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ABSTRACT

An analytic reversible Hamiltonian system with two degrees of freedom is studied in a neighborhood of its symmetric heteroclinic connection made up of a symmetric saddle-center, a symmetric orientable saddle periodic orbit lying in the same level of a Hamiltonian, and two non-symmetric heteroclinic orbits permuted by the involution. This is a codimension one structure; therefore, it can be met generally in one-parameter families of reversible Hamiltonian systems. There exist two possible types of such connections depending on how the involution acts near the equilibrium. We prove a series of theorems that show a chaotic behavior of the system and those in its unfoldings, in particular, the existence of countable sets of transverse homoclinic orbits to the saddle periodic orbit in the critical level, transverse heteroclinic connections involving a pair of saddle periodic orbits, families of elliptic periodic orbits, homoclinic tangencies, families of homoclinic orbits to saddle-centers in the unfolding, etc. As a by-product, we get a criterion of the existence of homoclinic orbits to a saddle-center.

The orbit structure of a non-integrable Hamiltonian system with two or more degrees of freedom is very complicated, and usually, it is impossible, except for some specific model situations, to describe its structure more or less completely. By this reason, a fruitful way to understand the orbit behavior in some parts of the phase space is to detect some simple invariant subsets (usually containing a finite number of orbits) whose neighborhoods can be understood from the viewpoint of their orbit structure. When it has been done, we try to find such structures in general systems and thus to describe partially the behavior of the system under study. This approach goes back to A. Poincaré. The problem investigated here follows these lines. It was inspired by the study of stationary waves in a nonlocal Whitham equation[22] that is reduced to the reversible Hamiltonian system with two degrees of freedom for which homoclinic orbits to different types of equilibria have to be detected. We rely in this research on earlier results on the behavior near a homoclinic orbit to a saddle-center equilibrium[14,15,19,20,28,43] as well as near homoclinic tangencies.[8,9,11–13,29,32] The results obtained demonstrate how much can be understood at this approach.

I. INTRODUCTION

Studying Hamiltonian dynamics is an interesting and hard problem attracting researchers from many branches of science since Hamiltonian systems serve as mathematical models in different problems in physics, chemistry, and engineering. The structure of such systems is usually rather complicated; therefore, one of a fruitful approach to these types of problems is the study of the given system near some invariant sets, which can be selected by simple conditions. Studying systems in neighborhoods of homoclinic orbits and heteroclinic connections is the problem of such a type. Investigations of dynamical phenomena near a homoclinic orbit to a saddle periodic orbit were the first such problem, and its setup and understanding the complexity of orbit behavior of the system near such a structure go back to Poincaré.[34] The real complexity of orbit behavior was understood due to works by Birkhoff,[2] Smale,[39] and finally Shilnikov.[36] Other problems, where complicated dynamics was detected, were studied in many papers by Shilnikov and coauthors; among them, the most influential are Afraimovich *et al.*,[1] Gavrilov and Silnikov,[10] Shilnikov,[37] and Shilnikov *et al.*[38] Homoclinic dynamics in Hamiltonian systems began studying in

Ref. 6 where Shilnikov results about the complicated dynamics near a saddle-focus homoclinic loop were carried over to the Hamiltonian case. The generalization of the Melnikov method onto the autonomous case for systems close to Hamiltonian integrable[25] allowed one to present examples of a complicated behavior both for Hamiltonian perturbation of an integrable Hamiltonian system with a saddle-focus skirt and for dissipative perturbations. The complicated dynamics near a bunch of homoclinic orbits to a saddle in a Hamiltonian system was detected in Ref. 42 (for generalizations and proofs of these results, see Refs. 16 and 41). A complex dynamics in a Hamiltonian system near a homoclinic orbit of an equilibrium not being hyperbolic was detected first in Ref. 23; there, the equilibrium supposed a saddle-center. The setup for this problem was earlier formulated in Ref. 5, but no essential results were found there. Results of Ref. 23 were later extended in different directions in Ref. 14, 15, 19, 20, 28, and 43.

In this paper, the dynamics is studied in reversible Hamiltonian systems with two degrees of freedom in a neighborhood of a symmetric heteroclinic connection that consists of a symmetric saddle-center and a symmetric saddle periodic orbit that is connected by two nonsymmetric heteroclinic orbits being permuted by the involution (see Fig. 1).

This type of connection is a codimension one phenomenon in the class of reversible Hamiltonian systems. Thus, such a structure can unavoidably appear in one-parameter families. Systems with such structures are met in applications. For instance, they were discovered in Ref. 22 where solitons in a nonlocal Whitham equation[27] were studied. Also, this structure can be found in a one parameter family of reversible Hamiltonian systems near a destruction of a homoclinic orbit to a saddle-center. Results of Refs. 14 and 19 suggest that saddle periodic orbits accumulate to the saddle-center loop in the singular level of the Hamiltonian, and therefore, after destruction, unstable separatrix of the symmetric saddle-center can lie on the stable manifold of a symmetric saddle periodic orbit. Due to reversibility, there is a pairing stable separatrix of the saddle-center that lies on the unstable manifold of the same periodic orbit as it is symmetric.

One more application of results obtained is a criterion of the existence of saddle-center homoclinic loops in a reversible Hamiltonian system (see Theorem 11).

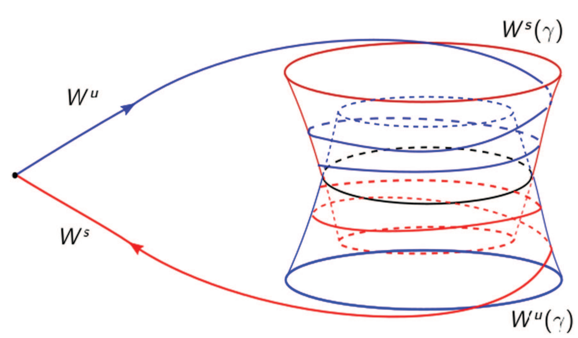

FIG. 1. Scheme of the heteroclinic connection.

Main results of the paper prove the existence of hyperbolic sets and elliptic periodic orbits near a heteroclinic connection. Existence of hyperbolic sets is based on the construction of families of transverse homoclinic orbits and heteroclinic connections involving two saddle periodic orbits and four transverse heteroclinic orbits for them. Existence of elliptic periodic orbits is proved by the following scheme. We look first for homoclinic orbits with quadratic tangencies to saddle periodic orbits in different situations and apply then results going back to Newhouse[32,33] and Gavrilov–Shilnikov[10] and many others later[8,9,11–13,29,32] on the existence of cascades of elliptic periodic orbits.

One should notice that connections involving an equilibrium and a saddle periodic orbit were studied earlier in several papers.[4,18,26] However, all these authors considered a situation when the equilibrium is hyperbolic, for instance, a saddle-focus. There, the dynamics differs from ours; moreover, the study of non-hyperbolic Hamiltonian dynamics is more involved.

II. SETTING UP AND MAIN NOTIONS

Let (M, Ω) be a real analytic four-dimensional symplectic manifold, Ω be its symplectic two-form, and H be a real analytic function (a Hamiltonian). Such a function defines a Hamiltonian vector field X_H on M. Henceforth, we assume X_H to have an equilibrium p of the saddle-center type, and without a loss of generality, we assume $H(p) = 0$. One more assumption we use is the existence in the same level of $H = 0$ a saddle periodic orbit γ. We use below the notation $V_c = \{x \in M | H(x) = c\}$.

Recall an equilibrium p of X_H on M is called to be a saddle-center[23] if the linearization operator of the vector field at p has a pair of pure imaginary eigenvalues $\pm i\omega$, $\omega \in \mathbb{R} \setminus \{0\}$, and a pair nonzero reals $\pm\lambda \neq 0$. In a neighborhood of such an equilibrium, the system has a unique local invariant smooth two-dimensional invariant symplectic submanifold W^c_p filled with closed orbits l_c (Lyapunov family of periodic orbits). For the case of analytic M, H, the submanifold W^c_p is real analytic. Near the point p in the level $H = c$ periodic orbit l_c is of saddle type and is located each on its own level V_c.

Also, in a neighborhood of p, the system has local 3-dimensional center-stable W^{cs} and center-unstable manifold W^{cu} containing both p. These submanifolds contain orbits being asymptotic, as $t \to \infty$ (for W^{cs}), to periodic orbits l_c, or as $t \to -\infty$ (for W^{cu}); here, $W^c = W^{cs} \cap W^{cu}$. Submanifold W^{cs} (respectively, W^{cu}) is foliated by levels V_c into local stable (unstable) manifolds of periodic orbits l_c; these submanifolds are diffeomorphic to cylinders $I \times S^1$. Besides, W^{cs} (respectively, W^{cu}), being a solid cylinder, contains as an axis, an analytic curve W^s (W^u), local stable (unstable) manifold of the equilibrium p; they consist of p and two semi-orbits tending p as $t \to \infty$ ($t \to -\infty$).

As was supposed above, V_0 contains a saddle periodic orbit γ. In the whole M orbit, γ belongs to a one-parameter family of such periodic orbits $\gamma_c \subset V_c$ forming an analytic 2-dimensional symplectic cylinder. Recall that in V_c, periodic orbit γ_c has two local analytic 2-dimensional Lagrangian submanifolds $W^s(\gamma)$, $W^u(\gamma)$ being its stable and unstable local submanifolds. All of them are topologically either cylinders (if the multipliers are positive) or Möbius strips (if their multipliers are negative).

Later on, in the paper, the vector field $v = X_H$ under consideration is supposed to be reversible as well. This means[7] that M acts a smooth involution $L : M \to M$, $L^2 = id_M$, and v obeys the identity $DL(v) = -v \circ L$. For its solutions, this property reads as follows: if $x(t)$ is a solution to v, then $x_1(t) = Lx(-t)$ is also its solution.

Orbit γ of a reversible v is called *symmetric* if it is invariant with respect to the action of L. In particular, an equilibrium p, $v(p) = 0$, is symmetric if $L(p) = p$,; i.e., p belongs to the fixed point set of L, $Fix(L) = \{x \in M | L(x) = x\}$. The following statement holds true.[7]

Proposition 1: *An orbit of a reversible vector field is symmetric iff it intersects Fix(L). A symmetric periodic orbit intersects Fix(L) at two points exactly. The inverse statement is also valid: if an orbit of a reversible vector field intersects Fix(L) at two different points x_1, x_2, then this orbit is symmetric periodic one and its period is equal to the doubled transition time from x_1 to x_2.*

For a Hamiltonian system, the reversibility requires clarification since the involution acts on the symplectic form

$$[L^*\Omega](\xi, \eta) = \Omega(DL(\xi), DL(\eta)).$$

We assume below that an analytic involutive diffeomorphism L is *anti-canonical* mapping; i.e., $L^*(\Omega) = -\Omega$ and L is concordant with H: $H \circ L = H$. In this case, the following identities hold:

$$DL(X_H) = -X_H \circ L, \text{ è } \Phi_t \circ L = L \circ \Phi_{-t}.$$

We also assume $Fix(L)$ to be an analytic two-dimensional submanifold in M (not obligatory connected).

Finally, we assume X_H to have a heteroclinic connection consisting of a symmetric saddle-center p, $H(p) = 0$, a symmetric periodic orbit γ in the same level of $H(\gamma) = 0$ and two heteroclinic orbits: Γ_1 going, as t increases, from γ to p, and $\Gamma_2 = L(\Gamma_1)$ going, as t increases, from p to γ. Our goal is to study the orbit behavior in a neighborhood of this heteroclinic connection. It is worth mentioning that the problem, by its setup, is a bifurcation problem since the orbit structure varies as values of the Hamiltonian c vary near a critical value $c = 0$. For instance, on the levels others than V_0, the equilibrium is absent; thus, the contour is destroyed and bifurcations are expected.

III. MOSER COORDINATES

To examine the orbit structure of the system near a connection, we shall use convenient coordinates in a neighborhood of p and in a neighborhood of γ. Corresponding results in the analytic case are due to Moser,[30,31] and a finite-smooth version for a saddle fixed point of a symplectic diffeomorphism exists in Ref. 11.

