

ITERATED HIGHER WHITEHEAD PRODUCTS IN TOPOLOGY OF MOMENT-ANGLE COMPLEXES

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Abstract: We give an example of a simplicial complex whose corresponding moment-angle complex is homotopy equivalent to a wedge of spheres, but there is a sphere that cannot be realized by any linear combination of iterated higher Whitehead products. Using two explicitly defined operations on simplicial complexes, we prove that there exists a simplicial complex that realizes any given iterated higher Whitehead product. Also we describe the smallest simplicial complex that realizes an iterated product with only two pairs of nested brackets.

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§ 1. Introduction

The moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is a cell complex built from products of polydisks and tori parametrized by simplices in a finite simplicial complex \mathcal{K} . There is torus action on $\mathcal{Z}_{\mathcal{K}}$, which plays an important role in toric topology (see [1]). In the case when \mathcal{K} is a triangulation of a sphere, $\mathcal{Z}_{\mathcal{K}}$ is a topological manifold with rich geometric structure. Moment-angle complexes are particular examples of the homotopy-theoretical construction of *polyhedral products*, which provides a wonderful “testing ground” for application of unstable homotopy theory techniques.

In this paper we study the topological structure of moment-angle complexes $\mathcal{Z}_{\mathcal{K}}$. Interest to higher Whitehead products in homotopy groups of moment-angle complexes and polyhedral products goes back to the work of Panov and Ray [2] where the first structure results were obtained. Some important results about the structure of higher Whitehead products were obtained by Grbić, Theriault [3] and Iriye, Kishimoto [4].

We consider the two classes of simplicial complexes: The first class B_{Δ} consists of simplicial complexes \mathcal{K} for which $\mathcal{Z}_{\mathcal{K}}$ is homotopy equivalent to a wedge of spheres. The second class W_{Δ} consists of $\mathcal{K} \in B_{\Delta}$ such that all spheres in the wedge are realized by iterated higher Whitehead products (see §2). Buchstaber and Panov asked in [1, Problem 8.4.5] whether it is true that $B_{\Delta} = W_{\Delta}$. In this paper we show that this is not so (see §7). Namely, we give an example of a simplicial complex whose corresponding moment-angle complex is homotopy equivalent to a wedge of spheres, but there is a sphere that cannot be realized by any linear combination of iterated higher Whitehead products.

On the other hand we show that W_{Δ} is large enough. Namely, we show that W_{Δ} is closed under two explicitly defined operations on simplicial complexes (see Proposition 5.1 and Theorem 5.2). Then using these operations we prove that there exists a simplicial complex that realizes every given iterated higher Whitehead product (see Theorem 5.3). We also describe the smallest simplicial complex that realizes an iterated product with only two pairs of nested brackets (see Theorem 6.1).

§ 2. Preliminaries

A *simplicial complex* \mathcal{K} on the set $[m] \stackrel{\text{def}}{=} \{1, 2, \dots, m\}$ is a collection of subsets $I \subset [m]$ closed under

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taking any subsets. We refer to $I \in \mathcal{K}$ as *simplices* or *faces* of \mathcal{K} , and always assume that $\emptyset \in \mathcal{K}$. Denote by Δ^{m-1} or $\Delta(1, \dots, m)$ the full simplex on the set $[m]$.

Assume we are given some set of m topological pairs

$$(\underline{X}, \underline{A}) = \{(X_1, A_1), \dots, (X_m, A_m)\},$$

where $A_i \subset X_i$. Given a simplex $I \in \mathcal{K}$, we put

$$(\underline{X}, \underline{A})^I = \{(x_1, \dots, x_m) \in X_1 \times \dots \times X_m \mid x_j \in A_j \text{ for } j \notin I\}.$$

The *polyhedral product* of $(\underline{X}, \underline{A})$ corresponding to \mathcal{K} is the following subset of $X_1 \times \dots \times X_m$:

$$(\underline{X}, \underline{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\underline{X}, \underline{A})^I \quad (\subset X_1 \times \dots \times X_m).$$

In case when all pairs (X_i, A_i) are (D^2, S^1) we use the notation $\mathcal{Z}_{\mathcal{K}}$ for $(\underline{X}, \underline{A})^{\mathcal{K}}$, and refer to $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$ as the *moment-angle complex*. Also, if all pairs (X_i, A_i) are (X, pt) we use the abbreviated notation $X^{\mathcal{K}}$ for $(\underline{X}, \underline{A})^{\mathcal{K}}$.

Theorem 2.1 [1, Chapter 4]. *The moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is the homotopy fiber of the canonical inclusion $(\mathbb{C}P^\infty)^{\mathcal{K}} \hookrightarrow (\mathbb{C}P^\infty)^m$.*

We will also need the following more explicit description of the mapping $\mathcal{Z}_{\mathcal{K}} \rightarrow (\mathbb{C}P^\infty)^{\mathcal{K}}$. Consider the mapping of pairs $(D^2, S^1) \rightarrow (\mathbb{C}P^\infty, pt)$ sending the interior of the disk homeomorphically onto the complement of the basepoint in $\mathbb{C}P^1$. By functoriality, we have the induced mapping of the polyhedral products $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}} \rightarrow (\mathbb{C}P^\infty)^{\mathcal{K}}$.

The general definition of higher Whitehead product can be found in [5]. We only describe Whitehead products in the space $(\mathbb{C}P^\infty)^{\mathcal{K}}$ and their lifts to $\mathcal{Z}_{\mathcal{K}}$. In this case the indeterminacy of higher Whitehead products can be controlled effectively because extension mappings can be chosen canonically.

Let μ_i be the mapping $(D^2, S^1) \rightarrow S^2 \cong \mathbb{C}P^1 \hookrightarrow (\mathbb{C}P^\infty)^{\vee m} \hookrightarrow (\mathbb{C}P^\infty)^{\mathcal{K}}$. Here the second mapping is the canonical inclusion of $\mathbb{C}P^1$ into the i th wedge summand. The third mapping is induced by embedding m disjoint points into \mathcal{K} . The *Whitehead product* (or *Whitehead bracket*) $[\mu_i, \mu_j]$ of μ_i and μ_j is the homotopy class of the mapping

$$S^3 \cong \partial D^4 \cong \partial(D^2 \times D^2) \cong D^2 \times S^1 \cup S^1 \times D^2 \xrightarrow{[\mu_i, \mu_j]} (\mathbb{C}P^\infty)^{\mathcal{K}},$$

where

$$[\mu_i, \mu_j](x, y) = \begin{cases} \mu_i(x) & \text{for } (x, y) \in D^2 \times S^1, \\ \mu_j(y) & \text{for } (x, y) \in S^1 \times D^2. \end{cases}$$

Every Whitehead product becomes trivial after composing with the embedding $(\mathbb{C}P^\infty)^{\mathcal{K}} \hookrightarrow (\mathbb{C}P^\infty)^m \simeq K(\mathbb{Z}^m, 2)$. This implies that the mapping $[\mu_i, \mu_j] : S^3 \rightarrow (\mathbb{C}P^\infty)^{\mathcal{K}}$ has a lift $S^3 \rightarrow \mathcal{Z}_{\mathcal{K}}$; we will use the same notation for it. Such a lift $[\mu_i, \mu_j]$ is given by the inclusion of the subcomplex

$$[\mu_i, \mu_j] : S^3 \cong D^2 \times S^1 \cup S^1 \times D^2 \hookrightarrow \mathcal{Z}_{\mathcal{K}}.$$

If the Whitehead product $[\mu_i, \mu_j]$ is trivial then the mapping $[\mu_i, \mu_j] : S^3 \rightarrow \mathcal{Z}_{\mathcal{K}}$ can be extended canonically to a mapping $D^4 \cong D_i^2 \times D_j^2 \hookrightarrow \mathcal{Z}_{\mathcal{K}}$.

