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An analog of Chern's conjecture for the Euler–Satake characteristic of affine orbifolds

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ABSTRACT

S.S. Chern conjectured that the Euler characteristic of every closed affine manifold has to vanish. We present an analog of this conjecture stating that the Euler–Satake characteristic of any compact affine orbifold is equal to zero. We prove that Chern's conjecture is equivalent to its analog for the Euler–Satake characteristic of compact affine orbifolds, not necessarily effective. This fact allowed us to extend to orbifolds sufficient conditions for Chern's conjecture proved by Klingler and Kostant–Sullivan. Thus, we prove that, if an n-dimensional compact affine orbifold $\mathcal{N}$ is complete or if its holonomy group belongs to the special linear group $SL(n, \mathbb{R})$, then the Euler–Satake characteristic of $\mathcal{N}$ has to vanish. An application to pseudo-Riemannian orbifolds is considered. We give examples of orbifolds from the class under investigation. In particular, we construct an example of a compact incomplete affine orbifold with vanishing Euler–Satake characteristic, the holonomy group of which is not contained in $SL(n, \mathbb{R})$.

1. Introduction. Main results

B. Klingler [12, Section 1.1] emphasizes that affine manifolds "are surprisingly very poorly understood". Since orbifolds can be considered as manifolds with singularities, the same can be said about affine structures on orbifolds.

S.S. Chern (in 1955) puts forward the following conjecture on topology of affine manifolds.

Chern's conjecture. *The Euler characteristic of every closed affine manifold has to vanish.*

In other words, Chern's conjecture states that non-zero Euler characteristic is an obstruction to the existence of an affine structure on a closed manifold. This conjecture is open today in general case.

Chern's conjecture was proved by B. Kostant and D. Sullivan for complete affine manifolds, i.e. for affine space forms [14].

An n-dimensional affine manifold is called special if its holonomy group belongs to the special linear group $SL(n, \mathbb{R})$. Recently, B. Klingler [12] proved Chern's conjecture for special affine manifolds. Some other sufficient conditions for Chern's conjecture have been found (see a short overview in [12]).

Orbifolds were introduced by I. Satake [16] as a generalization of the concept of manifold and they were called by him V-manifolds. The term orbifold was suggested by W. P. Thurston. He applied the classification of compact two-dimensional Riemannian orbifolds of constant curvature to the classification of closed three-dimensional manifolds [19].

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Orbifolds arise in different branches of mathematics and physics, including algebraic and differential geometry, topology (see an overview [1]). Orbifolds find applications in conformal field theory, deformation quantization, foliation theory. In particular, orbifolds are used in string theory as the spaces of string propagation [6].

The first articles devoted to differential geometry and topology of orbifolds belong to I. Satake [16,17]. Many concepts of smooth manifold theory such as de Rham cohomology, characteristic classes, and the Gauss–Bonnet theorem were generalized by I. Satake to orbifolds.

Differential geometry of orbifolds keeps developing. Automorphism groups of geometric structures on smooth orbifolds are investigated in [2,21] and others papers. The papers [4] and [11] are devoted to geometrization of three-dimensional orbifolds.

The aim of this paper is to find an analog of Chern's conjecture for affine orbifolds and extend known results for affine manifolds to affine orbifolds, mainly the results of B. Klingler [12] and B. Kostant and D. Sullivan [14].

We prove the following statement.

Theorem 1. *Chern's conjecture for affine manifolds is equivalent to the following analog for orbifolds:*
The Euler–Satake characteristic of an n-dimensional compact affine orbifold has to vanish.

The holonomy group of an effective n-dimensional affine orbifold is a subgroup of $GL(n, \mathbb{R})$ defined up to conjugacy (see Section 3.3).

Definition 1. Let $\mathcal{N}$ be an n-dimensional affine orbifold and let $\mathcal{N}^{ef}$ be the associated effective affine orbifold. The orbifold $\mathcal{N}$ is said to be *special* if the holonomy group of $\mathcal{N}^{ef}$ is contained in the special linear group $SL(n, \mathbb{R})$.

