. Kh., and Pochinka, O. V., A Three-Color Graph As a Complete Topological Invariant for Gradient-Like Diffeomorphisms of Surfaces, *Sb. Math.,* 2014, vol. 205, nos. 9–10, pp. 1387–1412; see also: *Mat. Sb.,* 2014, vol. 205, no. 10, pp. 19–46.

[10] Grines, V. Z. and Pochinka, O. V., On the Simple Isotopy Class of a Source-Sink Diffeomorphism on the 3-Sphere, *Math. Notes,* 2013, vol. 94, nos. 5–6, pp. 862–875; see also: *Mat. Zametki,* 2013, vol. 94, no. 6, pp. 828–845.

[11] Hirsch, M. W., Pugh, C. C., and Shub, M., *Invariant Manifolds,* Lecture Notes in Math., vol. 583, New York: Springer, 1977.

[12] Luk'janov, V. I. and Shil'nikov, L. P., Some Bifurcations of Dynamical Systems with Homoclinic Structures, *Soviet Math. Dokl.,* 1978, vol. 19, pp. 1314–1318; see also: *Dokl. Akad. Nauk SSSR,* 1978, vol. 243, no. 1, pp. 26–29.

[13] Matsumoto, Sh., There Are Two Isotopic Morse – Smale Diffeomorphisms Which Cannot Be Joined by Simple Arcs, *Invent. Math.,* 1979, vol. 51, no. 1, pp. 1–7.

[14] Munkres, J., Obstructions to the Smoothing of Piecewise-Differentiable Homeomorphisms, *Bull. Amer. Math. Soc.,* 1959, vol. 65, pp. 332–334.

[15] Newhouse, S., Palis, J., and Takens, F., Stable Arcs of Diffeomorphisms, *Bull. Amer. Math. Soc.,* 1976, vol. 82, no. 3, pp. 499–502.

[16] Newhouse, S., Palis, J., and Takens, F., Bifurcations and Stability of Families of Diffeomorphisms, *Inst. Hautes Études Sci. Publ. Math.,* 1983, No. 57, pp. 5–71.

[17] Newhouse, S. and Peixoto, M. M., There Is a Simple Arc Joining Any Two Morse – Smale Flows, in *Trois études en dynamique qualitative,* Astérisque, vol. 31, Paris: Soc. Math. France, 1976, pp. 15–41.

[18] Nozdrinova, E. V., Rotation Number As a Complete Topological Invariant of a Simple Isotopic Class of Rough Transformations of a Circle, *Russian J. Nonlinear Dyn.,* 2018, vol. 14, no. 4, pp. 543–551.

[19] Nozdrinova, E. and Pochinka, O., On the Existence of a Smooth Arc without Bifurcations Joining Source-Sink Diffeomorphisms on the 2-Sphere, *J. Phys.: Conf. Ser.,* 2018, vol. 990, no. 1, 012010, 7 pp.

[20] Nozdrinova, E. and Pochinka, O., Solution of the 33rd Palis – Pugh Problem for Gradient-Like Diffeomorphisms of a Two-Dimensional Sphere, *Discrete Contin. Dyn. Syst.,* 2021, vol. 41, no. 3, pp. 1101–1131.

[21] Palis, J. and Pugh, C. C., Fifty Problems in Dynamical Systems, in *Dynamical Systems: Proc. Sympos. Appl. Topology and Dynamical Systems (Univ. Warwick, Coventry, 1973/1974): Presented to E. C. Zeeman on His Fiftieth Birthday,* A. Manning (Ed.), Lecture Notes in Math., vol. 468, Berlin: Springer, 1975, pp. 345–353.