Higher Whitehead products are defined inductively as follows.

Let $\mu_{i_1}, \dots, \mu_{i_n}$ be a collection of mappings such that the $(n-1)$ -fold product $[\mu_{i_1}, \dots, \widehat{\mu}_{i_k}, \dots, \mu_{i_n}]$ is trivial for any k . Then for every $(n-1)$ -fold product there is a *canonical* extension $[\mu_{i_1}, \dots, \widehat{\mu}_{i_k}, \dots, \mu_{i_n}]$ to a mapping from $D^{2(n-1)}$ which is the composition

$$\overline{[\mu_{i_1}, \dots, \widehat{\mu}_{i_k}, \dots, \mu_{i_n}]} : D_{i_1}^2 \times \dots \times D_{i_{k-1}}^2 \times D_{i_{k+1}}^2 \times \dots \times D_{i_n}^2 \hookrightarrow \mathcal{Z}_{\mathcal{K}} \rightarrow (\mathbb{C}P^\infty)^{\mathcal{K}},$$

and all these extensions are compatible on the intersections. The n -fold product $[\mu_{i_1}, \dots, \mu_{i_n}]$ is defined as the homotopy class of the mapping¹⁾

$$S^{2n-1} \cong \partial(D_{i_1}^2 \times \dots \times D_{i_n}^2) \cong \bigcup_{k=1}^n D_{i_1}^2 \times \dots \times S_{i_k}^1 \times \dots \times D_{i_n}^2 \xrightarrow{[\mu_{i_1}, \dots, \mu_{i_n}]} (\mathbb{C}P^\infty)^{\mathcal{Z}_K},$$

which is given as follows:

$$[\mu_{i_1}, \dots, \mu_{i_n}](x_1, \dots, x_n) = \begin{cases} \overline{[\mu_{i_1}, \dots, \mu_{i_{n-1}}]}(x_1, \dots, x_{n-1}) & \text{for } x_n \in S_{i_n}^1, \\ \dots \\ \overline{[\mu_{i_1}, \dots, \widehat{\mu}_{i_k}, \dots, \mu_{i_n}]}(x_1, \dots, \widehat{x}_k, \dots, x_n) & \text{for } x_k \in S_{i_k}^1, \\ \dots \\ \overline{[\mu_{i_2}, \dots, \mu_{i_n}]}(x_2, \dots, x_n) & \text{for } x_1 \in S_{i_1}^1. \end{cases}$$

Alongside with higher Whitehead products of canonical mappings μ_i we will consider *general iterated* higher Whitehead products, i.e. higher Whitehead products whose arguments can be higher Whitehead products. For example, $[\mu_1, \mu_2, [\mu_3, \mu_4, \mu_5], [\mu_6, \mu_{13}, [\mu_7, \mu_8, \mu_9], \mu_{10}], [\mu_{11}, \mu_{12}]]$. In most cases we consider only *nested* iterated higher Whitehead products, i.e. products of the form

$$w = [\mu_{i_{01}}, \dots, \mu_{i_{0p_0}}, [\dots [\mu_{i_{q1}}, \dots, \mu_{i_{qp_q}}]]] : S^{d(w)} \rightarrow (\mathbb{C}P^\infty)^{\mathcal{K}}.$$

Here $d(w)$ denotes the dimension of w .

In the case of ordinary Whitehead products any iterated higher Whitehead product lifts to a mapping $S^{d(w)} \rightarrow \mathcal{Z}_K$ for dimensional reasons.

DEFINITION 2.2. We say that a simplicial complex \mathcal{K} *realizes* a Whitehead product w if \mathcal{Z}_K is homotopy equivalent to a wedge of spheres in which one of the wedge summands is realized by a lift $S^{d(w)} \rightarrow \mathcal{Z}_K$ of w .

NOTATION 2.3. Denote by W_Δ the class of simplicial complexes \mathcal{K} such that \mathcal{Z}_K is a wedge of spheres and each sphere in the wedge is a lift of a linear combination of iterated higher Whitehead products. The class W_Δ is not empty as it contains the boundary of the simplex $\partial\Delta^n$ for each $n > 0$. The moment-angle complex \mathcal{Z}_K corresponding to $\partial\Delta^n$ is homotopy equivalent to S^{2n+1} , which can be realized by the product $[\mu_0, \dots, \mu_n]$.

We consider the following decomposition of the disk D^2 into 3 cells: the point $1 \in D^2$ is the 0-cell; the complement to 1 in the boundary circle is the 1-cell, which we denote by S ; and the interior of D^2 is the 2-cell, which we denote by D . These cells are canonically oriented as subsets of \mathbb{R}^2 . By taking products we obtain a cellular decomposition of $(D^2)^m$ whose cells are parametrized by pairs of subsets $J, I \subset [m]$ with $J \cap I = \emptyset$: the set J parametrizes the S -cells in the product and I parametrizes the D -cells as we describe below. We denote the cell of $(D^2)^m$ corresponding to the pair J, I by $\chi(J, I)$:

$$\chi(J, I) = \{(x_1, \dots, x_m) \in (D^2)^m \mid x_j \in S \text{ for } j \in J \text{ and } x_l = 1 \text{ for } l \notin J \cup I\}.$$

Then \mathcal{Z}_K embeds as a cellular subcomplex in $(D^2)^m$; we have $\chi(J, I) \subset \mathcal{Z}_K$ whenever $I \in \mathcal{K}$.

The coproduct in the homology of a cell-complex X can be defined as follows. Consider the composite mapping of cellular cochain complexes

$$\mathcal{C}_*(X) \xrightarrow{\tilde{\Delta}_*} \mathcal{C}_*(X \times X) \xrightarrow{P} \mathcal{C}_*(X) \otimes \mathcal{C}_*(X). \quad (1)$$

¹⁾In all set-theoretic formulas in this paper we consider the product operation to be a higher priority than the union.

Here the mapping P sends the basis chain corresponding to a cell $e^i \times e^j$ to $e^i \otimes e^j$. The mapping $\tilde{\Delta}_*$ is induced by a cellular mapping $\tilde{\Delta}$ homotopic to the diagonal

$$\Delta : X \xrightarrow{x \mapsto (x,x)} X \times X.$$

In homology, (1) induces a coproduct $H_*(X) \rightarrow H_*(X) \otimes H_*(X)$ which does not depend on a choice of cellular approximation and is functorial. But (1) itself is not functorial because the choice of a cellular approximation is not canonical.