The following statement extends the result of B. Klingler [12, Theorem 1.5] to orbifolds.

Theorem 2. *Let $\mathcal{N}$ be an n-dimensional compact affine orbifold. If $\mathcal{N}$ is special, then its Euler–Satake characteristic is zero.*

An orbifold $\mathcal{N}$ equipped with a pseudo-Riemannian metric g is called a *pseudo-Riemannian* orbifold. As an application of Theorem 2 we obtain the following statement.

Theorem 3. *Let $(\mathcal{N}, g)$ be an n-dimensional compact flat pseudo-Riemannian orbifold of arbitrary signature. Then the Euler–Satake characteristic of $\mathcal{N}$ is zero.*

The following statement extends the corresponding result of B. Kostant and D. Sullivan [14] to affine orbifolds.

Theorem 4. *If $\mathcal{N}$ be an n-dimensional compact complete affine orbifold, then the Euler–Satake characteristic of $\mathcal{N}$ is zero.*

Remark 1. It is known from [12, Remark 1.2] that Chern's conjecture holds true for complex affine manifolds. This fact and Proposition 4 imply that the Euler–Satake characteristic of any compact complex affine orbifold is zero.

Remark 2. For closed surfaces Chern's conjecture is proved by Benzécri [3]. By Theorem 1, the analog of Chern's conjecture is true for all two-dimensional compact affine orbifolds.

Examples constructed below show that for the topological and orbifold Euler characteristics of orbifolds the direct analog of Chern's conjecture is not true (see Section 5).

The paper is organized as follows. In Section 2.1, we recall some facts, concerning the orbifold category.

In Section 2.2, we discuss regular coverings of orbifolds and their properties.

Section 2.3 is focused on different equivalent approaches to the concept of Euler–Satake characteristic for orbifolds and its properties. The definitions of the topological and orbifold Euler characteristics are recalled in Section 2.4.

In Section 3.1, we consider an affine structure as a Cartan geometry. We define the holonomy group of an effective affine orbifold in Section 3.3.

The structure of compact affine orbifolds is investigated in Section 3.4. The key result is the statement that every compact affine orbifold is very good in Thurston's terminology, i.e. it is the global quotient of an affine manifold by a finite automorphism group (Proposition 4).

Theorem 1 is proved in Section 3.5. Theorems 2–4 are proved in Section 4.

In Section 5, we construct examples of computations of the Euler–Satake characteristic, the topological and orbifold Euler characteristics of some orbifolds (Examples 2–3). We also construct an example of a compact affine orbifold with zero Euler–Satake characteristic that is neither complete nor special (Example 1).

Assumption. All manifolds, orbifolds, vector fields, forms are assumed to be smooth of class C^∞. Orbifolds (in particular, manifolds) are assumed to be connected if otherwise is not mentioned.

Notations. We denote by $\mathfrak{X}(M)$ the module of vector fields over the algebra of functions on a manifold M. If $\mathfrak{N}$ is a smooth distribution on M then $\mathfrak{X}_{\mathfrak{N}}(M) := \{X \in \mathfrak{X}(M) \mid X_u \in \mathfrak{N}_u \quad \forall u \in M\}$.

We extend the notations of [13, Chapter 2] to orbifolds. We denote by $P(\mathcal{N}, H)$ a principal H-bundle over an orbifold $\mathcal{N}$ which is also an orbifold in the general case [2].

We denote by $\cong$ an isomorphism in an appropriate category.

The quotient spaces of X with respect to both left and right actions of a group G are denoted by X/G.

The order of a finite group G is denoted by $|G|$.