Theorem 1: *Let X_H be an analytic Hamiltonian vector field and p its equilibrium of the saddle-center type. Then, there is a neighborhood U of p, analytic symplectic coordinates (x_1, y_1, x_2, y_2), $\Omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, and a real analytic function $h(\xi, \eta)$ such that H casts in the form*

$$H(x_1, y_1, x_2, y_2) = h(\xi, \eta) = \lambda\xi + \omega\eta + R(\xi, \eta),$$

$$R(\xi, \eta) = O(\xi^2 + \eta^2), \ \xi = x_1 y_1; \quad \eta = \frac{x_2^2 + y_2^2}{2}.$$

To get a symplectic change of coordinates in this theorem except for results of Ref. 30, one needs to use Rüssmann's paper.[35]

Remark 1: *By a linear scaling of time and, if necessary, a canonical transformation $y_1 \to x_1, x_1 \to -y_1$, one can obtain $\lambda = -1$ and $\omega > 0$ (new ω is up to the sign of the ratio $|\omega/\lambda|$). Later on, we utilize this normalization.*

Denote $\Phi^t : m \to \Phi^t(m)$ the flow generated by the vector field X_H. In the coordinates of Theorem 1, the system of differential equations is written down as

$$\dot{x_1} = -h_\xi x_1, \quad \dot{y_1} = h_\xi y_1, \quad \dot{x_2} = -h_\eta y_2, \quad \dot{y_2} = h_\eta x_2, \qquad (1)$$

and its flow Φ^t is

$$\begin{pmatrix} x_1(t) \\ y_1(t) \\ x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \exp[-t \cdot h_\xi^0] & 0 & 0 & 0 \\ 0 & \exp[t \cdot h_\xi^0] & 0 & 0 \\ 0 & 0 & \cos(t \cdot h_\eta^0) & -\sin(t \cdot h_\eta^0) \\ 0 & 0 & \sin(t \cdot h_\eta^0) & \cos(t \cdot h_\eta^0) \end{pmatrix} \times \begin{pmatrix} x_1^0 \\ y_1^0 \\ x_2^0 \\ y_2^0 \end{pmatrix}, \qquad (2)$$

where notations are used,

$$h_\xi^0 = h_\xi(\xi_0, \eta_0), \quad h_\eta^0 = h_\eta(\xi_0, \eta_0), \quad \xi_0 = x_1^0 y_1^0,$$
$$\eta_0 = ((x_2^0)^2 + (y_2^0)^2)/2.$$

The Hamiltonian system under consideration is reversible as well; hence, it is important to understand to which simplest form both the system and the involution can be reduced in a neighborhood of a saddle-center by means of the same symplectic coordinate change. This was done in Ref. 14. We recall the needed results.

Theorem 2: *Let X_H be an analytic Hamiltonian vector field and p its equilibrium of the saddle-center type. Suppose, in addition, X_H be reversible with respect to the analytic anti-canonical involution L and p is symmetric. Then, in some neighborhood U of p, there are analytic coordinates, as in Theorem 1, such that L has one of two forms,*

$$\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}$$

or

$$\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}.$$

IV. LOCAL ORBIT STRUCTURE NEAR THE SADDLE-CENTER

Using Moser coordinates is the easiest way to describe the local topology of levels V_c and the orbit behavior on each level.[24] The system locally near p takes the form (1). There are two invariant symplectic disks: $x_1 = y_1 = 0$ and $x_2 = y_2 = 0$. Quadratic functions $\xi = x_1 y_1$, $\eta = (x_2^2 + y_2^2)/2$ are local integrals of the system. Consider the momentum plane (ξ, η) in a neighborhood of the origin $(0, 0)$. The level V_c of the Hamiltonian corresponds to the analytic curve $\xi = -c + \omega\eta + O(\eta^2 + c^2) = a_c(\eta)$, $0 \le \eta \le \eta_*$. For c small enough, these curves form an analytic foliation of a neighborhood of the origin $(0, 0)$. In fact, as $\eta \ge 0$, one needs to consider the rectangle $|\xi| \le \xi_0$, $0 \le \eta \le \eta_*$ in the momentum plane.

Consider first the level V_0 and then we get the curve $\xi = \omega\eta + O(\eta^2) = a(\eta)$ on the momentum plane (ξ, η), $0 \le \eta \le \eta_*$. To construct a neighborhood of the origin in $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2), we choose four cross sections $|x_1| = d$, $|y_1| = d$. In the manifold M, a neighborhood U of the point p in coordinates (x_1, y_1, x_2, y_2) can be thought as the direct product of two disks $(x_1, y_1, 0, 0)$ and $(0, 0, x_2, y_2)$.

The local structure of V_0 is investigated via its foliation into invariant levels of integral η. At $\eta = 0$ [the origin on the disk $(0, 0, x_2, y_2)$], we have $\xi = a(0) = 0$; i.e., we get a "cross" on the disk $(x_1, y_1, 0, 0)$ (the union of two segments $y_1 = 0$ and $x_1 = 0$). In U, the cross coincides with the union of local stable and unstable curves of the saddle-center p. For $\eta > 0$, the value $\xi = a(\eta)$ is positive, and on the disk $(x_1, y_1, 0, 0)$, we get two pieces of the hyperbola $x_1 y_1 = a(\eta)$, which lie in the first and third quadrants of the plane $(x_1, y_1, 0, 0)$, respectively. In U, each piece of the hyperbola is multiplied by the circle $x_2^2 + y_2^2 = 2\eta$ on the plane $(0, 0, x_2, y_2)$. Varying η from zero until η_*, we get in U two solid cylinders that have a unique common point, the origin, i.e., the saddle-center itself. Each solid cylinder contains the angle made up of two gluing semi-segments of the cross ($x_1 \ge 0$, $y_1 = 0$ and $y_1 \ge 0$, $x_1 = 0$ for one cylinder and $x_1 \le 0$, $y_1 = 0$ and $y_1 \le 0$, $x_1 = 0$ for another cylinder). Each such angle is the topological limit, as $\eta \to +0$, of cylinders $x_1 y_1 = a(\eta)$, $x_2^2 + y_2^2 = 2\eta$, lying in the same solid cylinder. In particular, each of two solid cylinders contains one half of the stable curve (a stable separatrix) and one half of the unstable curve (unstable separtatrix) of the saddle-center. At the fixed $\eta > 0$, each two-dimensional cylinder is an invariant set and orbits on it go from one of two cross sections $|y_1| = d$ to another of two cross sections $|x_1| = d$ (see Fig. 2).

Remark 2: *It is worth remarking the property that will be used below. In Moser coordinates on the level $H = 0$, the cross sections for orbits in the solid cylinder, which is projected onto the first quadrant of the plane $x_2 = y_2 = 0$, are $y_1 = d > 0$ (for entering orbits) and $x_1 = d$ (for outgoing orbits), but for the second solid cylinder, which is projected onto the third quadrant, they are $y_1 = -d$ (for entering orbits) and $x_1 = -d$ (for outgoing orbits). This implies that for the case of an involution of the first type, the symmetry permutes the orbit $x_1 = x_2 = y_2 = 0, y_1 > 0$, with that $y_1 = x_2 = y_2 = 0, x_1 > 0$, and orbit on $y_1 < 0, x_1 = x_2 = y_2 = 0$ with that on $x_1 < 0$, $y_1 = x_2 = y_2 = 0$. Hence, for the involution of the first case, the symmetry permutes cross sections of the same solid cylinder.*

For the case of the involution of the second type, the symmetry permutes the orbit $y_1 > 0, x_1 = x_2 = y_2 = 0$ with that on $x_1 < 0$, $y_1 = x_2 = y_2 = 0$ and orbit on $y_1 < 0, x_1 = x_2 = y_2 = 0$ with that on $x_1 > 0$, $y_1 = x_2 = y_2 = 0$. Hence, such symmetry permutes cross sections from the different solid cylinders. This observation is used below for the classification of heteroclinic connections.

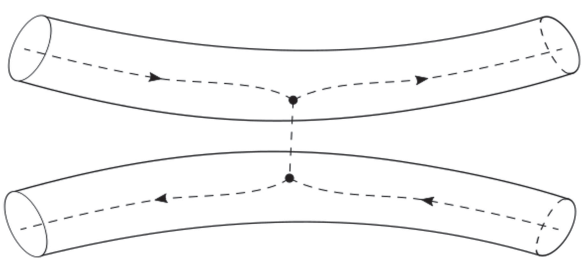

FIG. 2. Local topology of the level $c = 0$. The dashed lines mark separatrices of the saddle-center.

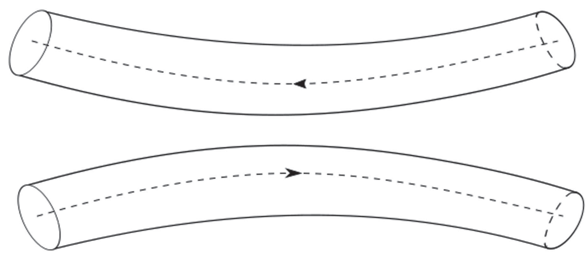

FIG. 3. Local topology of the level $c < 0$. Dashed lines mark some orbits.

A level V_c as $c < 0$ corresponds to the curve $\xi = -c + \omega\eta + O(\eta^2 + c^2) = a_c(\eta) > 0$ on the momentum plane (ξ, η) for all $0 \le \eta \le \eta_*$. Therefore, the level V_c as $c < 0$ consists of two disconnected solid cylinders, their projections onto the momentum plane form two curvilinear rectangles in the first and third quadrants bounded by related segments $|x_1| = d$, $|y_1| = d$, and pieces of hyperbolas $x_1 y_1 = a_c(0) = -c > 0$ and $x_1 y_1 = a_c(\eta_*)$. Each two-dimensional cylinder $\eta = \eta_0$ is foliated by flow orbits going from one cross section $|y_1| = d$ to another one $|x_1| = d$ (see Fig. 3).

For $c > 0$, the situation is more complicated since the curve $\xi = a_c(\eta)$ on the plane (ξ, η) for $0 \le \eta \le \eta_*$ corresponds to both positive and negative values of ξ (we consider $|c|$ small enough). Denote η_c the unique positive root of the equation $a_c(\eta) = 0$. Then, for $0 \le \eta \le \eta_c$, pieces of hyperbola $x_1 y_1 = a_c(\eta) < 0$ belong to the second and fourth quadrants of the plane $(x_1, y_1, 0, 0)$, but for $\eta > \eta_c$, they belong to the first and third quadrants of this plane. The topological type of the set V_c is a connected sum of two solid cylinders (i.e., balls). To see this, let us consider the projection of V_c on the plane $(x_1, y_1, 0, 0)$, and it lies inside of the quadrate $|x_1| \le d$, $|y_1| \le d$. Let us cut this set into two halves by the diagonal $y_1 = -x_1$. Over this diagonal in V_c, a two-dimensional sphere S is situated. Indeed, at extreme points of the diagonal in the second and fourth quadrants, two points of V_c correspond (for them $\eta = 0$), but for any point of the diagonal between extreme points in V_c, a circle lies since $\eta > 0$ for such points. In particular, over the point $(0, 0)$ of the diagonal, the circle $x_2^2 + y_2^2 = 2\eta_c$ lies, and it is the Lyapunov periodic orbit l_c. Segments $x_1 = 0$ and, respectively, $y_1 = 0$ of the plane in V_c correspond to stable and unstable manifolds of the periodic orbits.

Subset $V_c^+ \subset V_c$ lying over the half-plane $x_1 + y_1 > 0$ is composed of three parts. One part corresponds to the values $\eta_c \le \eta \le \eta_*$. For the strict inequality, we get a set being diffeomorphic to the direct product of an open annulus and a segment. As $\eta \to \eta_c + 0$, this set has as a topological limit the set being the direct

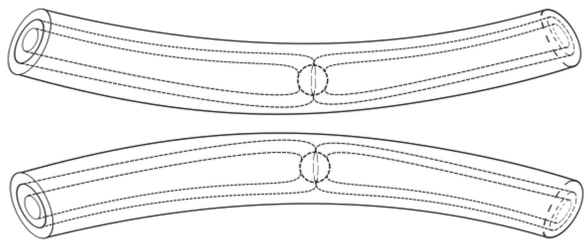

FIG. 4. Local topology of the level $c > 0$: two inner balls cut out, and two boundary spheres glue together. Look at the saddle Lyapunov periodic orbit and its stable and unstable manifolds.

product of an angle in the plane (x_1, y_1): $0 \le x_1 \le d$, $y_1 = 0$ and $0 \le y_1 \le d$, $x_1 = 0$, and a circle $x_2^2 + y_2^2 = 2\eta_c$. Two other parts of V_c^+ are two solid cylinders. These are those subsets in V_c^+ that are projected into the second and third quadrants of the plane (x_1, y_1), and they correspond to $\xi < 0$. Every such a solid cylinder is foliated into two-dimensional cylinders lying over pieces of hyperbola $\xi = a_c(\eta) < 0$, $0 \le \eta < \eta_c$, $x_1 + y_1 > 0$, respectively, in the second and fourth quadrants. One of the bounding circles of this cylinder lies over the point on the diagonal (for each cylinder, this point is own), and the second bounding circle lies over the point on the segment $y_1 = d$ or $x_1 = d$. The solid cylinder projecting onto the second quadrant is glued with its lateral boundary to the set lying over the first quadrant along the cylinder $x_1 = 0$, $y_1 > 0$, $x_2^2 + y_2^2 = 2\eta_c$, but the second solid cylinder that is projecting onto the fourth quadrant is glued by its lateral boundary to the set over the first quadrant along the cylinder $y_1 = 0$, $x_1 > 0$, $x_2^2 + y_2^2 = 2\eta_c$. Thus, the set is obtained that is homeomorphic to the solid cylinder from which an inner ball with the boundary S is cut.

Similarly, the second part V_c^- of the set V_c is obtained lying over half-plane $x_1 + y_1 \le 0$. The sets V_c^- and V_c^+ are glued along the sphere S, and the gluing corresponds to the same points on the diagonal $y_1 = -x_1$. Visually, this can be imagined in such a way that in each half (solid cylinder), we cut out by an inner ball and glue the halves obtained along the boundary of balls in accordance with their orientation. This is a particular case of a connected sum of two manifolds. Topologically, the set obtained is homeomorphic to a spherical layer $S^2 \times I$. Since each level is foliated into invariant cylinders $\eta = const$, we get a complete picture of the local orbit behavior near a saddle-center (see Figs. 2–4).