Nevertheless, in the case $X = \mathcal{Z}_{\mathcal{K}}$ we can use the following construction.

Construction 1. Consider the mapping $\tilde{\Delta} : D^2 \rightarrow D^2 \times D^2$ given in the polar coordinates $z = \rho e^{i\varphi} \in D^2$, $0 \leq \rho \leq 1$, $0 \leq \varphi \leq 2\pi$, by the formula

$$\rho e^{i\varphi} \mapsto \begin{cases} (1 - \rho + \rho e^{2i\varphi}, 1) & \text{for } 0 \leq \varphi \leq \pi, \\ (1, 1 - \rho + \rho e^{2i\varphi}) & \text{for } \pi \leq \varphi < 2\pi. \end{cases}$$

This is a cellular mapping homotopic to the diagonal $\Delta : D^2 \rightarrow D^2 \times D^2$, and its restriction to the boundary circle S^1 is a diagonal approximation for S^1 :

$$\begin{array}{ccc} S^1 & \longrightarrow & D^2 \\ \tilde{\Delta}|_{S^1} \downarrow & & \downarrow \tilde{\Delta} \\ S^1 \times S^1 & \longrightarrow & D^2 \times D^2. \end{array}$$

Taking the m -fold product we get a cellular approximation $\tilde{\Delta}^m \stackrel{\text{def}}{=} \tilde{\Delta} \times \cdots \times \tilde{\Delta} : (D^2)^m \rightarrow (D^2)^m \times (D^2)^m$ which restricts to a cellular approximation of the diagonal mapping of $\mathcal{Z}_{\mathcal{K}}$, as described in the following diagram:

$$\begin{array}{ccc} \mathcal{Z}_{\mathcal{K}} & \longrightarrow & (D^2)^m \\ \tilde{\Delta}^m|_{\mathcal{Z}_{\mathcal{K}}} \downarrow & & \downarrow \tilde{\Delta}^m \\ \mathcal{Z}_{\mathcal{K}} \times \mathcal{Z}_{\mathcal{K}} & \longrightarrow & (D^2)^m \times (D^2)^m. \end{array}$$

The diagonal approximation $\tilde{\Delta}$ is functorial with respect to mappings of moment-angle complexes induced by simplicial mappings.

Further, we denote $\tilde{\Delta}^m|_{\mathcal{Z}_{\mathcal{K}}}$ simply by $\tilde{\Delta}$.

§ 3. Algebraic Constructions

Let $\Lambda\langle u_1, \dots, u_m \rangle$ and $\mathbb{Z}\langle \mathcal{K} \rangle$ denote respectively the exterior coalgebra and the Stanley–Reisner coalgebra of a simplicial complex \mathcal{K} , which is a subcoalgebra of the symmetric coalgebra $\mathbb{Z}\langle v_1, \dots, v_m \rangle$. The Stanley–Reisner coalgebra $\mathbb{Z}\langle \mathcal{K} \rangle$ is generated as a \mathbb{Z} -module by monomials whose support is a simplex of \mathcal{K} [1, § 8.4]. Consider the submodule $\mathcal{R}_*(\mathcal{K})$ of $\Lambda\langle u_1, \dots, u_m \rangle \otimes \mathbb{Z}\langle \mathcal{K} \rangle$ additively generated by monomials not containing $u_i v_i$ and v_i^2 . Clearly, $\mathcal{R}_*(\mathcal{K}) \subset \Lambda\langle u_1, \dots, u_m \rangle \otimes \mathbb{Z}\langle \mathcal{K} \rangle$ is a subcoalgebra. We endow it with the differential $\partial = \sum_{i=1}^m u_i \frac{\partial}{\partial v_i}$ of degree -1 .

The following statements are obtained by dualization of the corresponding statements from [1, § 4.5] for cellular cochains and cohomology.

Lemma 3.1. *The mapping*

$$g : \mathcal{R}_*(\mathcal{K}) \xrightarrow{u_j v_i \mapsto \chi(J,I)} \mathcal{C}_*(\mathcal{Z}_{\mathcal{K}})$$

is an isomorphism of chain complexes. Hence, there is an additive isomorphism $H(\mathcal{R}_*(\mathcal{K}), \partial) \cong H_*(\mathcal{Z}_{\mathcal{K}})$.

Lemma 3.2. *The cellular chain coalgebra $\mathcal{C}_*(\mathcal{Z}_{\mathcal{K}})$ with the product defined via the diagonal approximation $\tilde{\Delta} : \mathcal{Z}_{\mathcal{K}} \rightarrow \mathcal{Z}_{\mathcal{K}} \times \mathcal{Z}_{\mathcal{K}}$ (see Construction 1) is isomorphic to the coalgebra $\mathcal{R}_*(\mathcal{K})$. So, there is an isomorphism of homology coalgebras $H((\mathcal{R}_*(\mathcal{K}), \partial)) \cong H_*(\mathcal{Z}_{\mathcal{K}})$.*

Given $J \subset [m]$ and a simplicial complex on $[m]$, denote by $\mathcal{K}_{\mathcal{J}}$ the full subcomplex on the vertex set J , i.e. $\mathcal{K}_{\mathcal{J}} = \{\mathcal{I} \in \mathcal{K} \mid \mathcal{I} \subset \mathcal{J}\}$.

Theorem 3.3. *The homomorphisms*

$$C_{p-1}(\mathcal{K}_{\mathcal{J}}) \xrightarrow{L \mapsto \varepsilon(L, \mathcal{J}) \chi(J \setminus L, L)} C_{p+|\mathcal{J}|}(\mathcal{Z}_{\mathcal{K}})$$

induce the injective homomorphisms

$$\tilde{H}_{p-1}(\mathcal{K}_{\mathcal{J}}) \hookrightarrow H_{p+|\mathcal{J}|}(\mathcal{Z}_{\mathcal{K}}),$$

which are functorial with respect to simplicial inclusions. Here $L \in \mathcal{K}_{\mathcal{J}}$ is a simplex and $\varepsilon(L, \mathcal{J})$ is the sign of the shuffle (L, \mathcal{J}) . The inclusions above induce the isomorphism of the abelian groups

$$h : \bigoplus_{\mathcal{J} \subset [m]} \tilde{H}_*(\mathcal{K}_{\mathcal{J}}) \xrightarrow{\cong} H_*(\mathcal{Z}_{\mathcal{K}}).$$

§ 4. The Hurewicz Homomorphism for Moment-Angle Complexes

We will use the notation S_i and D_i for the 1-cell and the 2-cell in the i th factor of $(D^2)^m$. We denote the product cells in $(D^2)^m$ by the records like $D_i S_j D_k$.

In this section we consider the iterated higher Whitehead products of the following form:

$$\left[\mu_{i_{01}}, \dots, \mu_{i_{0p_0}}, [\dots [\mu_{i_{n1}}, \dots, \mu_{i_{np_n}}] \dots] \right] : S^{2(p_0 + \dots + p_n) - (n+1)} \rightarrow \mathcal{Z}_{\mathcal{K}},$$

where $i_{kl} \in [m]$ for all k and l . The existence of a simplicial complex $\mathcal{K} \in W_{\Delta}$ realizing each product above will be proved in the next section (see Theorem 5.3).