2. Orbifolds

2.1. The category of orbifolds

Denote by $\mathcal{N}$ a connected paracompact Hausdorff topological space and let $n \in \mathbb{N}$. Let $\tilde{U}$ be a connected open subset of $\mathbb{R}^n$, Γ_U a finite group acting on $\tilde{U}$, $\varphi_U : \tilde{U} \to \mathcal{N}$ a Γ_U-invariant map which induces a homeomorphism q_U of $\tilde{U}/\Gamma_U$ onto the open subset $U = \varphi_U(\tilde{U})$ of $\mathcal{N}$. Then the triple $(\tilde{U}, \Gamma_U, \varphi_U)$ is called an *orbifold chart* on $\mathcal{N}$ *with the coordinate neighborhood* U.

Let $(\tilde{U}, \Gamma_U, \varphi_U)$ and $(\tilde{V}, \Gamma_V, \varphi_V)$ be two orbifold charts with the coordinate neighborhoods U and V respectively, and $U \subset V$. An *embedding* of the chart $(\tilde{U}, \Gamma_U, \varphi_U)$ into the chart $(\tilde{V}, \Gamma_V, \varphi_V)$ is a pair $(\varphi_{VU}, \lambda_{VU})$, where $\lambda_{VU} : \Gamma_U \to \Gamma_V$ is a group monomorphism inducing an isomorphism of the kernels of the actions and $\varphi_{VU} : \tilde{U} \to \tilde{V}$ is a smooth embedding, which is equivariant in the sense that $\varphi_{VU}(\gamma(x)) = \lambda_{VU}(\gamma)(\varphi_{VU}(x))\ \forall \gamma \in \Gamma_U,\ \forall x \in \tilde{U}$.

An *orbifold atlas* on $\mathcal{N}$ is a family $\mathcal{A} = \{(\tilde{U}, \Gamma_U, \varphi_U)\}$ of orbifold charts, which cover $\mathcal{N}$ and are *locally compatible* in the following sense: for any two charts $(\tilde{U}, \Gamma_U, \varphi_U)$ and $(\tilde{V}, \Gamma_V, \varphi_V)$ with the coordinate neighborhoods U and V and for any $x \in U \cap V$, there exist an open neighborhood $W \subset U \cap V$ of x and a chart $(\tilde{W}, \Gamma_W, \varphi_W)$ with the coordinate neighborhood W such that there are embeddings $(\varphi_{UW}, \lambda_{UW})$ and $(\varphi_{VW}, \lambda_{VW})$ of the chart $(\tilde{W}, \Gamma_W, \varphi_W)$ into $(\tilde{U}, \Gamma_U, \varphi_U)$ and $(\tilde{V}, \Gamma_V, \varphi_V)$ respectively.

A connected paracompact Hausdorff topological space $\mathcal{N}$ equipped with a maximal (with respect to the inclusion) orbifold atlas $\mathcal{A}$ is called an *n-dimensional orbifold*, which will be also denoted by $\mathcal{N}$.

For an orbifold chart $(\tilde{U}, \Gamma_U, \varphi_U)$, let K_U^{ef} be the subgroup of Γ_U, which consists of all $\gamma \in \Gamma_U$ such that $\gamma \cdot \tilde{x} = \tilde{x}$ for any $\tilde{x} \in \tilde{U}$. As observed in [7, Section 2.2], for any two charts $(\tilde{U}, \Gamma_U, \varphi_U)$ and $(\tilde{V}, \Gamma_V, \varphi_V)$ with the coordinate neighborhoods U and V such that $U \cap V \neq \emptyset$, the subgroups K_U^{ef} and K_V^{ef} are isomorphic. This observation allows us to give the following definition.

Definition 2. The *inefficiency group* $K_{\mathcal{N}}^{ef}$ of a connected orbifold $\mathcal{N}$ with an atlas $\mathcal{A}$ is the group K_U^{ef} for an arbitrary chart $(\tilde{U}, \Gamma_U, \varphi_U) \in \mathcal{A}$. It is uniquely determined up to group isomorphism.