V. POINCARÉ MAP IN A NEIGHBORHOOD OF γ

To describe the orbit behavior in a neighborhood of a periodic orbit γ in the level V_0, we consider a two-dimensional symplectic analytic Poincaré map generated by the flow of X_H on some analytic cross section Σ to γ. For the case under consideration, the vector field is reversible; hence, the cross section can be chosen in such a way that the reversibility would preserve for the Poincaré map as well.

A symmetric periodic orbit intersects submanifold $Fix(L)$ at two points m_1, m_2. Take one of them, $m = m_1$, and consider a three-dimensional analytic cross section N for γ containing m. N can be chosen in such a way that m belongs to N along with some sufficiently small analytic disk from $Fix(L)$ and N is invariant with respect to the action of L. We assume further such choice of the cross section.

In a sufficiently small neighborhood of m levels, V_c forms an analytic foliation into three-dimensional submanifolds since $dH_m \neq 0$. The level V_0 contains the curve γ, but N is transversal to the curve; hence, V_0 and N intersect each other transversely at the point m; therefore, they intersect in M along an analytic two-dimensional disk $\Sigma \subset N$. Σ is a cross section to γ in the level V_0, and we get an analytic Poincaré map $S : \Sigma \to \Sigma$ with a saddle fixed point m.

To study orbit behavior of a system in a neighborhood of γ, we use the Moser theorem[31] on the normal form of a two-dimensional analytic symplectic diffeomorphism near its saddle fixed point. As γ is orientable by the assumption, its multipliers ν, ν^{-1} are positive.

Theorem 3 (Moser): *In a neighborhood of a saddle fixed point of a real analytic symplectic diffeomorphism S, there are analytic symplectic coordinates (u, v) and an analytic function $f(\zeta)$, $\zeta = uv$, $f(0) = \nu$, such that S takes the following form:*

$$\bar{u} = u/f(\zeta), \bar{v} = vf(\zeta), \text{ where } f(\zeta) = \nu + O(\zeta),\ 0 < \nu < 1. \quad (3)$$

For our case, S is also reversible with respect to the restriction of the involution on Σ, and involution permutes stable and unstable manifolds (here—curves) of the fixed point m. As $X_H(m) \neq 0$, then intersection of $Fix(L)$ and V_0 is transverse at m and it is an analytic curve $l \subset \Sigma$ being the symmetry line containing m. It is not hard to prove, following Ref. 3 that symplectic coordinates (u, v) in the Moser theorem can be chosen in such a way that the restriction of the involution on Σ would act as $(u, v) \to (v, u)$. Then, the fixed point set of the involution near point m coincides with the diagonal $u = v$. We assume henceforth this to hold.

The orbit Γ_1 is nonsymmetric and approaches γ as $t \to -\infty$; hence, it intersects Σ at a countable set of points tending to m, but not lying on l. These points belong to the analytic curve w_u being the trace on the disk Σ of manifold $W^u(\gamma)$. Similarly, the orbit Γ_2 intersects Σ at a countable set of points approaching m at positive iterations of S, and the points do not lie on l. These points belong to the analytic curve w_s being the trace of the manifold $W^s(\gamma)$ on Σ. In Moser coordinates, a local stable curve coincides with the axis v (it is given as $u = 0$) and a local unstable one with the axis u $(v = 0)$. Therefore, the point p_s, the trace of Γ_2, has coordinates $(0, v_+)$, and $p_u = L(p_s)$ is the trace of Γ_1 and has coordinates $(u_-, 0)$. To be definite, we assume $v_+ > 0$. Then, due to reversibility, one has $u_- = v_+$.

We choose neighborhoods Π^s, Π^u of point p_s, p_u defined by inequalities $\Pi^s : |v - v_+| < \varepsilon$, $|u| < \delta$, è $\Pi^u : |v| < \delta$, $|u - u_-| < \varepsilon$, and the quantities δ, ε are small enough. The set of points from Π^s that is transformed to Π^u by some iteration of the map S, as is known,[36,39] consists of the countable set of strips in Π^s accumulating to the stable curve $u = 0$. Due to a convenient normal form, these strips are easily found. The following assertion holds

Lemma 1: *Equations $u = f^k(\zeta)(u_- \pm \varepsilon)$ define functions $u = s_k^{\pm}(v)$ whose domain is $|v - v_+| < \varepsilon$. For them, inequalities $s_k^+(v) > s_k^-(v)$ hold true $s_k^-(v) > s_{k+1}^+(v)$, and $s_k^+(v)$ uniformly tend to zero as $k \to \infty$.*

Proof. The proof of this lemma is obvious. The lateral boundaries of strips σ_k^s are segments $|v - v_+| = \pm\varepsilon$. The curves $s_k^+(v)$ provided by the solution of the equation $u_k = u_- + \varepsilon$ serve their upper boundaries, and their lower boundaries are the curves given by solutions of the equations $u_k = u_- - \varepsilon$. To prove the lemma, we take an arbitrary v, $|v - v_+| \le \varepsilon$ and find the related values $u = s_k^+(v)$ and $u = s_k^-(v)$ from the equations,

$$u = f^k(\zeta)(u_- + \varepsilon),\ \ u = f^k(\zeta)(u_- - \varepsilon).$$

Consider, for example, the first equation. Since the value of ζ preserves along the orbit of S, then multiplying both sides of the first equation at v, we get $g_k(\zeta) = \zeta/f^k(\zeta) = v(u_- + \varepsilon)$. For $k \ge k_0 > 0$, this sequence of complex functions has the inverse one in each, and all of them are defined in the same disk $|\zeta| \le \sigma$ of the complex plane $\mathbb{C}$. This follows from the complex inverse function theorem since $g_k(0) = 0, g_k'(0) = \nu^{-k}$. Thus, we get $s_k^+(v) = g^{-1}(v(u_- + \varepsilon))/v$ and $s_k^+(v) \to 0$ as $k \to \infty$ uniformly in v. □

Functions $s_k^{\pm}(v)$ are upper and lower boundaries of the strip σ_k^s. It implies the existence of a countable set of such strips,

$$k > k_0 = E\left\{\frac{\ln((\varepsilon + u_-)/\delta)}{ln(\nu^{-1})}\right\}.$$

Here, it is assumed $\varepsilon + u_- > \delta$ (this is the first restriction on the quantities ε, δ).

The restriction of L on Σ acts as $L : (u, v) \to (v, u)$; hence, we get $u_- = v_+ = r$. Thus, we have the same condition on k_0 for strips σ_k^u,

$$k > k_0 = E\left\{\frac{\ln((\varepsilon + r)/\delta)}{ln(\nu^{-1})}\right\}.$$

One more restriction on these quantities provides the requirement that the neighborhood Π^s would not intersect with its image $S(\Pi^s)$, Π^u with its pre-image $S^{-1}(\Pi^u)$, and $\Pi^s \cap \Pi^u = \emptyset$. These conditions lead to the inequalities,

$$\delta < r\frac{1-\nu}{1+\nu}, \quad \delta < r - \varepsilon.$$

Now, we can assert, due to construction, that all orbits of the map S, which pass through the points of the neighborhood Π^s and reach the neighborhood Π^u for positive iterations, have to pass through one of strips $\sigma_k^s, k \ge k_0$. These strips have, as its topological limit, the segment $u = 0$ in Π^s. Its points belong to the stable manifold, and they tend under S^n to the fixed point m as $n \to \infty$.

From the reversibility of S and the same considerations, we get that images of strips σ_k^s in Π^u, i.e., strips σ_k^u, accumulate as $k \to \infty$ to the points of the segment $v = 0$ in Π^u. These points under negative iterations of S tend to m.

Remark 3: *It is worth remarking a useful fact. In a neighborhood of the saddle periodic orbit γ for a given small value of $|c|$, the level of the Hamiltonian $V_c = \{H = c\}$ is the analytic invariant three-dimensional submanifold. In V_c, the only saddle periodic orbit γ_c lies being the continuation in c of the orbit γ. The family γ_c makes up an analytic two-dimensional local symplectic cylinder containing the orbit $\gamma_0 = \gamma$. Three-dimensional cross section N chosen above, under intersecting with V_c, gives an analytic two-dimensional symplectic disk Σ_c being the local cross section for the restriction of the flow on V_c. The Poincaré map S_c on Σ_c is symplectic analytic having the saddle fixed point m_c. For this map, the Moser theorem is also valid and the map can be transformed to the form (3). Moreover, since the dependence on c is analytic, the change of variables can be done for all small enough c at once, and the function f in (3) will depend on c analytically. This utilizes below to study the dynamics on V_c for c close to $c = 0$.*

VI. GLOBAL MAPS

Now, we derive the representations of the global maps T_1 generated by the flow near Γ_1 acting as $T_1 : \Pi^u \to D_1$. It is analytic symplectic diffeomorphism and is written as $x_2 = f(u, v)$, $\quad y_2 = g(u, v)$; here, symplecticity is equivalent to the identity $f_u g_v - f_v g_u \equiv 1$ (area preservation).

Linearization of this map at the point $(u_-, 0)$ has the form $x_2 = \alpha(u - u_-) + \beta v$, $y_2 = \gamma(u - u_-) + \delta v$, where $\alpha = f_u$, $\beta = f_v$, $\quad \gamma = g_u$, $\quad \delta = g_v$; all derivatives are calculated at the point $(u_-, 0)$. Thus, a general form of T_1 is

$$x_2 = \alpha(u - u_-) + \beta v + \cdots, \quad y_2 = \gamma(u - u_-) + \delta v + \cdots.$$

With the system under study being reversible and cross sections being chosen consistently with the action of involution, then the global map $T_2 : D_2 \to \Pi^s$ near $\Gamma_2 = L(\Gamma_1)$ is expressed as $T_2 = L \circ T_1^{-1} \circ L$ or in coordinates,

$$\begin{aligned} u_1 &= \gamma\bar{x}_2 + \alpha\bar{y}_2 + \cdots, \\ v_1 - v_+ &= -\delta\bar{x}_2 - \beta\bar{y}_2 + \cdots. \end{aligned}$$

Below, when studying the orbit behavior on the levels V_c for $c \ne 0$, we shall need to know the form of the global maps in these cases. As was mentioned above, without loss of generality, we can regard as coordinates on the disks $D_1(c), D_2(c)$ symplectic coordinates (x_2, y_2) and $(\bar{x}_2, \bar{y}_2)$, respectively, and on the disk $\Sigma(c)$ symplectic coordinates (u, v). Global maps are analytic symplectic diffeomorphisms analytically depending on c. Thus, they have the form

$$\begin{aligned} T_1(c):\ x_2 &= a(c) + \alpha(c)(u - u_-) + \beta(c)v + \cdots, \\ y_2 &= b(c) + \gamma(c)(u - u_-) + \delta(c)v + \cdots, \\ T_2(c):u_1 &= a_1(c) + \gamma(c)\bar{x}_2 + \alpha(c)\bar{y}_2 + \cdots, \\ v_1 - v_+ &= b_1(c) - \delta(c)\bar{x}_2 - \beta(c)\bar{y}_2 + \cdots, \end{aligned} \tag{4}$$

where $a_1(c) = \gamma(c)a(c) - \alpha(c)b(c),\ b_1(c) = -\delta(c)a(c) + \beta(c)b(c)$.

VII. TWO TYPES OF SYMMETRIC CONNECTIONS

The symmetric periodic orbit γ in the level V_0 lies outside of a neighborhood of point p; therefore, one needs to conform the location of this orbit and its symmetry with the action of involution in U relative to coordinates. This concordance is relied on the existence of connecting orbits Γ_1 and $\Gamma_2 = L(\Gamma_1)$.

Near the point p involution, L permutes local stable and unstable curves of p. Recall (see above) that in a neighborhood of point p, the local topological type of the level V_0 is a pair of three-dimensional solid cylinders with two of their inner points glued chosen by one in each cylinder (after gluing this is the point p) (see

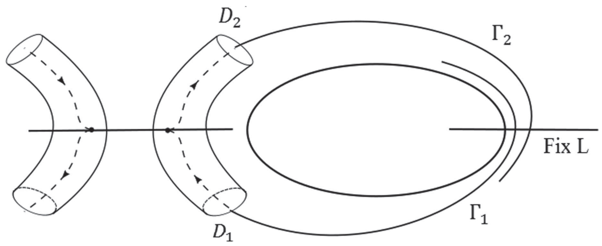

FIG. 5. Type one connection. Solid segments are traces of $Fix(L)$.

Fig. 2). The lateral boundary of each solid cylinder is a smooth two-dimensional invariant cylinder, and two other boundaries are two disks ("lids"). For each solid cylinder, orbits enter through one lid and leave the cylinder through the other lid.

By assumption on the connection, we know that $\Gamma_2 = L(\Gamma_1)$ globally. Since heteroclinic orbits contain pieces of these curves adherent to p, two cases are possible here. In the first case, these two heteroclinic orbits can belong locally near p to the same solid local cylinder (type one connection); in the second case, they belong locally to different solid cylinders (type two connection).