The following lemma is a generalization of Lemma 3.1 in [6].

Lemma 4.1. *The Hurewicz image*

$$h([\mu_{i_{01}}, \dots, \mu_{i_{0p_0}}, [\dots, [\mu_{i_{(n-1)1}}, \dots, \mu_{i_{(n-1)p_{n-1}}}, [\mu_{i_{n1}}, \dots, \mu_{i_{np_n}}] \dots]]) \in H_{2(p_0 + \dots + p_n) - (n+1)}(\mathcal{Z}_{\mathcal{K}})$$

is represented by the cellular chain

$$\prod_{k=0}^n \left(\sum_{j=1}^{p_k} D_{i_{k1}} \dots D_{i_{k(j-1)}} S_{i_{kj}} D_{i_{k(j+1)}} \dots D_{i_{kp_k}} \right). \quad (2)$$

PROOF. We induct on the number n of nested higher products.

For $n = 0$ we have the single higher product

$$\begin{aligned} & [\mu_1, \dots, \mu_k] : S^{2k-1} \cong \partial(D_1^2 \times \dots \times D_k^2) \\ & \cong D_1^2 \times \dots \times D_{k-1}^2 \times S_k^1 \cup \dots \cup S_1^1 \times D_2^2 \times \dots \times D_k^2 \rightarrow (\mathbb{C}P^\infty)^{\mathcal{K}}, \end{aligned}$$

which lifts to the inclusion of a subcomplex

$$[\mu_1, \dots, \mu_k] : D_1^2 \times \dots \times D_{k-1}^2 \times S_k^1 \cup \dots \cup S_1^1 \times D_2^2 \times \dots \times D_k^2 \rightarrow \mathcal{Z}_{\mathcal{K}}.$$

Therefore, the Hurewicz image is represented by the cellular chain (2).

Let $n = 1$; i.e., we have a product of the form $[\mu_{i_1}, \dots, \mu_{i_p}, [\mu_{j_1}, \dots, \mu_{j_q}]]$, $q > 1$.

By the definition of higher Whitehead products, the class

$$[\mu_{i_1}, \dots, \mu_{i_p}, [\mu_{j_1}, \dots, \mu_{j_q}]] \in \pi_{2(p+q-1)}((\mathbb{C}P^\infty)^\mathcal{K})$$

is represented by the composite mapping

$$\begin{aligned} S^{2(p+q-1)} &\cong D_{i_1}^2 \times \dots \times D_{i_p}^2 \times \partial D_{j_1 \dots j_q}^{2q-1} \\ \cup \left(\left(\bigcup_{k=1}^p D_{i_1}^2 \times \dots \times D_{i_{k-1}}^2 \times S_{i_k}^1 \times D_{i_{k+1}}^2 \times \dots \times D_{i_p}^2 \right) \times D_{j_1 \dots j_q}^{2q-1} \right) \\ &\rightarrow D_{i_1}^2 \times \dots \times D_{i_p}^2 \times pt \\ \cup \left(\left(\bigcup_{k=1}^p D_{i_1}^2 \times \dots \times D_{i_{k-1}}^2 \times S_{i_k}^1 \times D_{i_{k+1}}^2 \times \dots \times D_{i_p}^2 \right) \times S_{j_1 \dots j_q}^{2q-1} \right) \\ &\rightarrow S_{i_1}^2 \times \dots \times S_{i_p}^2 \times pt \\ \cup \left(\left(\bigcup_{k=1}^p S_{i_1}^2 \times \dots \times S_{i_{k-1}}^2 \times pt \times S_{i_{k+1}}^2 \times \dots \times S_{i_p}^2 \right) \times S_{j_1 \dots j_q}^{2q-1} \right) &\rightarrow (\mathbb{C}P^\infty)^\mathcal{K}. \end{aligned}$$

By Theorem 2.1, the composite mapping above lifts to an inclusion of a cell subcomplex as described by the diagram:

$$\begin{array}{ccc} \mathcal{Z}_\mathcal{K} & \xrightarrow{\quad} & (\mathbb{C}P^\infty)^\mathcal{K} \\ & \swarrow \text{---} & \nearrow [\mu_{i_1}, \dots, \mu_{i_p}, [\mu_{j_1}, \dots, \mu_{j_q}]] \\ D_{i_1}^2 \times \dots \times D_{i_p}^2 \times pt \cup \left(\left(\bigcup_{k=1}^p D_{i_1}^2 \times \dots \times S_{i_k}^1 \times \dots \times D_{i_p}^2 \right) \times \left(\bigcup_{k=1}^q D_{j_1}^2 \times \dots \times S_{j_k}^1 \times \dots \times D_{j_q}^2 \right) \right) & & \\ & \parallel & \\ D_{i_1}^2 \times \dots \times D_{i_p}^2 \times pt \cup \left(\left(\bigcup_{k=1}^p D_{i_1}^2 \times \dots \times S_{i_k}^1 \times \dots \times D_{i_p}^2 \right) \times S_{j_1 \dots j_q}^{2q-1} \right). & & \end{array}$$

In the union above, all cells but $D_{i_1}^2 \times \dots \times D_{i_p}^2 \times pt$ have dimension $2(p+q-1)$ while $D_{i_1}^2 \times \dots \times D_{i_p}^2 \times pt$ has dimension $2p < 2(p+q-1)$. Thus, the Hurewicz image of the lifted iterated higher Whitehead product in $H_{2(p+q-1)}(\mathcal{Z}_\mathcal{K})$ is represented by the cellular chain

$$\left(\sum_{k=1}^p D_{i_1} \dots D_{i_{k-1}} S_{i_k} D_{i_{k+1}} \dots D_{i_p} \right) \cdot \left(\sum_{k=1}^q D_{j_1} \dots D_{j_{k-1}} S_{j_k} D_{j_{k+1}} \dots D_{j_p} \right).$$

A similar argument applies in the general case. Consider the iterated higher Whitehead product $[\mu_{i_{01}}, \dots, \mu_{i_{0p_0}}, w] : S^{2p_0+d(w)-1} \rightarrow (\mathbb{C}P^\infty)^\mathcal{K}$. Here

$$w = [\mu_{i_{11}}, \dots, \mu_{i_{1p_1}}, [\dots, [\mu_{i_{(n-1)1}}, \dots, \mu_{i_{(n-1)p_{n-1}}}, [\mu_{i_{n1}}, \dots, \mu_{i_{np_n}}]] \dots]].$$

The sphere $S^{2p_0+d(w)-1}$ is decomposed into the union

$$D_{i_{01}}^2 \times \dots \times D_{i_{0p_0}}^2 \times \partial D^{d(w)} \cup \left(\left(\bigcup_{k=1}^p D_{i_{01}}^2 \times \dots \times D_{i_{0(k-1)}}^2 \times S_{i_{0k}}^1 \times D_{i_{0(k+1)}}^2 \times \dots \times D_{i_{0p_0}}^2 \right) \times D_w^{d(w)} \right).$$