The inefficiency group $K_{\mathcal{N}}^{ef}$ is defined up to isomorphisms. If $K_{\mathcal{N}}^{ef}$ is trivial, then the group Γ_U acts effectively on $\tilde{U}$ for all charts of the atlas $\mathcal{A}$. In this case the orbifold $\mathcal{N}$ is called *effective*.

We emphasize that $K_{\mathcal{N}}^{ef}$ is a finite group.

Let $\mathcal{N}$ be an ineffective orbifold with atlas $\mathcal{A} = \{(\tilde{U}, \Gamma_U, \varphi_U)\}$. Then the group $\Gamma_U^{ef} = \Gamma_U/K_U^{ef}$ acts effectively on $\tilde{U}$. The atlas $\mathcal{A}^{ef} = \{(\tilde{U}, \Gamma_U^{ef}, \varphi_U)\}$ on the topological space $\mathcal{N}$ defines an orbifold $\mathcal{N}^{ef}$ which is called the effective orbifold associated to $\mathcal{N}$.

For each point $x \in \mathcal{N}$ of an n-dimensional effective orbifold $(\mathcal{N}, \mathcal{A})$, there exists a chart $(\tilde{U}, \Gamma_U, \varphi_U) \in \mathcal{A}$ such that $\tilde{U}$ is the n-dimensional arithmetic space $\mathbb{R}^n$, $\varphi_U(0) = x$ with $0 = (0, \dots, 0) \in \mathbb{R}^n$, and Γ_U is a finite group of orthogonal transformations of $\mathbb{R}^n$. Such a chart is called a *linearized chart at* x.

For orbifold charts $(\tilde{U}, \Gamma_U, \varphi_U)$ and $(\tilde{V}, \Gamma_V, \varphi_V)$ in $\mathcal{A}$ with the coordinate neighborhoods, containing $x \in \mathcal{N}$, the isotropy subgroups $(\Gamma_U)_y$ and $(\Gamma_V)_z$ of the points $y \in \varphi_U^{-1}(x)$ and $z \in \varphi_V^{-1}(x)$ are isomorphic. Therefore, for every point x of $\mathcal{N}$, there is a uniquely determined up to group isomorphism group $\Gamma_{(x)}$, called *the orbifold group* at x. A point x is said to be *regular* if its orbifold group $\Gamma_{(x)}$ is isomorphic to inefficiency group $K_{\mathcal{N}}^{ef}$ of $\mathcal{N}$; otherwise, it is called *singular*.

Let $\mathcal{N}$ and $\mathcal{N}'$ be smooth orbifolds with atlases $\mathcal{A}$ and $\mathcal{A}'$ respectively. A continuous map $f : \mathcal{N} \to \mathcal{N}'$ is called *smooth* if, for each $x \in \mathcal{N}$, there exist charts $(\tilde{U}, \Gamma_U, \varphi_U) \in \mathcal{A}$ and $(\tilde{U}', \Gamma_{U'}, \varphi_{U'}) \in \mathcal{A}'$ such that $x \in U = \varphi_U(\tilde{U})$, $f(U) \subset U' = \varphi_{U'}(\tilde{U}')$, a smooth map $f_{U'U} : \tilde{U} \to \tilde{U}'$, satisfying the identity $f \circ \varphi_U = \varphi_{U'} \circ f_{U'U}$, and for any $\gamma \in \Gamma_U$, there is $\gamma' \in \Gamma_{U'}$ such that $f_{U'U}(\gamma(x)) = \gamma'(f_{U'U}(x))\ \forall x \in \tilde{U}$. The map $f_{U'U}$ is called *the representative* of f in the charts $(\tilde{U}, \Gamma_U, \varphi_U)$ and $(\tilde{U}', \Gamma_{U'}, \varphi_{U'})$. It is defined up to composition with elements in Γ_U and $\Gamma_{U'}$ respectively.

A smooth map $f : \mathcal{N} \to \mathcal{N}'$ is called a *submersion*, if each of its representative $f_{U'U} : \tilde{U} \to \tilde{U}'$ is a submersion of manifolds.