Near p in Moser coordinates, disks $y_1 = \pm d$ and $x_1 = \pm d$ can be taken as cross sections (lids) to Γ_1 and Γ_2, where the sign is determined by the intersection with the related heteroclinic orbits. We can always assume, as above, that Γ_1 enters to its solid cylinder through $y_1 = d$. We denote it as D_1. Coordinates on the disk are (x_2, y_2) since a coordinate conjugated to y_1 is found from the equality $H = 0$. For the connection of the first type, the second cross section (for Γ_2, in particular) is the disk $D_2 = \{x_1 = d\}$. For the connection of the second type, the second cross section (for Γ_2, in particular) is the disk $x_1 = -d$.

In Moser coordinates, the type one connection corresponds to the local action $L : (x_1, y_1, x_2, y_2) \to (y_1, x_1, -x_2, y_2)$, but the type two does to the action $L : (x_1, y_1, x_2, y_2) \to (-y_1, -x_1, -x_2, y_2)$. The type one means the invariance of the related cylinder with respect to the involution and the type two means their permutability (one cylinder transforms to another one). In case if L preserves the cylinder, the intersection of $Fix(L)$ with the cylinder is a curve, but if L permutes cylinders, this intersection is the only point p (see Figs. 5 and 6).

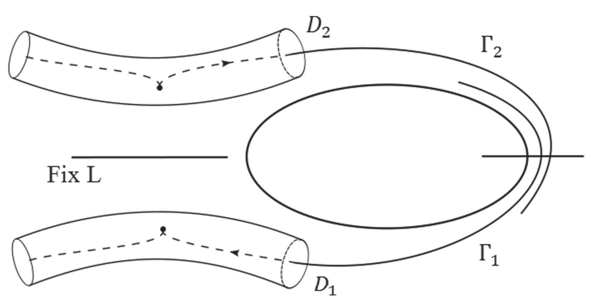

FIG. 6. Type two connection. Solid segments are traces of $Fix(L)$.

Now, consider those orbits of the vector field that enter through D_1 near Γ_1 but distinct from Γ_1. As t increases, they enter into the related solid cylinder, pass it, and leave the cylinder (the semi-orbit Γ_1 itself tends to p and stays in the cylinder). These orbits either intersect D_2 (case 1) or leave the cylinder without intersecting D_2 (case 2). In case 2, the Poincaré map is not defined in a neighborhood of the connection for $c \leq 0$ since orbits close to Γ_1 do not return on D_2 and if a system under consideration does not fulfill some additional global conditions (of the type the existence of a homoclinic orbit joining two remaining lids).

Remark 4: *For $c > 0$ small enough for case 2, the Poincaré map on the related disk $D_1(c)$ becomes defined only inside some small disk centered at $(0, 0)$ whose boundary circle is the trace of stable manifold $W^s(l_c)$ of the Lyapunov periodic orbit l_c. Then, for some sequence of positive $c_n \to 0$, the trace on the disk $D_1(c_n)$ of $W^s(\gamma_{c_n})$ is tangent to the trace of $W^u(\gamma_{c_n})$, which is accompanied with the appearance of elliptic periodic orbits. This will be considered below.*

VIII. IN THE SINGULAR LEVEL V_0

The study of the orbit behavior in V_0 is performed by us mainly in case 1 since for the case two all nearby orbits to Γ_1 leave this level. In the neighborhood U of p, we have in Moser coordinates the representation $h = -\xi + \omega\eta + R(\xi, \eta)$; hence, the manifold W^{cs} is given as $x_1 = 0$, W^{cu} as $y_1 = 0$, W^s by the equalities $x_1 = x_2 = y_2 = 0$, and W^u by $y_1 = x_2 = y_2 = 0$. Suppose, to be definite, that in U heteroclinic orbit Γ_1 approaches to p for values $y_1 > 0$; i.e., in the level V_0, disk D_1 is defined by the equality $y_1 = d$ and disk D_2 by the equality $x_1 = d$. In U, signs of variables x_1 and y_1 preserve by the flow and three-dimensional cross section $N^s : y_1 = d > 0, |x_1| \leq \delta, \eta \leq \eta_0$, is transverse to Γ_1 and to all orbits close to Γ_1 due to the inequality $h_\xi = -1 + \cdots \neq 0$ in U. Similar assertions are valid for the cross section $N^u = L(N^s) : x_1 = d, |y_1| \leq \delta, \eta \leq \eta_0$. Each cross section is foliated by levels $H = c$ into disks, one of which is $D_1 = V_0 \cap N^s$ and, respectively, $D_2 = V_0 \cap N^u$.

Denote as $a(\eta)$ the solution of the equation $h(\xi, \eta) = 0$ with respect to ξ, $\xi = a(\eta) = \omega\eta + O(\eta^2)$. One may regard that when variables (x, y) vary in $U \cap V_0$, corresponding solutions of the equation $h(\xi, \eta) = 0$ lie on the graph of function a. Then, two-disk D_1 in N^s is the graph of the function $x_1 = a(\eta)/d$, and two-disk D_2 in N^u is the graph of function $y_1 = a(\eta)/d$. Both D_1, D_2 are analytic disks being symplectic with respect to the restriction of 2-form Ω on D_1 and D_2, respectively, and the local map $T : D_1 \to D_2$ generated by the flow Φ^t is symplectic.

Remark 5: *For type 2 of the involution, the cross sections are $y_1 = d$ (for N^s) and $x_2 = -d$ (for N^u). Therefore, orbits from $D_1 = N^s \cap V_0$ hit $D_2 = N^u \cap V_0$ only if $a(\eta) < 0$.*

Let us find an explicit representation of the map T in coordinates (x_2, y_2). The passage time τ for orbits from N^s to N^u is found from (1), where $x_1(\tau) = d$, $y_1(0) = d$: $\tau = -(h_\xi)^{-1} \ln(d/x_1)$, $x_1 = a(\eta)/d$. From (1), it follows that T has the form

$$\bar{x}_2 = x_2 \cos\Delta(\eta) - y_2 \sin\Delta(\eta), \quad \bar{y}_2 = x_2 \sin\Delta(\eta) + y_2 \cos\Delta(\eta), \tag{5}$$

with

$$\Delta(\eta) = -\frac{h_\eta}{h_\xi}\ln(d/x_1) = a'(\eta)\ln(d^2/a(\eta))$$
$$= (\omega + O(\eta))\ln(d^2/a(\eta)). \tag{6}$$

Remark 6: *For type 2 of the involution, the formula is modified as* $\Delta(\eta) = a'(\eta)\ln(-d^2/a(\eta))$.

Our first result is the following theorem.

Theorem 4: *If an analytic reversible Hamiltonian system has a heteroclinic connection of type 1, then the saddle periodic orbit γ has a countable set of one-round transverse homoclinic orbits. In the case of type 2 connection, no other orbits exist in a sufficiently small neighborhood of the connection in V_0 except for orbits of the connection itself.*

To be precise, let us make more exact the notion of a one-round homoclinic orbit for γ. To this end, consider in V_0 a sufficiently small tubular neighborhood of the orbit γ. Since V_0 is orientable, this neighborhood is homeomorphic to a solid torus $D^2 \times S^1$. The union of points of the orbit Γ_1, point p and points of the orbit Γ_2 gives a simple non-closed curve without self-intersections in V_0. One may regard that this infinite curve consists of three connected pieces, one of which, R, lies outside of the tubular neighborhood of γ, and two remaining ones are inside of this tubular neighborhood (recall that orbits Γ_1, Γ_2 tend asymptotically to γ). Now, consider a homoclinic orbit to γ, whose global part outside of the tubular neighborhood of γ belongs to a small neighborhood of the curve R and is simply connected, but two remaining parts are inside of the tubular neighborhood. Such a homoclinic orbit for γ will be called a one-round one.

Proof. To prove the theorem, we will show that a segment of the unstable separatrix $w_u \cap \Pi^u$ of the saddle fixed point m on Σ is transformed by the map $T_2 \circ T \circ T_1$ into an analytic curve that intersects transversely at the countable set of points at the segment $w_s \cap \Pi^s$ of the stable separatrix w_s of the same fixed point. The scheme of constructing the related Poincaré map is shown in Fig. 7.

Consider in $\Pi^u: v = 0,\ |u - u_-| \le \varepsilon_1 < \varepsilon$, a segment (A, B) of the curve w_u. Its image under the action of T_1 is a parameterized curve on the disk D_1: $x_2(\tau) = \alpha\tau + \cdots,\ y_2(\tau) = \gamma\tau + \cdots$, and its parameter is $\tau = u - u_-$. Since T_1 is a diffeomorphism, we get a smooth curve in D_1 passing through $(0, 0)$, and its tangent vector

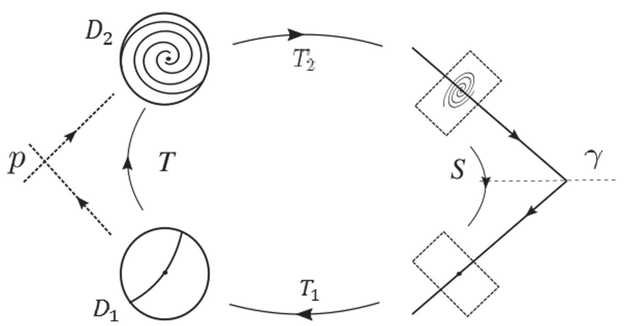

FIG. 7. Poincaré map as $c = 0$.

at $(0, 0)$ is a nonzero vector (α, γ). Boundary points of this curve denote as A_1, B_1 and the curve obtained as $[A_1, B_1]$.

The curve $[A_1, B_1]$ by the map T is transformed to the spiral-shape curve on the disk D_2,

$$\bar{x}_2 = x_2(\tau)\cos\Delta(\eta(\tau)) - y_2(\tau)\sin\Delta(\eta(\tau)),$$
$$\bar{y}_2 = x_2(\tau)\sin\Delta(\eta(\tau)) + y_2(\tau)\cos\Delta(\eta(\tau)).$$

In symplectic polar coordinates on D_1, D_2, respectively,

$$x_2 = \sqrt{2\eta}\cos\phi, \quad y_2 = \sqrt{2\eta}\sin\phi, \bar{x}_2 = \sqrt{2\bar{\eta}}\cos\theta,$$
$$\bar{y}_2 = \sqrt{2\bar{\eta}}\sin\theta;$$

the map T has the form

$$\bar{\eta} = \eta, \quad \theta = \phi + \Delta(\eta) \pmod{2\pi}.$$

This map is defined for values $\eta > 0$. Under the action of T, the curve $[A_1, B_1]$ transforms into two infinite spirals corresponding to $\tau > 0$ and $\tau < 0$,

$$\bar{\eta} = \eta(\tau), \quad \theta = \phi(\tau) + \Delta(\eta(\tau)),$$

where for $|\tau|$ small enough, we have for $\alpha \neq 0$,

$$\eta(\tau) = (x_2^2(\tau) + y_2^2(\tau))/2 = \frac{\alpha^2 + \gamma^2}{2}\tau^2 + O(\tau^3),$$
$$\tan\phi(\tau) = \frac{\gamma}{\alpha} + O(\tau),$$

and for $\alpha = 0$, the angle is defined via $\cot\phi$; here, the values ϕ as $\tau \to +0$ and $\tau \to -0$ differ by π. Since ϕ is bounded as $\tau \to \pm 0$, but function $\Delta(\eta(\tau))$ monotonically increases to ∞, then each of spirals, as $|\tau| \to 0$, tends to $(0, 0)$ on D_2, making an infinite number of rotations in angle: $\theta(\tau) \to \infty$. Take a segment on $u = 0$ symmetric to $[A, B]$ and its T_2-pre-image $[A_2, B_2]$ on D_2, it is an analytic segment through the point $(0, 0)$ symmetric to $[A_1, B_1]$. Therefore, it intersects each spiral at the countable set of points through which orbits pass tending to γ as $t \to \pm\infty$; that is, they are Poincaré homoclinic orbits.[36] To complete the proof, we need to show transversality of intersections spirals and $[A_2, B_2]$. Alternatively, we shall prove the transversality of T_2-images of spirals and the segment $u = 0$ on Π^s.

Consider, for instance, one of spirals, defined by inequality $\tau > 0$ and apply T_2,

$$u_1 = \gamma\bar{x}_2 + \alpha\bar{y}_2 + \cdots = \sqrt{2\eta(\tau)}\sqrt{\alpha^2 + \gamma^2}$$
$$\times\left[\sin(\varphi(\tau) + \Delta(\eta(\tau)) + \sigma) + O(\sqrt{2\eta(\tau)})\right],$$
$$v_1 - v_+ = -\delta\bar{x}_2 - \beta\bar{y}_2 + \cdots = \sqrt{2\eta(\tau)}\sqrt{\beta^2 + \delta^2}$$
$$\times\left[\sin(\varphi(\tau) + \Delta(\eta(\tau)) + \sigma_1) + O(\sqrt{2\eta(\tau)})\right].$$

The map T_2 transforms the spiral and the point $(0, 0)$ from D_2 to some spiral-shape curve and the point $(0, v_+)$ in Π^s. We need to prove that the spiral obtained does not tangent to the segment $u = 0$ at any common point. To this end, we show that the derivative $u_1'(\tau)$ does not vanish at the intersection points of the spiral with the segment $u = 0$ in Π^s. As $\eta(\tau) \neq 0$, zeros of the function $u_1(\tau)$

are determined by zeros of the function $\sin(\varphi(\tau) + \Delta(\eta(\tau)) + \sigma)$, and one needs to check the inequality $u_1'(\tau) \neq 0$ for those τ where $u_1 = 0$.