By contracting $\partial D^{d(w)}$ to a point we obtain some cell subcomplex $X \subset \mathcal{Z}_{\mathcal{K}}$. The inclusion $X \hookrightarrow \mathcal{Z}_{\mathcal{K}}$ is a lift of the Whitehead product $[\mu_{i_{01}}, \dots, \mu_{i_{0p_0}}, w]$. Arguing by induction we find that the Hurewicz image of the mapping $S_w^{d(w)} \rightarrow \mathcal{Z}_{\mathcal{K}}$ is represented by the cellular chain

$$\prod_{k=1}^n \left(\sum_{j=1}^{p_k} D_{i_{k1}} \cdots D_{i_{k(j-1)}} S_{i_{kj}} D_{i_{k(j+1)}} \cdots D_{i_{kp_k}} \right).$$

By dimensional reasons,

$$h([\mu_{i_{01}}, \dots, \mu_{i_{0p_0}}, w] |_{D_{i_{01}}^2 \times \cdots \times D_{i_{0p_0}}^2 \times pt}) = 0 \in H_{2p_0+d(w)-1}(\mathcal{Z}_{\mathcal{K}}).$$

Thus,

$$h([\mu_{i_{01}}, \dots, \mu_{i_{0p_0}}, w]) = \prod_{k=0}^n \left(\sum_{j=1}^{p_k} D_{i_{k1}} \cdots D_{i_{k(j-1)}} S_{i_{kj}} D_{i_{k(j+1)}} \cdots D_{i_{kp_k}} \right). \quad \square$$

EXAMPLE 4.2. Consider the Whitehead product $[\mu_1, \mu_2, [\mu_3, \mu_4, \mu_5]]$. The simplicial complex $\mathcal{K} \in W_{\Delta}$ realizing this product is described in Example 5.4 below. By Lemma 4.1, the homology class $h([\mu_1, \mu_2, [\mu_3, \mu_4, \mu_5]]) \in H_8(\mathcal{Z}_{\mathcal{K}})$ is represented by the cellular chain

$$\begin{aligned} & D_1 S_2 D_3 D_4 S_5 + D_1 S_2 D_3 S_4 D_5 + D_1 S_2 S_3 D_4 D_5 \\ & + S_1 D_2 D_3 D_4 S_5 + S_1 D_2 D_3 S_4 D_5 + S_1 D_2 S_3 D_4 D_5. \end{aligned}$$

§ 5. Operations on W_{Δ} and Realization of Whitehead Products

Let $\mathcal{K} = \mathcal{K}_1 \cup_I \mathcal{K}_2$ be the simplicial complex obtained by gluing \mathcal{K}_1 and \mathcal{K}_2 along a common face I (we allow $I = \emptyset$, in which case $\mathcal{K}_1 \cup_I \mathcal{K}_2 = \mathcal{K}_1 \sqcup \mathcal{K}_2$).

Proposition 5.1 (cp. [1, Theorem 8.2.1]). *If $\mathcal{K}_1, \mathcal{K}_2 \in W_{\Delta}$ then $\mathcal{K} = \mathcal{K}_1 \cup_I \mathcal{K}_2 \in W_{\Delta}$ for any common face I of simplicial complexes \mathcal{K}_1 and \mathcal{K}_2 .*

PROOF. Consider a full subcomplex \mathcal{K}_J in $\mathcal{K} = \mathcal{K}_1 \cup_I \mathcal{K}_2$. Let $V(\mathcal{K}_1)$ and $V(\mathcal{K}_2)$ be the sets of vertices of the simplicial complexes \mathcal{K}_1 and \mathcal{K}_2 . Put $J_1 = V(\mathcal{K}_1) \cap J$, $J_2 = V(\mathcal{K}_2) \cap J$. Consider the two cases. If $J \cap I = \emptyset$, then \mathcal{K}_J is the disjoint union $\mathcal{K}_{J_1} \sqcup \mathcal{K}_{J_2}$, and we have $H_p(\mathcal{K}_J) \cong H_p(\mathcal{K}_{J_1}) \oplus H_p(\mathcal{K}_{J_2})$. If $J \cap I \neq \emptyset$ then \mathcal{K}_J is homotopy equivalent to $\mathcal{K}_{J_1} \vee \mathcal{K}_{J_2}$, and $\tilde{H}_p(\mathcal{K}_J) \cong \tilde{H}_p(\mathcal{K}_{J_1}) \oplus \tilde{H}_p(\mathcal{K}_{J_2})$. In both cases generators of each summand mapping to generators of the corresponding homology groups $H_*(\mathcal{Z}_{\mathcal{K}})$ under the homomorphism of Theorem 3.3.

Let $\{\sigma_{\alpha}(J_1, p)\}_{\alpha \in A}$ and $\{\sigma_{\beta}(J_2, p)\}_{\beta \in B}$ be sets of simplicial chains which represent bases of the free abelian groups $\tilde{H}_p((\mathcal{K}_1)_{J_1})$ and $\tilde{H}_p((\mathcal{K}_2)_{J_2})$ respectively. Let $\{\chi_{\alpha}(J_1, p)\}_{\alpha \in A}$ and $\{\chi_{\beta}(J_2, p)\}_{\beta \in B}$ be the images of these bases under the mapping $C_{p-1}(\mathcal{K}_{J_l}) \rightarrow C_{p+|J_l|}(\mathcal{Z}_{\mathcal{K}_l})$, $l = 1, 2$, of Theorem 3.3. Considering the same bases as elements of $\tilde{H}_*(\mathcal{K}_J)$, we see that the homomorphism $C_{p-1} \rightarrow C_{p+|J|}$ sends them to the cellular chains

$$\prod_{j \in J \setminus J_2} S_j \cdot \{\chi_{\alpha}(J_1, p)\}_{\alpha \in A} \quad \text{and} \quad \prod_{j \in J \setminus J_1} S_j \cdot \{\chi_{\beta}(J_2, p)\}_{\beta \in B} \quad (3)$$

respectively. When $J_1 \neq \emptyset$ and $J_2 \neq \emptyset$, but $J \cap I = \emptyset$, we get the new generator of $\tilde{H}_0(\mathcal{K}_J)$ which is represented by a simplicial chain $j_1 + j_2$ with $j_1 \in J_1$ and $j_2 \in J_2$; this one is different from the generators of the homology of \mathcal{K}_{J_1} and \mathcal{K}_{J_2} . The corresponding cellular chain in $C_*(\mathcal{Z}_{\mathcal{K}})$ is

$$\prod_{j \neq j_1, j_2} S_j \cdot (D_{j_1} S_{j_2} + S_{j_1} D_{j_2}). \quad (4)$$

Let $\{w_\alpha(J_1, p)\}_{\alpha \in A}$ and $\{w_\beta(J_2, p)\}_{\beta \in B}$ be the Whitehead products corresponding to the bases $\{\sigma_\alpha(J_1, p)\}_{\alpha \in A}$ and $\{\sigma_\beta(J_2, p)\}_{\beta \in B}$. The Hurewicz images of the products

$$\begin{aligned} & [\mu_{k_1}, [\mu_{k_2}, [\dots [\mu_{k_{r-1}}, [\mu_{k_r}, w_\alpha(J_1, p)]] \dots]] \text{ for } J \setminus J_2 = \{k_1, \dots, k_r\}, \\ & [\mu_{l_1}, [\mu_{l_2}, [\dots [\mu_{l_{s-1}}, [\mu_{l_s}, w_\beta(J_2, p)]] \dots]] \text{ for } J \setminus J_1 = \{l_1, \dots, l_s\} \end{aligned}$$

are represented by chains (3). Chain (4) represents the Hurewicz image of the product

$$[\mu_{j_3}, [\mu_{j_4}, [\dots [\mu_{j_{|J|}}, [\mu_{j_1}, \mu_{j_2}]] \dots]], \quad J = \{j_1, \dots, j_{|J|}\}.$$