Let $\mathfrak{Orb}$ denote the category of orbifolds, whose objects are smooth orbifolds and morphisms are smooth maps of orbifolds. Note that the category of manifolds is a complete subcategory in $\mathfrak{Orb}$.

Two points in an n-dimensional orbifold $\mathcal{N}$ are said to have the *same orbifold type*, if there exist neighborhoods of these points, which are isomorphic in the category $\mathfrak{Orb}$. The subset of points of the same orbifold type with the induced topology has a natural structure of a smooth manifold, which is disconnected in general [2]. Let Δ_k be the union of these manifolds of dimension k. The manifolds of points of different types may have the same dimension. Emphasize that every connected component $\Delta_k^{c_j}$ of Δ_k consists of points of the same type.

It is possible that $\Delta_k = \emptyset$ for some $k \in \{0, \dots, n-1\}$.

The family $\Delta(\mathcal{N}) = \{\Delta_k^{c_j} \mid j \in J_k,\ k \in \{0, \ldots, n\}\}$ is called the *stratification* of the n-dimensional orbifold $\mathcal{N}$, and $\Delta_k^{c_j}$ is called its *stratum*. This stratification is natural [15].

It is well known that the set of regular points is the stratum Δ_n. It is a connected, open, dense subset in $\mathcal{N}$, which is an n-dimensional manifold with the induced smooth structure.

2.2. Regular coverings of orbifolds

Recall the notion of a covering orbifold [19]. A smooth map $f: \mathcal{N}' \to \mathcal{N}$ of orbifolds $(\mathcal{N}', \mathcal{A}')$ and $(\mathcal{N}, \mathcal{A})$ is called a *covering* of the orbifold $\mathcal{N}$, if, for any $x \in \mathcal{N}$, there exists a chart $(\tilde{U}, \Gamma_U, \varphi_U) \in \mathcal{A}$ at x such that $f^{-1}(U)$ is a disjoint union of open subsets $V_i \subset \mathcal{N}'$, $i \in I$ (I is countable), and, for each $i \in I$, there is an isomorphism $\psi_i: \tilde{U}/\Gamma_i \to V_i$, where Γ_i is a subgroup of Γ_U with the same kernel K_U^{ef} of the action, such that $f|_{V_i} \circ \psi_i \circ p_i = \varphi_U$, where $p_i: \tilde{U} \to \tilde{U}/\Gamma_i$ is the quotient map.

Remark that the subgroups Γ_i are pairwise conjugate in the group Γ_U. The orbifold $\mathcal{N}'$ is called a *covering orbifold* of $\mathcal{N}$.

A deck transformation of a covering $f: \mathcal{N}' \to \mathcal{N}$ is a diffeomorphism $h: \mathcal{N}' \to \mathcal{N}'$ in the category $\mathfrak{Orb}$ such that $f \circ h = f$. The set $G(f)$ of all deck transformations of a covering f forms a group called the *deck transformation group*. A covering $f: \mathcal{N}' \to \mathcal{N}$ is called *regular*, if $\mathcal{N} = \mathcal{N}'/G(f)$. Remark that $\mathcal{N} = \mathcal{N}'/G(f)$ if and only if the group $G(f)$ acts transitively on every fiber $f^{-1}(x)$, $x \in \mathcal{N}$.

A diffeomorphism $\tilde{h}: \mathcal{N}' \to \mathcal{N}'$ of an orbifold $\mathcal{N}'$ is said to *lie over* a diffeomorphism $h: \mathcal{N} \to \mathcal{N}$ with respect to a covering $f: \mathcal{N}' \to \mathcal{N}$, if $f \circ \tilde{h} = h \circ f$.

A *universal cover* $k: \mathcal{N}_0 \to \mathcal{N}$ of an orbifold $\mathcal{N}$ is an initial object in the category of orbifold coverings, i.e., for any covering $f: \mathcal{N}' \to \mathcal{N}$, there is a unique covering $\nu: \mathcal{N}_0 \to \mathcal{N}'$, satisfying the equality $f \circ \nu = k$ [10].