The derivative $u_1'(\tau)$ at points where $\sin(\varphi(\tau) + \Delta(\eta(\tau)) + \sigma) = 0$ is equal up to a nonzero multiplier

$$\cos(\varphi(\tau) + \Delta(\eta(\tau)) + \sigma)(\varphi'(\tau) + \Delta'(\eta(\tau))\eta'(\tau)).$$

Thus, the first multiplier is nonzero and the principal term in the bracket for small enough τ is $\Delta'(\eta(\tau))\eta'(\tau)$, which tends to infinity as $\tau \to 0$. Indeed, in accordance to formula (6) for Δ, we have

$$\begin{aligned}\Delta'(\eta) &= a''(\eta)\ln(d^2/a(\eta)) - \frac{a'^2(\eta)}{a(\eta)}\\ &= \frac{-a'^2(\eta) + a''(\eta)a(\eta)\ln(d^2/a(\eta))}{a(\eta)};\end{aligned}$$

hence, the numerator is negative and separated from zero for small η, but the denominator tends to zero as $\eta \to +0$. The ratio $\eta'(\tau)/a(\eta(\tau))$ is of the order $1/\tau$. Therefore, the existence of a countable set of transverse homoclinic orbits has been proved.

For case 2 orbits of the system passing on Π^u through the points of an unstable curve $v = 0$, $|u - u_-| < \varepsilon$, as t increases, intersect disk D_1 and after that leave V_0 (see Fig. 6). The same holds true, due to symmetry, as t decreases, for orbits passing on Π^s through the points of a stable curve $u = 0, |v - v_+| < \varepsilon$. □

The proven theorem allows one to use results[36,39] about the orbit structure near a transverse homoclinic orbit of a two-dimensional diffeomorphism. Specifically, near each homoclinic orbit, there exists its neighborhood such that orbits of a diffeomorphism passing through this neighborhood make up an invariant hyperbolic subset whose dynamics is conjugated with the shift on a transitive Markov chain (see, for instance, Ref. 17). On Π^s, we have a countable set of different homoclinic orbits accumulating at the trace of heteroclinic orbit Γ_2. It is clear for a fixed homoclinic point from the set, the size of a neighborhood, where the description holds, tends to zero as homoclinic points approach to the trace of Γ_2. If we consider an only finite number of homoclinic points outside of a small neighborhood of the trace of Γ_2, then we get a uniformly hyperbolic set generated by these homoclinic orbits. Therefore, the entire region where the hyperbolic set for our case exists should be of a two-horn shape bounded by two parabola-like curves, which are tangent at the point $(0, v_+)$ (see Fig. 8) The strips near homoclinic orbits (see above) for different homoclinic points interact each other under iterations of the Poincaré map that lead to a Markov chain with a countable set of states, but the invariant set obtained in this way is not uniformly hyperbolic but only non-uniformly hyperbolic.

IX. HYPERBOLICITY AND ELLIPTICITY IN LEVELS $c < 0$

In this section, we consider levels V_c, $c < 0$, near the connection for case 1. For case 2 and $c < 0$, all orbits, entering through $D_1(c)$ to a neighborhood, leave it, and the same is true for the orbits entering a neighborhood, as t decreasing, through $D_2(c)$.

We prove (1) the existence of a hyperbolic set constructed on a finite number of transverse homoclinic orbits to the saddle periodic orbit γ_c and (2) the existence of a countable set of intervals of values $c < 0$ accumulating at zero whose values of c correspond to levels

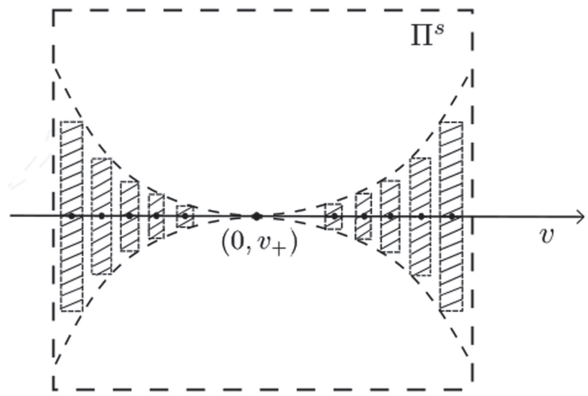

FIG. 8. The shape of the non-uniform hyperbolicity region in Π^s.

where in V_c, a one-round elliptic periodic orbit exists. Remind that for case 1 and negative c small enough, all orbits in a neighborhood of p passing through one "lid" $D_1(c) = N^s \cap V_c$ of the solid cylinder, as t increases, intersect its second lid $D_2(c) = N^u \cap V_c$.

Existence of finitely many transverse homoclinic orbits to a periodic orbit γ_c is almost evident and follows from their existence at $c = 0$. For any negative c small enough, we consider the solid cylinder of the local part of the level V_c near p whose lids are $D_1(c), D_2(c)$. Then, global maps transform: a segment $w_u(c)$ on $\Pi^u(c)$ (i.e., $v = 0$) is mapped by $T_1(c)$ onto a curvilinear segment on $D_1(c)$ passing near point $(0, 0)$ at the distance of the order $|c|^l$, $l \geq 1$. The same holds true for a symmetric curve on $D_2(c)$ being the pre-image with respect to $T_2(c)$ of the segment of $w_s(c)$ (i.e., $u = 0$). Let us cut out on $D_1(c), D_2(c)$ small disks of the radius of an order $O(\sqrt{|c|})$ centered at points $(0, 0)$. Since the map $T(c)$ preserves η, we cut out thereby by one interval on each curvilinear segment of each disk. After cutting out, two remaining segments stay on each disk. Consider the images of the remaining segments on $D_2(c)$ under the map $T_2(c) \circ T(c)$.

Theorem 5: *For the type one connection, if $|c|$ is small enough, the image of each remaining segment is a finite spiral that intersects transversely at a finite number of points of the curve $u = 0$ in $\Pi^s(c)$.*

Proof. We follow the lines of the case $c = 0$. The difference is that we first cut out the disk on $D_1(c)$ of the radius $O(\sqrt{|c|})$, $\eta \leq \eta(c)$, with the center at $(0, 0)$. The image of the segment $v = 0$ on $\Pi^u(c)$ under the action of the map $T_1(c)$ is a smooth curve passing at the distance of the order c^l, $l \geq 1$, from the point $(0, 0)$ [the case when this curve passes through the point $(0, 0)$ is not excluded]. Therefore, the circle of the radius $\eta = \eta(c) \sim |c|$ intersects this curve at two points; i.e., parts of this curve lying outside of the circle are two smooth segments and their images with respect to $T(c)$ are two finite spirals that intersect transversely the $T_2(c)$-pre-image of the segment $u = 0$ from $\Pi^s(c)$. Thus, we get a finite number of transverse homoclinic orbits for γ_c. Obviously, the less $|c|$, the more number of transverse homoclinic orbits can be found. □

To prove the existence of elliptic points in some neighborhood of the connection in the whole M, we first find a countable set of values $c_n < 0$ for which the system in the level V_{c_n} has a non-transverse homoclinic orbit with quadratic tangency for periodic orbit γ_{c_n}. This allows one to apply results on the existence of cascades of elliptic periodic orbits on the levels close to V_{c_n} (see, for instance, Refs. 8 and 11).

Theorem 6: *Suppose type 1 connection is considered and an inequality $a_1'(0) \neq 0$ holds. There is a sequence $c_n \to -0$ such that in the level V_{c_n}, periodic orbit γ_{c_n} possesses a homoclinic orbit along which stable and unstable manifolds $W^s(\gamma_{c_n})$, $W^u(\gamma_{c_n})$ have a quadratic tangency. For every such c_n, there exists a countable set of c-intervals I_{nm}, $I_{nm} \to c_n$ as $m \to \infty$, whose values $c \in I_{nm}$ represent levels where the system has a one-round elliptic periodic orbit in this level of the Hamiltonian.*

Proof. The inequality $a_1'(0) \neq 0$ is of the general position condition. It is the analog of the condition C in Ref. 11. This guarantees that for $|c|$ small enough, the $T_2(c)$-image of the point $(0,0)$ in $\Pi^s(c)$ is an analytic curve intersecting transversely stable manifold $u = 0$ of the saddle fixed point. To be precise, we remind that we assume coordinates (u, v) not depending on c but only maps do. As a corollary of this inequality, by reversibility, there is a $c_0 < 0$ such that for $c \in (c_0, 0)$, the $T_1(c)$-image of the segment $v = 0$ is a smooth curve that does not pass through the point $(0,0)$ on $D_1(c)$ and the distance from $(0,0)$ to this curve is of the order $|c|$.

To prove the first assertion of the theorem, we consider level V_c for small negative c and find the image of the segment $v = 0$ from $\Pi^u(c)$ under the map $T(c) \circ T_1(c)$. This is an analytic curve in $D_2(c)$. One needs to show that this curve for a countable set of c-values touches the $T_2(c)$-pre-image of the segment $u = 0$ from $\Pi^s(c)$.

Let us write down the representation of the map $T(c)$. It is similar to (5), but for $c < 0$, function $\Delta_c(\eta)$ is analytic and has the form

$$\Delta_c(\eta) = a_c'(\eta) \ln \frac{d^2}{a_c(\eta)}, \; a_c(\eta) = -c + \omega\eta + O_2(c, \eta) > 0.$$

The positivity of the function $a_c(\eta)$ implies the map $T(c)$ be a local analytic symplectic diffeomorphism in some neighborhood of the point $(x_2, y_2) = (0,0)$ for all sufficiently small in modulus negative c.

The map $T_1(c)$ is also analytic; hence, the $T_1(c)$-image of the segment $v = 0, |u - u_-| \leq \varepsilon$ is an analytic curvilinear segment in $D_1(c)$ passing near point $(x_2, y_2) = (0,0)$ at the distance of the order $|c|$. This follows from the genericity assumption $a_1'(0) \neq 0$ and symmetry of $T_1(c)$ and $T_2(c)$. By symmetry, the $T_2(c)$-pre-image of the segment $u = 0, |v - v_+| \leq \varepsilon$ is also an analytic curvilinear segment in $D_2(c)$ being symmetric with respect to L to the segment in $D_1(c)$ and passing near the point $(\bar{x}_2, \bar{y}_2) = (0,0)$ at the same distance of the order $|c|$.

In polar coordinates on disks $D_1(c)$, $D_2(c)$, the map T_c has the form

$$\overline{\eta} = \eta, \quad \overline{\theta} = \varphi + \Delta_c(\eta),$$

with $\Delta_c(\eta) = (\omega + \cdots) \ln[d^2/(-c + \omega\eta + \cdots)]$.

Expanding in formulas for $T_1(c)$ coefficients by the Taylor formula up to the terms of the first order in c, we get $a(c) = ac + \cdots$, $a \neq 0$, $b(c) = bc + \cdots$. Then, one has

$$\Delta_c(\eta) = -\omega \ln \frac{d^2}{-c + \omega[(ac + \alpha(u - u_-))^2 + (bc + \gamma(u - u_-))^2]/2} + O_2(c, \eta).$$

On the disk $D_1(c)$, the curvilinear segment under consideration is an analytic smooth curve at the distance of the order $|c|$ from $(0,0)$; therefore, there is a circle $\eta = \eta_c$ such that this circle and the curve have a common point and they are tangent at this point. In principle, this point can be not unique. Other points of this curve are outside of this circle.

Local map $T(c)$ preserves η; hence, the $T(c)$-image on $D_2(c)$ of the curve is a spiral-shape curve that lies outside of the circle $\eta = \eta_c$ on $D_2(c)$. By symmetry, on the same circle on $D_2(c)$, there are other points of tangency with the curve being $T_2(c)$-pre-image of the segment $u = 0$ from $\Pi^s(c)$. An important observation is the following assertion.

Lemma 2: *For c small enough, the only point of tangency is the circle and the curve on $D_1(c)$ exists. The tangency at this point is quadratic.*

Proof. Denote σ_c^s, σ_c^u circles $\eta = \eta_c$ on $D_1(c)$, $D_2(c)$, respectively. By symmetry, it is sufficient to prove the assertion for the closed curve $T_2(c)(\sigma_c^u)$; i.e., this curve is quadratically tangent to $u = 0$ at exactly one point as $|c|$ small enough. The circle σ_c^u has the representation in polar coordinates $\bar{x}_2 = \sqrt{2\eta_c} \cos\theta$, $\bar{y}_2 = \sqrt{2\eta_c} \sin\theta$, $r(c) = \sqrt{2\eta_c} \sim |c|$. Thus, its $T_2(c)$-image is (4),

$$u_1 = a_1(c) + r(c)[\gamma(c)\cos\theta + \alpha(c)\sin\theta + O(r)],$$
$$v_1 - v_+ = b_1(c) - r(c)\,[\delta(c)\cos\theta + \beta(c)\sin\theta + O(r)].$$

First we find the points where a tangent to this curve is collinear with the vector $(0,1)$; i.e., $u_1'(\theta) = 0$. This gives the equation $-\gamma \sin\theta + \alpha\cos\theta + O(r) = 0$. It has two roots defined up to $O(r)$ as $\theta_1 = \rho, \theta_2 = \rho + \pi$, where $\sin\rho = \alpha/\sqrt{\alpha^2 + \gamma^2}, \cos\rho = \gamma/\sqrt{\alpha^2 + \gamma^2}$. Equating $u_1(\theta_i) = 0$, we come to the relations relative r: $r(c) = \pm a_1(c)/\sqrt{\alpha^2 + \gamma^2} + O(c^2)$, where the sign is determined by that θ_i for which $r(c) > 0$. Due to assumption $a_1'(0) \neq 0$, we get a unique root providing the tangency of an even order.