The wedge sum of the Whitehead products above

$$\left(\bigvee_{\substack{J_1, J_2 \\ J_1 \cap I = J_2 \cap I}} \left(\bigvee_{p \geq 0} \left(\bigvee_{\alpha \in A} S^{p+|J|+1} \vee \bigvee_{\beta \in B} S^{p+|J|+1} \right) \right) \right) \vee \bigvee_{\substack{J_1, J_2 \neq \emptyset \\ J_1 \cap I = J_2 \cap I = \emptyset}} S^{|J|+1} \rightarrow \mathcal{Z}_{\mathcal{K}}$$

induces an isomorphism in homology. As all spaces involved are simply connected, it is a homotopy equivalence. \square

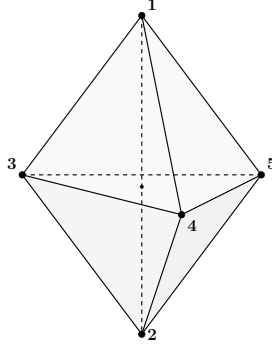


Fig. 1. $\mathcal{K} = \mathcal{J}_1(\partial\Delta^2)$

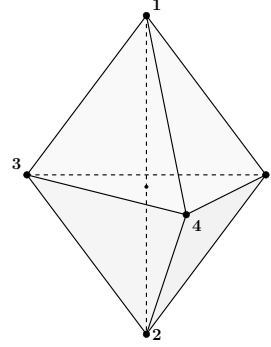


Fig. 2. $\mathcal{L} = \mathcal{J}_1(\partial\Delta(3, 4, 5)) \cup \{1, 2, 3\}$

Theorem 5.2. *Let $\mathcal{K} \in W_\Delta$ be a simplicial complex. Then the simplicial complex*

$$\mathcal{J}_n(\mathcal{K}) = (\partial\Delta^n * \mathcal{K}) \cup \Delta^n$$

belongs to W_Δ .

Note that $\mathcal{J}_n(\mathcal{K}) \simeq \Sigma^n(\mathcal{K}) \vee S^n$.

The case $\mathcal{K} = \partial\Delta^2$ and $n = 1$ is shown in Fig. 1.

PROOF. Put $\mathcal{L} = \mathcal{J}_n(\mathcal{K})$ and let $V(\mathcal{K}) = I$ and $V(\Delta^n) = I_1$ be the sets of vertices. By Theorem 3.3, the homology of $\mathcal{Z}_{\mathcal{L}}$ comes from the homology of the full subcomplexes $\mathcal{L}_{J_1, J} = ((\partial\Delta^n)_{J_1} * \mathcal{K}_J) \cup \Delta^n_{J_1}$ where $J_1 \subset I_1$ and $J \subset I$.

If $J_1 \subsetneq I_1$ is a nonempty proper subset, then the complex $\mathcal{L}_{J_1, J}$ is topologically contractible. So, in this case $\tilde{H}_*(\mathcal{L}_{J_1, J}) = 0$. If $J_1 = \emptyset$, then the corresponding full subcomplex is $\mathcal{L}_{\emptyset, J} = \mathcal{K}_J$. Hence, $\tilde{H}_*(\mathcal{L}_{\emptyset, J}) \cong \tilde{H}_*(\mathcal{K}_J)$. Finally, when $J_1 = I_1$, we have

$$\mathcal{L}_{I_1, J} = (\partial\Delta^n * \mathcal{K}_J) \cup \Delta^n \simeq \Sigma^n(\mathcal{K}_J) \vee S^n. \quad (5)$$

Hence, $\tilde{H}_*(\mathcal{L}_{I_1, J}) \cong \tilde{H}_{*-n}(\mathcal{K}_J) \oplus \tilde{H}_*(S^n)$, where the generator of the second summand is represented by the boundary of the $(n+1)$ -simplex $\Delta(I_1, j)$.

We will show that all generators of $H_*(\mathcal{Z}_{\mathcal{L}})$ are represented by the cellular chains (2), and therefore are the Hurewicz images of iterated higher Whitehead products by Lemma 4.1. The generator of $H_*(\mathcal{Z}_{\mathcal{L}})$ corresponding to the wedge summand S^n in (5) is represented by the cellular chain

$$\prod_{k=1}^{m-1} S_{j_k} \cdot \left(D_{i_1} \dots D_{i_{n+1}} S_j + \left(\sum_{k=1}^{n+1} D_{i_1} \dots S_{i_k} \dots D_{i_{n+1}} \right) D_j \right),$$

where $I_1 = \{i_1, \dots, i_{n+1}\}$ and $J = \{j_1, \dots, j_{m-1}, j\}$. It is the Hurewicz image of the Whitehead product

$$[\mu_{j_1}, [\mu_{j_2}, [\dots [\mu_{j_{m-1}}, [\mu_{i_1}, \dots, \mu_{i_{n+1}}, \mu_j]] \dots]].$$

Finally, every generator of $H_*(\mathcal{Z}_{\mathcal{L}})$ corresponding to the wedge summand $\Sigma^n(\mathcal{K}_J)$ in (5) is represented by the cellular chain

$$\left(\sum_{k=1}^{n+1} D_{i_1} \dots S_{i_k} \dots D_{i_{n+1}} \right) h(w), \quad (6)$$

where w is the Whitehead product that goes to the corresponding generator of $\tilde{H}_{*-n}(\mathcal{K}_J) \subset H_*(\mathcal{Z}_{\mathcal{K}})$. The chain (6) represents the class $h([\mu_{i_1}, \dots, \mu_{i_{n+1}}, w])$.

The wedge sum of the Whitehead products described above is a mapping from a wedge of spheres to $\mathcal{Z}_{\mathcal{L}}$ that induces an isomorphism of homology groups. Hence, it is a homotopy equivalence. \square

Theorem 5.3. *For every iterated higher Whitehead product*

$$w = [\mu_{i_{01}}, \dots, \mu_{i_{0p_0}}, [\dots [\mu_{i_{n1}}, \dots, \mu_{i_{np_n}}] \dots]] \quad (7)$$

there exists a simplicial complex $\mathcal{K} \in W_{\Delta}$ that realizes w .