To prove that the tangency is quadratic, one needs to check that for c small enough, the derivative $u_1''(v_1) \neq 0$ at the tangency point. This derivative in the parametric form is given as (we omit subscript 1 in this calculation)

$$u''(v) = \frac{u_\theta'' v_\theta' - u_\theta' v_\theta''}{v_\theta'^3} = \sqrt{\alpha^2 + \gamma^2} \frac{[-(\alpha\delta - \beta\gamma) + O(r)]}{\pm r(1 + O(r))}$$
$$= \pm\sqrt{\alpha^2 + \gamma^2}\, r(c)^{-1}(1 + O(r(c))).$$

Since we saw that $r(c) \sim |c|$ as $|c| \to 0$, this derivative is as larger in modulus as smaller $|c|$ is. Thus, we conclude that the tangency is quadratic. □

Therefore, we have on $D_2(c)$ two analytic curves: a spiral and a curve; both touch the circle $\eta = \eta_c$ at only points (generally speaking, different ones). Now, let us follow a mutual position of these two points on the circle as $c \to -0$. The point on the curve tends to the point $(0,0)$ as $c \to -0$ with a definite tangent. However, the tangency point of the spiral, as we shall prove below, rotates monotonically as $c \to -0$ performing infinitely many full revolutions in the angle. This implies that the point of tangency for the spiral infinitely many times c_n passes through the point of tangency for the curve giving quadratic tangency of the spiral and the curve (see Fig. 9).

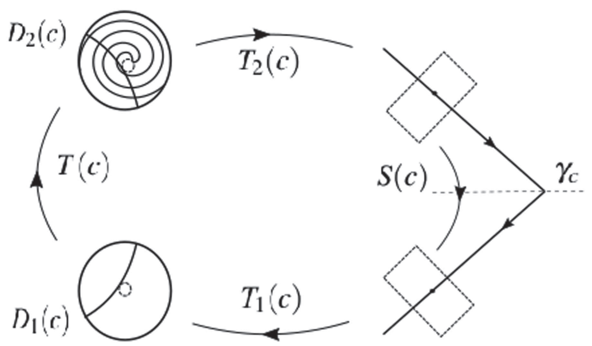

FIG. 9. Case 1: Poincaré map as $c < 0$, a mechanism of homoclinic tangency.

Let us call the unique point of tangency of the circle and the spiral on $D_2(c)$ a nose of the spiral. Near this point, due to a quadratic tangency, the spiral is located out of the disk bounded by the circle. Let us show that the nose of the spiral moves monotonically in θ as $c \to -0$.

The coordinates of the nose correspond to that point of the segment $v = 0$ where the $T_1(c)$-pre-image of the circle $\eta = \eta_c$ in $D_1(c)$ touches the segment. The angle $\theta(c)$ corresponding to the nose of the spiral is calculated using the formula $\theta(c) = \varphi(c) + \Delta_c(\eta_c)$ where the values $(\varphi(c), \eta(c))$ have to be inserted. As we saw when proving Lemma 2, the angle $\varphi(c)$ has a definite limit as $c \to -0$ since the point of tangency of the curvilinear segment and the circle $\eta = \eta_c$ on $D_1(c)$ and the point of tangency of the circle $\eta = \eta_c$ and a curvilinear segment on $D_2(c)$ are connected by the symmetry relation $L : (x_2, y_2) \to (\bar{x}_2, \bar{y}_2)$ $\bar{x}_2 = -x_2, \bar{y}_2 = y_2$. We shall show that the value $\Delta_c(\eta_c)$ tends monotonically to infinity as $c \to -0$. If so, this gives a necessary conclusion on the infinite number of full revolutions in the angle θ.

The value η_c at the tangency point is equal to

$$\eta_c = \frac{1}{2}(x_2^2(c) + y_2^2(c)) = r^2(c)/2 = \frac{a_1^2(c)}{2(\alpha^2(c) + \gamma^2(c))} + O(c^3). \tag{7}$$

Therefore, we have

$$\Delta_c(\eta_c) = (\omega + O(c^2)) \ln \frac{d^2}{-c + \omega\frac{1}{2(\alpha^2+\gamma^2)}a_1^2(c) + O(c^3)} \sim -\ln(-c).$$

Thus, $\theta(c)$ depends on c monotonically and increases unboundedly as $c \to -0$. Therefore, the point in $D_2(c)$, being the nose of the spiral, infinitely many times c_n coincides with the point on the same circle where the pre-image of the segment $u = 0$ from $\Pi^s(c)$ touches the circle.

The second assertion of the theorem follows from the theorem on the existence of elliptic points near a homoclinic tangency for a symplectic map (see, for instance, Refs. 8 and 11).

Theorem 7: *Let f be a smooth (at least C^4) symplectic map having a saddle fixed point p and a homoclinic orbit $f^n(q)$ through the point $q \neq p$. Suppose stable and unstable curves of p are quadratically tangent at q. Then, for any generic smooth one-parametric family of smooth symplectic maps f_μ that coincides as $\mu = 0$ with f on any segment $[-\mu_0, \mu_0]$, there is an integer $k_0 \in \mathbb{Z}$ and infinitely many open intervals I_k, $k \geq k_0$, such that $I_k \to 0$, as $k \to \infty$, and the map f_μ, $\mu \in I_k$ has a one-round elliptic homoclinic orbit (of the period $q + k$).* □

X. TYPE 1 CONNECTION: HYPERBOLICITY AND ELLIPTICITY AS $c > 0$

As we know, any level V_c for $c > 0$ small enough contains a Lyapunov saddle periodic orbit l_c (remind that we assume $\omega > 0$). Its local stable $W^s(l_c)$ and unstable $W^u(l_c)$ manifolds belong to V_c and their extension by the flow occurs near $W^s(p)$ and $W^u(p)$, respectively. Therefore, they intersect cross sections $D_1(c)$ and $D_2(c)$ along the circles $\sigma_s(c)$, $\sigma_u(c)$.

As was indicated above, all flow orbits cutting $D_1(c)$ inside the circle $\sigma_s(c)$ go out from U not intersecting $D_2(c)$. That is why, we do not track for these orbits. However, flow orbits cutting $D_1(c)$ outside of $\sigma_s(c)$, as time increases, do intersect $D_2(c)$ and further the cross section Σ_c. We construct here a hyperbolic set that is formed near a heteroclinic connection that involves a pair of saddle periodic orbits l_c, γ_c and four transverse heteroclinic orbits, of which two go, as time increases, from l_c to γ_c (near Γ_2) and two others from γ_c to l_c (near Γ_1). Besides, there is a countable set of transverse homoclinic orbits for every periodic orbits l_c and γ_c. All of this is the basis for constructing the hyperbolic set.

Theorem 8: *For type 1 connection and $c > 0$ small enough, $W^s(l_c)$ and $W^u(\gamma_c)$ intersect transversely each other along two heteroclinic orbits $\Gamma_{11}(c)$, $\Gamma_{12}(c)$, and, by symmetry, $W^u(l_c)$ and $W^s(\gamma_c)$ intersect transversely each other along two heteroclinic orbits $\Gamma_{21}(c) = L(\Gamma_{11}(c))$, $\Gamma_{22}(c) = L(\Gamma_{12}(c))$, forming thereby a transverse heteroclinic connection.*

The trace of $W^u(\gamma_c)$ in $\Pi^s(c)$ [the image of the segment $v = 0$ under the action of the map $T_2(c) \circ T(c) \circ T_1(c)$] consists of a pair of spiral-shape analytic curves that wind both on the closed curve $T_2(c)(\sigma_u(c))$ and intersect transversely the segment $u = 0$ at a countable set of points being traces of transverse Poincaré homoclinic orbits of the periodic orbit γ_c.

By Smale's λ-lemma, for n large enough, the closed curve $T_2(c)(\sigma_u(c))$ contains two of its segments with end points on the segment $u = 0$ whose n-iterations under map $S(c)$ give a countable family of analytic curves smoothly accumulating, as $n \to \infty$, to the segment $v = 0$. There is an integer $n_0(s)$ such that for $n > n_0(s)$, these curves transversely intersect the closed curve $T_1^{-1}(c)(\sigma_s(c))$ giving a countable set of points being traces of transverse homoclinic orbits for l_c (see Fig. 10).

Proof. Consider first the $T_2(c)$-image of the circle $\sigma_u(c)$ on the disk Σ_c. Recall that the radius of the circle $\sigma_u(c)$ is of the order $\sqrt{c}$ since it is defined by the root of the equation $a_c(\eta) = -c + \omega\eta + O_2(c, \eta) = 0$. Hence, we get $\eta(c) = c/\omega + O(c^2)$. On the other hand, due to an analytic dependence of $T_2(c)$ in c, the $T_2(c)$-image of the center $(0, 0)$ analytically depends on c. Therefore, the distance from the point $(0, 0)$ to the curve being the $T_2(c)$-pre-image of the segment $u = 0$ in $\Pi^s(c)$ has the order c^l, $l \geq 1$. Moreover, $l = 1$ if the inequality $a_1'(0) \neq 0$ holds (see above). This implies that the curve $T_2(c)(\sigma_u(c))$ intersects, for c small enough, segment $u = 0$ transversely at two points. Indeed, the curve $T_2(c)(\sigma_u(c))$ can be written

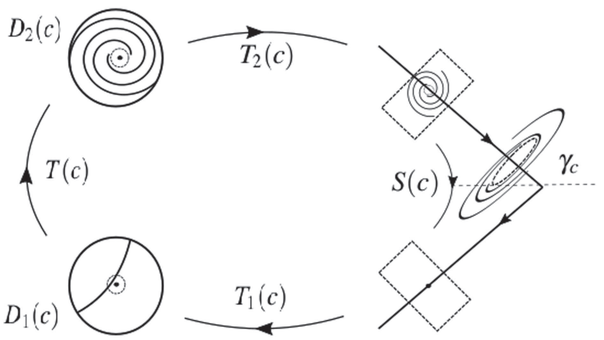

FIG. 10. Poincaré map as $c > 0$, hyperbolic set and tangency.

in a parametric form with parameter θ as

$$u = a_1(c) + r(c)[\gamma(c)\cos\theta + \alpha(c)\sin\theta + O(r)],$$
$$v - v_+ = b_1(c) - r(c)[\delta(c)\cos\theta + \beta(c)\sin\theta + O(r)],$$

where $r(c) = \sqrt{2\eta(c)} = \sqrt{c/\omega + O(c^2)}$. Equating $u_1 = 0$ to find intersection points with segment $u = 0$ and dividing both sides at $r(c)\sqrt{\alpha^2 + \gamma^2}$, we come to the equation with respect to θ,

$$A(c) + \cos(\theta - \rho) + O(c) = 0,\ A(c) = a_1(c)/r(c) \sim \sqrt{c},$$

which has two simple roots for c small enough. These simple roots correspond to two transverse intersection points. The flow orbits through these points are just $\Gamma_{21}(c)$, $\Gamma_{22}(c)$. By the reversibility of the map, the closed curve $T_1^{-1}(c)(\sigma_s(c))$ intersects transversely at two points of the segment $v = 0$ in $\Pi^u(c)$ as well. The flow orbits through these intersection points are $\Gamma_{11}(c)$, $\Gamma_{12}(c)$.

Now, consider the curvilinear segment being the $T_1(c)$-image of $v = 0$ in $D_1(c)$. As was proved, this curve intersects transversely at two points of the circle $\sigma_s(c)$ that divide the curve into three pieces. The flow orbits, passing through the middle piece, leave the neighborhood of the connection, but two remaining pieces give two analytic curves whose $T(c)$-images are two infinite spirals on $D_2(c)$, which wind up the circle $\sigma_u(c)$. Their $T_2(c)$-images give two countable families of transverse homoclinic orbits for γ_c.