PROOF. Consider the simplicial complex $\mathcal{L} = \mathcal{J}_{p_0-1} \circ \mathcal{J}_{p_1-1} \circ \dots \circ \mathcal{J}_{p_{n-1}-1}(\partial\Delta^{p_n-1})$. By Theorem 5.2, the complex \mathcal{L} belongs to W_{Δ} . The maximal dimensional sphere in the wedge $\mathcal{Z}_{\mathcal{L}}$ has dimension $2(p_0 + \dots + p_n) - (n+1)$ and it is realized by w , as shown by considering the Hurewicz homomorphism. \square

Table 1. Homology of $\mathcal{Z}_{\mathcal{K}}$ for $\mathcal{K} = \mathcal{J}_1(\partial\Delta^2)$ (see Fig. 1)

$H_5(\mathcal{Z}_{\mathcal{K}})$	\mathbb{Z}	$D_3D_4S_5 + D_3S_4D_5 + S_3D_4D_5$	$[\mu_3, \mu_4, \mu_5]$
	\mathbb{Z}	$D_1D_2S_3 + D_1S_2D_3 + S_1D_2D_3$	$[\mu_1, \mu_2, \mu_3]$
	\mathbb{Z}	$D_1D_2S_4 + D_1S_2D_4 + S_1D_2D_4$	$[\mu_1, \mu_2, \mu_4]$
	\mathbb{Z}	$D_1D_2S_5 + D_1S_2D_5 + S_1D_2D_5$	$[\mu_1, \mu_2, \mu_5]$
$H_6(\mathcal{Z}_{\mathcal{K}})$	\mathbb{Z}	$S_4(D_1D_2S_3 + D_1S_2D_3 + S_1D_2D_3)$	$[\mu_4, [\mu_1, \mu_2, \mu_3]]$
	\mathbb{Z}	$S_5(D_1D_2S_3 + D_1S_2D_3 + S_1D_2D_3)$	$[\mu_5, [\mu_1, \mu_2, \mu_3]]$
	\mathbb{Z}	$S_5(D_1D_2S_4 + D_1S_2D_4 + S_1D_2D_4)$	$[\mu_5, [\mu_1, \mu_2, \mu_4]]$
$H_7(\mathcal{Z}_{\mathcal{K}})$	\mathbb{Z}	$S_5S_4(D_1D_2S_3 + D_1S_2D_3 + S_1D_2D_3)$	$[\mu_4, [\mu_5, [\mu_1, \mu_2, \mu_3]]]$
$H_8(\mathcal{Z}_{\mathcal{K}})$	\mathbb{Z}	$(D_1S_2 + S_1D_2)(D_3D_4S_5 + D_3S_4D_5 + S_3D_4D_5)$	$[\mu_1, \mu_2, [\mu_3, \mu_4, \mu_5]]$

EXAMPLE 5.4. The Whitehead product $[\mu_1, \mu_2, [\mu_3, \mu_4, \mu_5]]$ is realized by the complex \mathcal{K} which is the minimal triangulation of a 2-sphere with diameter (see Fig. 1). Note that $\mathcal{K} = \mathcal{J}_1(\partial\Delta(3, 4, 5))$, and so $\mathcal{K} \in W_{\Delta}$.

Using Theorem 3.3 and Lemma 4.1 we identify the homology of $\mathcal{Z}_{\mathcal{K}}$, cellular chains representing the homology generators and Whitehead products which mapping to the generators under the Hurewicz homomorphism (see Table 1).

The wedge sum of the Whitehead products from the right column of Table 1 gives a mapping $(S^5)^{\vee 4} \vee (S^6)^{\vee 3} \vee S^7 \vee S^8 \rightarrow \mathcal{Z}_{\mathcal{K}}$ that induces an isomorphism of homology groups. Thus, we have $\mathcal{Z}_{\mathcal{K}} \simeq (S^5)^{\vee 4} \vee (S^6)^{\vee 3} \vee S^7 \vee S^8$.

Note that this is the first example of \mathcal{K} with a nontrivial iterated Whitehead product in which one of the arguments of a higher product is again a higher product.

§ 6. The Smallest Complex Realizing a Given Whitehead Product

Theorem 6.1. *The simplicial complex $\mathcal{J}_{p-1}(\partial\Delta^{q-1}) \in W_\Delta$ (see Theorem 5.2) is the smallest complex realizing the product*

$$[\mu_{i_1}, \dots, \mu_{i_p}, [\mu_{j_1}, \dots, \mu_{j_q}]]. \quad (8)$$

PROOF. Assume that \mathcal{K} realizes (8).

For (8) to be defined it is necessary that the product $[\mu_{j_1}, \dots, \mu_{j_q}]$ is defined and the products $[\mu_{i_1}, \dots, \mu_{i_p}]$ and

$$[\mu_{i_1}, \dots, \widehat{\mu}_{i_k}, \dots, \mu_{i_p}, [\mu_{j_1}, \dots, \mu_{j_q}]] \quad \text{for } k = 1, \dots, p \quad (9)$$

are trivial.

For the existence of $[\mu_{j_1}, \dots, \mu_{j_q}]$ it is necessary that each $[\mu_{j_1}, \dots, \widehat{\mu}_{j_k}, \dots, \mu_{j_q}]$ with $k = 1, \dots, q$ be trivial; i.e., $\{\widehat{j}_1, \dots, \widehat{j}_k, \dots, j_q\} \in \mathcal{K}$ is a simplex. Thus, the existence of the $[\mu_{j_1}, \dots, \mu_{j_q}]$ gives the inclusion $\partial\Delta^{q-1} \hookrightarrow \mathcal{K}$ and the triviality of $[\mu_{i_1}, \dots, \mu_{i_p}]$ gives the inclusion $\Delta^{p-1} = \{i_1, \dots, i_p\} \hookrightarrow \mathcal{K}$.

We will show that the triviality of products (9) is equivalent to the existence of inclusions

$$\{i_1, \dots, \widehat{i}_k, \dots, i_p\} * \partial\Delta^{q-1} \hookrightarrow \mathcal{K} \quad \text{for } k = 1, \dots, p. \quad (10)$$

Without loss of generality we can assume that $k = p$. Since $[\mu_{i_1}, \dots, \mu_{i_{p-1}}, [\mu_{j_1}, \dots, \mu_{j_q}]]$ is trivial, we have the inclusion

$$D_{i_1}^2 \times \dots \times D_{i_{p-1}}^2 \times \left(\bigcup_{k=1}^q D_{j_1}^2 \times \dots \times S_{j_k}^1 \times \dots \times D_{j_q}^2 \right) \hookrightarrow \mathcal{Z}_{\mathcal{K}};$$

see the proof of the case $n = 1$ in Lemma 4.1. Therefore, we have (10) for $k = p$.

It follows that $\mathcal{J}_{p-1}(\partial\Delta^{q-1})$ embeds into each simplicial complex \mathcal{K} that realizes (8). By Theorem 5.3, the complex $\mathcal{K} = \mathcal{J}_{p-1}(\partial\Delta^{q-1})$ realizes (8), and so it is the smallest complex with this property. \square

In Theorem 6.1 we showed that if the simplicial complex \mathcal{L} realizes (8) then there exists an embedding $\mathcal{J}_{p-1}(\partial\Delta^{q-1}) \hookrightarrow \mathcal{L}$. However, the next example shows that $\mathcal{J}_{p-1}(\Delta^{q-1})$ is not necessarily a full subcomplex of \mathcal{L} .