In order to find transverse homoclinic orbits to l_c, we remark that the segment in $\Pi^s(c)$ given as $u = \kappa > 0$ for κ small enough intersects the closed curve $T_2(c)(\sigma_u(c))$ transversely at two points. The same holds true for all pieces of $T_2(c)$-images of both spirals winding up at $\sigma_u(c)$ in $D_2(c)$. Thus, we have two countable families of curvilinear segments smoothly accumulating to two segments of the curve $T_2(c)(\sigma_u(c))$. By Smale's λ-lemma,[40] there is an integer $n_0 > 0$ such that all $S^n(c)$-images of curves of both countable families intersect transversely the closed curve $T^{-1}(c)(\sigma_s(c))$ in $\Pi^u(c)$. Thus, we have an invariant hyperbolic set in each level V_c, $c > 0$ small enough. □

The hyperbolic set, we have constructed, does not exhaust invariant sets in the level V_c. One can mention the wild hyperbolic sets existing near the quadratic homoclinic tangencies.[9] Here, we only prove the existence of elliptic periodic orbits for intervals of c-values accumulating at $c = 0$. To that end, we first prove the following.

Theorem 9: *For type 1 connection, there is a sequence of $c_n \to 0$ such that in the level V_{c_n}, the Lyapunov saddle periodic orbit l_{c_n} possesses a homoclinic orbit along which $W^s(l_{c_n})$ and $W^u(l_{c_n})$ have quadratic tangency.*

Proof. As the system is reversible and the related Poincaré map is also reversible, it is sufficient to prove that there exist c_n such that the trace on Σ_{c_n} of $W^u(l_{c_n})$ is a convex closed curve being quadratically tangent to the line of $Fix(L)$—the diagonal $u = v$.

As was proved above, for c_0 sufficiently small as $0 < c \le c_0$, the trace of $W^u(l_c)$ in $\Pi^s(c)$ is a closed curve that intersects the segment $u = 0$ at two points. The $S(c)$-pre-images of the line $u = v$ are a sequence of analytic curvilinear segments, given as $u = u_k(v, c)$, which tend in $\Pi^s(c)$ to $u = 0$ in C^2-topology uniformly with respect to c, as $k \to \infty$. Indeed, the inverse iterations of S are given as $u_{-n} = f^n(\zeta)u_{-n+1}$, $v_{-n} = v_{-n+1}/f^n(\zeta)$, $\zeta = u_0 v_0 = u_{-1}v_{-1} = \cdots = u_{-n}v_{-n}$. The function $u_{-n} = g_n(v_{-n})$ is found as a solution with respect to u of the equation $u = vf^{2n}(uv)$. Multiplying both sides on v, we get the equation $\zeta = v^2 f^{2n}(\zeta)$. Due to the form of $f(\zeta) = v + O_1(\zeta)$, we have an estimate $|f| \le (1 + v)/2 < 1$ for sufficiently small $|\zeta|$. Thus, for any fixed v, $|v - v_+| \le \delta$, we find the unique solution of the equation $\zeta_n(v)$. This function is analytic and family tends uniformly to zero as $n \to \infty$. This gives functions $u_n(v, c) = vf^{2n}(\zeta_n(v))$. This family tends to zero as $[(1 + v)/2]^{2n}$. Since they approach to zero in C^2-topology (in fact, in any C^k, $k \ge 2$), this implies that for n large enough, the intersection of the graph of function $u_n(v, c)$ with the closed curve $T_2(c)(\sigma_u(c))$ occurs in two points similar as for $u = 0$.

Fix some $0 < c \le c_0$. For c_0 is small enough, the closed curve is, up to third order terms, an ellipse in $\Pi^s(c)$ whose center approach to the line $u = 0$ with the order c and its principal axes have lengths of the order $\sqrt{c}$ and their rotation angle depends as c and has a limit defined by the matrix of the linearized map $T_2(0) = T_2$. This implies this family of closed curve intersects, as $c \to 0$, all graphs of the functions $u_n(v, c)$, and this intersection for the individual curve is either transversal or quadratically tangent or no intersection points at all. Those values of c when the related curve of the family is tangent to a fixed $u_n(v, c)$ give the values c_n we search for. □

Now, we can again apply Theorem 7 on the existence of the elliptic periodic point in a generic one-parameter unfolding of two-dimensional symplectic diffeomorphisms that contain a diffeomorphism with a quadratic homoclinic tangency.[8,11]

XI. TYPE 2 CONNECTION: $c > 0$

As was shown above, for case 2, all orbits in the level V_0 passing through a small neighborhood of the connection, other than those of the connection itself, leave this neighborhood and no orbits exist, which stay forever in this neighborhood. Levels V_c, $c < 0$, contain no orbits at all, which stay wholly in these levels since orbits entering the solid cylinder through $D_1(c)$ exit from the neighborhood of the point p without intersecting $D_2(c)$. That is why we consider levels V_c for $c > 0$, where orbits arise lying wholly in a neighborhood of the connection. On the corresponding transversal disk $D_1(c)$, these orbits enter the solid cylinder through points lying inside the circle $\sigma_s(c)$ and they exit through $D_2(c)$ inside the circle $\sigma_u(c)$ from another solid cylinder (see Remark 6).

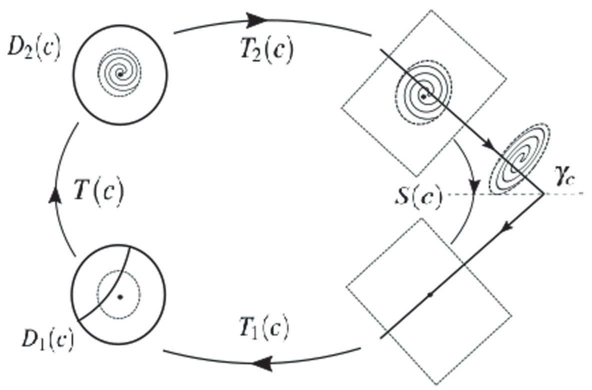

FIG. 11. Case 2: Poincaré map as $c > 0$, hyperbolic set and tangency.

Consider the image with respect to the map $T_2(c) \circ T(c) \circ T_1(c)$ of the trace of the unstable manifold $W^u(\gamma_c)$ [i.e., the segment $v = 0$ from $\Pi^u(c)$]. As was discussed above, generally, $T_1(c)$-image of this segment on the disk $D_1(c)$ is an analytic curvilinear segment whose distance from the center of the disk $(0, 0)$ has the order c^l, $l \geq 1$, due to the analytic dependence of the map $T_1(c)$ in c. If the genericity assumption above $a_1'(0)$ holds, then $l = 1$. On the disk $D_1(c)$, there is a circle $\sigma_s(c)$ defined as $\eta = \eta(c) = c/\omega + O(c^2)$, being the trace of the stable manifold $W^s(l_c)$. Thus, its radius is of the order $\sim \sqrt{c}$. This implies, as above, that $\sigma_s(c)$ and the curvilinear segment [trace of $W^s(\gamma_c)$] intersect each other transversely at two points for c small enough.

Consider now the interval of the curvilinear segment that lies on $D_1(c)$ inside of the circle $\sigma_s(c)$. Keeping in mind the modification of the formula for $\Delta_c(\eta)$ (see Remark 6), we see that this interval [without its two extreme points on the circle $\sigma_s(c)$] is transformed by the map $T(c)$ on $D_2(c)$ where it forms an infinite spiraling analytic curve that winds up by its both ends on the circle $\sigma_u(c)$ (see Fig. 11). On the same disk $D_2(c)$, there is an analytic curvilinear segment being the $T_2(c)$-pre-image of the segment $u = 0$ from $\Pi^s(c)$. The curvilinear segment intersects transversely the circle $\sigma_u(c)$, and this follows from its symmetry with the related curve in $D_1(c)$. Since the double spiral winds up by its both ends on the circle $\sigma_s(c)$ and the segment is transverse to the circle, we get, as above, a countable set of intersection points through which transverse homoclinic orbits of γ_c pass.

Here, we also have a countable set of intervals of c on which elliptic periodic orbits exist in V_c. Their proof is done by exactly the same manner as for the case 1 and $c > 0$. The crucial point here is again to find a sequence of $c_n \to 0$ such that in V_{c_n}, a tangent symmetric homoclinic orbit of l_c exists. We again iterate by the maps $S^n(c)$ on the disk $\Sigma(c)$ the closed curve $T_2(c)(\sigma_u(c))$ and find its tangency with the line $u = v$ of the trace $Fix(L)$. The consideration is the same as in Sec. X. Thus, we obtain

Theorem 10: *For case 2, there exists $c_0 > 0$ small enough such that on the interval $(0, c_0)$, a countable set of intervals exists whose values of c correspond to levels V_c containing a one-round elliptic periodic orbit in a four-dimensional neighborhood of the initial heteroclinic connection.*

XII. ONE-PARAMETER FAMILY OF REVERSIBLE HAMILTONIAN SYSTEMS: HOMOCLINICS OF THE SADDLE-CENTER

We consider in this section a generic 1-parameter family of reversible Hamiltonian systems X_{H_μ} being an unfolding of a system that has at $\mu = 0$ a heteroclinic connection studied in Secs. I–XI. The main result here is a theorem on the existence of a countable set of parameter values μ accumulating to $\mu = 0$ for which the related system has a homoclinic orbit of the saddle-center. Here, also, it will be shown that emerging homoclinic orbits of the saddle-center satisfy the general position conditions found in Refs. 20 and 23. These conditions guarantee the existence of complicated dynamics in the system and its non-integrability.[21] It is worth emphasizing that the result does not depend on what type of the connection is, the first or second one.

Recall that in the class of C^2-smooth Hamiltonian systems, the existence of a saddle-center equilibrium is a generic phenomenon. The existence of a saddle periodic orbit is also a generic phenomenon. However, for general reversible perturbations, heteroclinic orbits joining a saddle-center and a saddle-periodic orbit can be destroyed. We want to prove that generically in such a family, homoclinic orbits of a perturbed saddle-center emerge. The theorem we prove can serve as a criterion of the existence of homoclinic orbits of the saddle-center. It is worth remarking that finding homoclinic orbits of a saddle-center is a rather delicate problem generally. Such a theorem can also be useful in the case when one deals with a two-parameter family of reversible Hamiltonian systems where for some specific values of parameters, the related system has a heteroclinic connection studied above. Then, we can find a countable set of curves in the parameter space such that for parameters on this curve, the system has a homoclinic orbit of the saddle-center.

Theorem 11: *Let X_{H_μ} be a generic one-parameter family of reversible analytic Hamiltonian systems, and at $\mu = 0$, the system has a heteroclinic connection of type 1 or 2. Then, there exists a sequence of parameter values μ_n accumulating to $\mu = 0$ such that the related system of the family has a homoclinic orbit of the general type of a saddle-center. Related values μ_n have the same sign.*

Proof. Because of the reversibility, it is sufficient to prove the existence of a sequence $\mu_n \to 0$ for which the Hamiltonian system $X_{H_{\mu_n}}$ has a symmetric unstable separatrix of the saddle-center intersecting the cross section $\Sigma(\mu_n)$ at a point on the line $Fix(L)$.

For the case of a smooth one-parameter family of reversible analytic Hamiltonian systems being an unfolding of a system with a heteroclinic connection, all objects under consideration: a saddle-center, a periodic orbit in the singular level of the Hamiltonian, their stable and unstable manifolds smoothly depend on μ. As was indicated above, Moser theorems hold also for systems depending on parameters; therefore, there is a smooth one depending on a parameter change of variables such that in new coordinates (x_1, y_1, x_2, y_2) in a neighborhood of equilibrium p_μ, the Hamiltonian has the form of the analytic functions h_μ in variables $\xi = x_1y_1$, $\eta = (x_2^2 + y_2^2)/2$. The difference with the parameterless case is a smooth dependence of coefficients of the function h_μ on the parameter μ. We work further in these coordinates; thus, cross sections D_1 and D_2 to separatrices in the singular level of the Hamiltonian (i.e., containing a saddle-center) and N to the saddle periodic orbit $\gamma(\mu)$ can be regarded fixed and not depending on the parameter. However, in global maps, zero order terms smoothly depending on μ do appear, as separatrices of the perturbed saddle-center do not lie, generally speaking, on manifolds of the saddle periodic orbit.

The form of the perturbed maps is as follows:

$$T_1(\mu): \ x_2 = b(\mu) + \alpha(\mu)(u - u_-) + \beta(\mu)v + \cdots,$$
$$y_2 = a(\mu) + \gamma(\mu)(u - u_-) + \delta(\mu)v + \cdots,$$
$$T_2(\mu): \ u_1 = a(\mu) + \gamma(\mu)\bar{x}_2 + \alpha(\mu)\bar{y}_2 + \cdots,$$
$$v_1 - v_+ = b(\mu) - \delta(\mu)\bar{x}_2 - \beta(\mu)\bar{y}_2 + \cdots,$$
$$S(\mu): \ u_1 = u/f_\mu(\zeta), \ v_1 = vf_\mu(\zeta), \ \zeta = uv, \ f_\mu = \nu(\mu) + O_1(\zeta).$$

In coordinates used, the genericity condition for the family means the inequality $a'(0) \neq 0$ to hold. In virtue of the assumptions on the family, we have $a(0) = 0$, $b(0) = 0$, $\nu(0) = \nu < 1$. Geometrically, the genericity condition means that for $\mu \neq 0$, the trace of the unstable separatrix of the saddle-center intersects the trace of the stable manifold of the saddle periodic orbit transversely, as μ varies. More spectacularly, this can be seen in the space (u, v, μ), where the segment $(0, 0, \mu)$ represents the one-parameter family of saddle fixed points of the maps, the rectangular $u = 0$ represents the union of traces of stable manifolds for fixed points, and $v = 0$ corresponds to the union of traces of unstable manifolds for fixed points. The curve of traces on the cross sections of unstable separatrices of the saddle-centers for every small μ intersects transversely the rectangular $u = 0$ at that unique point where $\mu = 0$.

Consider now on the disk D_2 the point $(x_2, y_2) = (0, 0)$ being the trace of the unstable separatrix of the perturbed saddle-center. This point under the action of the map $S^n(\mu) \circ T_2(\mu)$ transforms into $u_n = a(\mu)/f^n_\mu(\zeta)$, $\quad v_n = (v_+ + b(\mu))f^n_\mu(\zeta)$. The condition this point belongs to $Fix(L)$ gives the equality $u_n = v_n$; i.e., $a(\mu)/f^n_\mu(\zeta) = (v_+ + b(\mu))f^n_\mu(\zeta)$, $\zeta = a(\mu)(v_+ + b(\mu))$. This equation for searching μ is written down in the form

$$\frac{a(\mu)}{v_+ + b(\mu)} = f^{2n}_\mu(a(\mu)(v_+ + b(\mu))) > 0.$$

Function $r(\mu)$ in the left side is defined on some neighborhood of $\mu = 0$, $r(0) = 0$, $r'(0) = a'(0)/v_+ \neq 0$; therefore, this function is strictly monotone. The smooth functions of the sequence in the right hand side are defined for any μ in a fixed small neighborhood of $\mu = 0$, and they tend uniformly to zero, as $n \to \infty$. Thus, for n large enough, for values of μ, where $a(\mu)$ is positive, the equation for every such n has a unique solution $\mu_n \to 0$ as $n \to \infty$. Asymptotics of values μ_n, for which homoclinic orbits of the saddle-center exist, is as follows:

$$\mu_n = \frac{\nu^{2n} v_+}{a'(0)}.$$

Now, suppose an analytic Hamiltonian system with two degrees of freedom has a saddle-center with a homoclinic orbit to it. In Ref. 23, it was proved that if some genericity condition holds for this homoclinic orbit, any small Lyapunov saddle periodic orbit on the center manifold of the saddle-center has in the related level of the Hamiltonian four transverse homoclinic orbits and, therefore, the system possesses a chaotic behavior. In Ref. 15, the genericity condition was formulated for some particular class of systems as an inequality on the coefficients of the scattering matrix of an auxiliary linear scattering problem. Later,[19] this genericity condition was reformulated as some genericity property of a scattering map derived for the linearized system at the homoclinic solution. It becomes applicable to any smooth Hamiltonian system, not only an analytic one. This allowed us[20] to extend these results onto multidimensional Hamiltonian systems with a homoclinic orbit of an equilibrium with one pair of simple pure imaginary eigenvalues and any number of pairs of other eigenvalues with nonzero real parts.

Let us show now that for found values μ_n, corresponding symmetric homoclinic orbits of the saddle-center satisfy the genericity condition from Ref. 23. In Moser coordinates in a neighborhood of the saddle-center, this condition means that on the singular level of the Hamiltonian, the linearization matrix of the global map calculated at the trace of the homoclinic orbit differs from a rotation matrix.

At $\mu = \mu_n$, the global map is defined in a neighborhood of the related homoclinic orbit. This map is the composition of maps $T_1(\mu) \circ S^{2n}(\mu) \circ T_2(\mu)$, and it can be written as $[T_1(\mu) \circ S^n(\mu)] \circ S^n(\mu) \circ T_2(\mu)$ and, in view of the reversibility and symmetry of cross sections, with $T_1(\mu) = L \circ T_2^{-1}(\mu) \circ L$, $S^n(\mu) = L \circ S^{-n}(\mu) \circ L$.

Zero order terms when expanding the map $T_1(\mu) \circ S^{2n}(\mu) \circ T_2(\mu)$ in $(\bar{x}_2, \bar{y}_2) = (0, 0)$ are equal to zero since we calculate the map at the trace $(0, 0)$ of a homoclinic orbit. The trace of the homoclinic orbit on D_1 is $(x_2, y_2) = (0, 0)$. Thus, the global map can be written as follows:

$$x_2 = A\bar{x}_2 + B\bar{y}_2 + \cdots, \qquad y_2 = C\bar{x}_2 + D\bar{y}_2 + \cdots, \ R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Entries A, B, C, D of the matrix R depend, of course, on n, but we omit this in order not to complicate notations. Since maps $T_1(\mu), S^{2n}(\mu), T_2(\mu)$ are symplectic, their composition is also a symplectic map. In coordinates we use, the symplecticity means the area preservation. This implies the equality $AD - BC = 1$.

Similar to (4), we denote

$$T_2(\mu): \ u = a(\mu) + \alpha(\mu)\bar{x}_2 + \beta(\mu)\bar{y}_2 + \cdots,$$
$$v - v_+ = b(\mu) + \gamma(\mu)\bar{x}_2 + \delta(\mu)\bar{y}_2 + \cdots, \tag{8}$$

$$S^n(\mu): u_n = u/f^n(\zeta, \mu), \ u_n = vf^n(\zeta, \mu), \ f(\zeta, \mu) = \nu(\mu) + O(\zeta). \tag{9}$$

The involution action on $\Sigma(\mu)$ displays as $L(u, v) = (v, u)$ and on D_1, D_2 as $L(\bar{x}_2, \bar{y}_2) = (-x_2, y_2)$. Therefore, we get for $T_1(\mu)$,

$$T_1(\mu): \begin{cases} x_2 = \delta(\mu)a(\mu) - \beta(\mu)b(\mu) + \beta(\mu)(u - u_-) - \delta(\mu)v + \cdots, \\ y_2 = \gamma(\mu)a(\mu) - \alpha(\mu)b(\mu) + \alpha(\mu)(u - u_-) - \gamma(\mu)v + \cdots. \end{cases}$$

We have to show that matrix R, being the linearization matrix for $T_1(\mu)\circ S^{2n}(\mu)\circ T_2(\mu)$ at $(0,0)$, is not a rotation matrix. The following assertion has been proved, in fact, in Ref. [20]. $\square$

Lemma 3: *Let a standard symplectic plane $\mathbb{R}^2$ with coordinates (x,y) and two-form $dx\wedge dy$ be given and R be a symplectic matrix $R^*IR=I$, where R^* is a transpose matrix and*

$$I=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ R=\begin{pmatrix}A & B\\ C & D\end{pmatrix},\ \begin{pmatrix}x\\ y\end{pmatrix}\to\begin{pmatrix}Ax+By\\ Cx+Dy\end{pmatrix}.$$

Then, R is not a rotation matrix, iff the ellipse, being R-image of the circle $S^1:\{x^2+y^2=1\}$, intersects S^1 at four points.

Proof. Denote $X=(A,C)^*$ and $Y=(B,D)^*$ as two vectors on the plane and set $Y^\perp=(D,-B)^*$. The R-image of the unit circle $x^2+y^2=1$ in symplectic polar coordinates $x=\sqrt{2\rho}\cos\theta$, $y=\sqrt{2\rho}\sin\theta$ is given as $x=A\cos\theta+B\sin\theta$, $x=C\cos\theta+D\sin\theta$. Since R is symplectic (here, it preserves area $dx\wedge dy$), these two curves have the same area and common center; thus, these two curves either have four intersection points or they coincide. In the former case, their intersection points are defined by simple zeroes of the equation

$$1=(A\cos\theta+B\sin\theta)^2+(C\cos\theta+D\sin\theta)^2$$

or via X, Y,

$$2X\cdot Y\sin 2\theta+(X^2-Y^2)\cos 2\theta=2-(X^2+Y^2).$$

This equation has four simple roots if the inequality holds,

$$4(X\cdot Y)^2+(X^2-Y^2)^2>(2-(X^2+Y^2))^2.$$

Because of equality $X\cdot Y^\perp=AD-BC=1$, the identity holds $(X\cdot Y)^2-X^2Y^2=-1$. Thus, we come to the valid inequality if the equality does not hold,

$$\begin{aligned}4(X\cdot Y)^2+(X^2-Y^2)^2&=4[(X\cdot Y)^2-X^2Y^2]+(X^2+Y^2)^2\\ &=(X^2+Y^2)^2-4>(X^2+Y^2-2)^2.\end{aligned}$$

If the last inequality in this string becomes equality, this is equivalent to $X^2+Y^2-2=0$ or $X=Y^\perp$ since $Y^2=(Y^\perp)^2$ and $X\cdot Y^\perp=1$. However, equality $X=Y^\perp$ provides

$$X^2=(Y^\perp)^2=1;\ \text{hence, } R=\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}. \qquad \square$$

It remains to verify if the inequality holds for the matrix R determined by the global map. As R is a symplectic matrix, it is sufficient to verify inequality $A^2+C^2\neq 1$. Differentiation of the map $T_1(\mu)\circ S^{2n}(\mu)\circ T_2(\mu)$ with account of (9) for $2n$-iteration gives the following expressions for entries of R:

$$A=\beta f^{-2n}\left[\alpha-2nu\frac{f'}{f}(\gamma u+\alpha v)\right]-\delta f^{2n}\left[\gamma+2nv\frac{f'}{f}(\gamma u+\alpha v)\right],$$

$$B=\beta f^{-2n}\left[\beta-2nu\frac{f'}{f}(\delta u+\beta v)\right]-\delta f^{2n}\left[\delta+2nv\frac{f'}{f}(\delta u+\beta v)\right],$$

$$C=\alpha f^{-2n}\left[\alpha-2nu\frac{f'}{f}(\gamma u+\alpha v)\right]-\gamma f^{2n}\left[\gamma+2nv\frac{f'}{f}(\gamma u+\alpha v)\right],$$

$$D=\alpha f^{-2n}\left[\beta-2nu\frac{f'}{f}(\delta u+\beta v)\right]-\gamma f^{2n}\left[\delta+2nv\frac{f'}{f}(\delta u+\beta v)\right].$$

Observe the following limits exist, as $n\to\infty$: $u\to 0$, $v\to v_+$, $f^{2n}\to 0$. Therefore, for the value A^2+C^2, we get

$$\begin{aligned}A^2+C^2=&(\alpha^2+\beta^2)f^{-4n}\left[\alpha-2nu\frac{f'}{f}(\gamma u+\alpha v)\right]^2+(\gamma^2+\delta^2)f^{4n}\left[\gamma+2nv\frac{f'}{f}(\gamma u+\alpha v)\right]^2\\ &-2(\alpha\gamma+\beta\delta)\left[\alpha-2nu\frac{f'}{f}(\gamma u+\alpha v)\right]\left[\gamma+2nv\frac{f'}{f}(\gamma u+\alpha v)\right],\end{aligned}$$

and for B^2+D^2, respectively,

$$\begin{aligned}B^2+D^2=&(\alpha^2+\beta^2)f^{-4n}\left[\beta-2nu\frac{f'}{f}(\delta u+\beta v)\right]^2+(\gamma^2+\delta^2)f^{4n}\left[\delta+2nv\frac{f'}{f}(\delta u+\beta v)\right]^2\\ &-2(\alpha\gamma+\beta\delta)\left[\beta-2nu\frac{f'}{f}(\delta u+\beta v)\right]\left[\delta+2nv\frac{f'}{f}(\delta u+\beta v)\right].\end{aligned}$$

Expressions for A^2+C^2 and B^2+D^2 show that their dependence on n, as $n\to\infty$, is determined by the value of $(\alpha^2+\beta^2)f^{-4n}\alpha$ or $(\alpha^2+\beta^2)f^{-4n}\beta$. Coefficients α and β depend on μ and they approach to values $\alpha(0),\beta(0)$ as $\mu\to 0$. Since the matrix of the linearized map T_2 at the point $(0,0)$ is non-degenerate (it is symplectic), then $\alpha^2+\beta^2\neq 0$ for n large enough, though separately, values $\alpha(\mu)$, $\beta(\mu)$ can vanish. Recall that the initial point for acting map S^{2n} is the point (u,v) with $u=a(\mu)$, $v=v_++b(\mu)$, $a(0)=b(0)=0$. Due to assumption $a'(0)\neq 0$ and exponential decay of the sequence μ_n, the values in square brackets $\alpha-2nu\frac{f'}{f}(\gamma u+\alpha v)$ or $\beta-2nu\frac{f'}{f}(\delta u+\beta v)$ do not vanish for n large enough. This proves the map $T_1\circ S\circ T_2$ to have its linear part different from the rotation matrix.

XIII. CONCLUSION

In this paper, we study some dynamical phenomena in a one-parameter unfolding of a reversible Hamiltonian system, which contains a system with a symmetric heteroclinic connection involving a symmetric saddle-center, a symmetric saddle periodic orbit in the same level of the Hamiltonian, and a pair of heteroclinic orbits joining the saddle-center and periodic orbit and permutable by the reversible involution. We found hyperbolic sets of several types, cascades of elliptic periodic orbits, and countable sets of the unfolding parameter for which homoclinic orbits to the saddle-center exist. All this characterized the chaotic orbit behavior of the related systems.

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DATA AVAILABILITY

The data that support the findings of this study are available within the article. If some extra requirements appear, they should be addressed to the corresponding author.

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