Table 2. Homology of $\mathcal{Z}_{\mathcal{L}}$ for $\mathcal{L} = \mathcal{J}_1(\partial\Delta(3, 4, 5)) \cup \{1, 2, 3\}$ (see Fig. 2)

$H_5(\mathcal{Z}_{\mathcal{K}})$	\mathbb{Z}	$D_3D_4S_5 + D_3S_4D_5 + S_3D_4D_5$	$[\mu_3, \mu_4, \mu_5]$
	\mathbb{Z}	$D_1D_2S_4 + D_1S_2D_4 + S_1D_2D_4$	$[\mu_1, \mu_2, \mu_4]$
	\mathbb{Z}	$D_1D_2S_5 + D_1S_2D_5 + S_1D_2D_5$	$[\mu_1, \mu_2, \mu_5]$
$H_6(\mathcal{Z}_{\mathcal{K}})$	\mathbb{Z}	$S_5(D_1D_2S_4 + D_1S_2D_4 + S_1D_2D_4)$	$[\mu_5, [\mu_1, \mu_2, \mu_4]]$
$H_8(\mathcal{Z}_{\mathcal{K}})$	\mathbb{Z}	$(D_1S_2 + S_1D_2)(D_3D_4S_5 + D_3S_4D_5 + S_3D_4D_5)$	$[\mu_1, \mu_2, [\mu_3, \mu_4, \mu_5]]$

EXAMPLE 6.2. Consider the simplicial complex $\mathcal{L} = \mathcal{J}_1(\partial\Delta(3, 4, 5)) \cup \Delta(1, 2, 3)$ (see Fig. 2). Reasoning as in Example 5.4 shows that $\mathcal{Z}_{\mathcal{K}} \simeq (S^5)^{\vee 3} \vee S^6 \vee S^8$ (see Table 2). Here S^8 is a lift of the product $[\mu_1, \mu_2, [\mu_3, \mu_4, \mu_5]]$. We can see that $\mathcal{J}_1(\partial\Delta(3, 4, 5))$ is not a full subcomplex of \mathcal{L} .

§ 7. An Example of Unrealizability

In this section we give an example of a simplicial complex \mathcal{K} such that the corresponding moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is homotopy equivalent to a wedge of spheres, but $\mathcal{K} \notin W_\Delta$. In other words, there is a sphere in the wedge which is not realizable by any linear combination of iterated higher Whitehead products (in the sense of Definition 2.2). This implies that the answer to [1, Problem 8.4.5] is negative.

Proposition 7.1. *Let \mathcal{K} be the simplicial complex $(\partial\Delta^2 * \partial\Delta^2) \cup \Delta^2 \cup \Delta^2$. The moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is homotopy equivalent to a wedge of spheres.*

PROOF. Note that the 2-skeleton of \mathcal{K} coincides with the 2-skeleton of a 5-dimensional simplex. Therefore, $\mathcal{Z}_{\mathcal{K}}$ is 6-connected. Note that $|\mathcal{K}| \simeq S^3 \vee S^2 \vee S^2$. Using Theorem 3.3, we can describe the homology of $\mathcal{Z}_{\mathcal{K}}$ (see Table 3).

Table 3. Homology of $\mathcal{Z}_{\mathcal{K}}$ for $\mathcal{K} = (\partial\Delta^2 * \partial\Delta^2) \cup \Delta^2 \cup \Delta^2$

$H_7(\mathcal{Z}_{\mathcal{K}})$	\mathbb{Z}^6
$H_8(\mathcal{Z}_{\mathcal{K}})$	\mathbb{Z}^6
$H_9(\mathcal{Z}_{\mathcal{K}})$	\mathbb{Z}^2
$H_{10}(\mathcal{Z}_{\mathcal{K}})$	\mathbb{Z}

By [7] (see also [1, Corollary 8.3.6]), there is a homotopy equivalence

$$f : (S^8)^{\vee 6} \vee (S^9)^{\vee 6} \vee (S^{10})^{\vee 2} \vee S^{11} \xrightarrow{\simeq} \Sigma\mathcal{Z}_{\mathcal{K}}.$$

Put $X = (S^7)^{\vee 6} \vee (S^8)^{\vee 6} \vee (S^9)^{\vee 2} \vee S^{10}$. As both spaces X and $\mathcal{Z}_{\mathcal{K}}$ are 6-connected, by the Freudenthal Theorem, the suspension homomorphism $\Sigma : \pi_n \rightarrow \pi_{n+1}$ for X and $\mathcal{Z}_{\mathcal{K}}$ is an isomorphism for $n < 13$. Consider the commutative diagram for $n < 13$:

$$\begin{array}{ccc} \pi_{n+1}(\Sigma X) & \xrightarrow{f_*} & \pi_{n+1}(\Sigma\mathcal{Z}_{\mathcal{K}}) \\ \cong \uparrow \Sigma_X & & \cong \uparrow \Sigma_{\mathcal{Z}_{\mathcal{K}}} \\ \pi_n(X) & \xrightarrow{\Sigma_{\mathcal{Z}_{\mathcal{K}}}^{-1} \circ f_* \circ \Sigma_X} & \pi_n(\mathcal{Z}_{\mathcal{K}}). \end{array}$$

The class $[i_j^n]$ of the inclusion of the j th n -sphere $i_j^n : S^n \hookrightarrow X$ mappings to the class of a mapping $S^n \rightarrow \mathcal{Z}_{\mathcal{K}}$ under the composite $\Sigma_{\mathcal{Z}_{\mathcal{K}}}^{-1} \circ f_* \circ \Sigma_X$. The wedge sum of these mappings gives a mapping $g : X \rightarrow \mathcal{Z}_{\mathcal{K}}$. By construction, $\Sigma g : \Sigma X \rightarrow \Sigma\mathcal{Z}_{\mathcal{K}}$ induces an isomorphism in homology. Thus, g also induces an isomorphism in homology, so it is a homotopy equivalence. \square

Proposition 7.2. *The sphere $S^{10} \subset \mathcal{Z}_{\mathcal{K}}$ cannot be realized by any linear combination of general iterated higher Whitehead products.*

PROOF. By dimensional reasons, if there is a general iterated higher Whitehead realizing the sphere S^{10} ; then it contains exactly two nested brackets. The internal bracket may have size 2, 3, 4, or 5. Since the 2-skeleton of \mathcal{K} coincides with the 2-skeleton of Δ^5 , all Whitehead 2- and 3-products are trivial. We are left with the following two iterated products with the internal bracket of size 5 and 4:

$$[\mu_{i_1}, [\mu_{i_2}, \mu_{i_3}, \mu_{i_4}, \mu_{i_5}, \mu_{i_6}]], \quad [\mu_{i_1}, \mu_{i_2}, [\mu_{i_3}, \mu_{i_4}, \mu_{i_5}, \mu_{i_6}]].$$

The first product is not defined because \mathcal{K} does not contain $\partial\Delta(i_2, i_3, i_4, i_5, i_6)$. For the second product, without loss of generality we can consider the two cases $\{i_1, i_2\} = \{1, 2\}$ and $\{i_1, i_2\} = \{1, 4\}$. In both cases the product is not defined because \mathcal{K} does not contain $\Delta(2, 4, 5, 6)$. \square

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