As known, for a connected orbifold $\mathcal{N}$, a universal cover always exists. It is unique up to covering isomorphisms and it is a regular covering ([19, Proposition 5.3.3]).

The deck transformation group $G(k)$ of a universal cover $k: \mathcal{N}_0 \to \mathcal{N}$ is called *the orbifold fundamental group* of $\mathcal{N}$, it is denoted by $\pi_1^{orb}(\mathcal{N})$.

In the terminology of Thurston [19], if an orbifold $\mathcal{N}$ has a covering $f: \mathcal{N}' \to \mathcal{N}$ such that $\mathcal{N}'$ is a manifold, then $\mathcal{N}$ is called a *good orbifold*. If, moreover, the group $G(f)$ is finite, then $\mathcal{N}$ is called a *very good orbifold.* An orbifold $\mathcal{N}$ is called *bad,* if it is not good.

Now we establish orbifold analogs of some classical results on regular coverings, which we will need in the proof of Proposition 4. Their proofs are quite similar to the proofs of the classical results. We will give them for the sake of completeness.

Lemma 1. *Let $k: \mathcal{N}_0 \to \mathcal{N}$ be a universal cover of an orbifold $\mathcal{N}$. Then, for any diffeomorphism $h \in Diff(\mathcal{N})$, there is a diffeomorphism $\tilde{h} \in Diff(\mathcal{N}_0)$, lying over h.*

Proof. Let $h \in Diff(\mathcal{N})$. Then $h \circ k: \mathcal{N}_0 \to \mathcal{N}$ is a universal cover of $\mathcal{N}$. Since a universal cover is unique up to covering isomorphisms, there is a diffeomorphism $\tilde{h} \in Diff(\mathcal{N}_0)$ satisfying the equality $k \circ \tilde{h} = h \circ k$, i.e. $\tilde{h}$ lies over h. □

Let $f: \mathcal{N}' \to \mathcal{N}$ be a covering of an orbifold $\mathcal{N}$. Let $k: \mathcal{N}_0 \to \mathcal{N}$ be a universal cover of $\mathcal{N}$. In this case, there exists a covering $\nu: \mathcal{N}_0 \to \mathcal{N}'$ satisfying the equality $f \circ \nu = k$. Thus, $\mathcal{N}_0$ is a common universal cover orbifold of $\mathcal{N}$ and $\mathcal{N}'$. By definition, $\pi_1^{orb}(\mathcal{N}) = G(k)$ and $\pi_1^{orb}(\mathcal{N}') = G(\nu)$. For any $\tilde{h} \in G(\nu)$, we have $\nu \circ \tilde{h} = \nu$, hence $f \circ (\nu \circ \tilde{h}) = f \circ \nu$ if and only if $k \circ \tilde{h} = k$, i. e. $G(\nu)$ is a subgroup of $G(k)$.

Proposition 1. *A covering $f: \mathcal{N}' \to \mathcal{N}$ is regular if and only if $\pi_1^{orb}(\mathcal{N}')$ is a normal subgroup of $\pi_1^{orb}(\mathcal{N})$. Moreover, if f is a regular covering, the quotient group $\pi_1^{orb}(\mathcal{N})/\pi_1^{orb}(\mathcal{N}')$ is isomorphic to the deck transformation group $G(f)$.*

Proof. Let $f: \mathcal{N}' \to \mathcal{N}$ be a regular covering. We will use the notations introduced above. Since $\nu: \mathcal{N}_0 \to \mathcal{N}'$ is a universal cover, by Lemma 1, for any $h \in G(f)$, there exists an element $\tilde{h} \in Diff(\mathcal{N}_0)$ lying over h, i.e. $\nu \circ \tilde{h} = h \circ \nu$. The following chain of equalities $k \circ \tilde{h} = (f \circ \nu) \circ \tilde{h} = f \circ (\nu \circ \tilde{h}) = f \circ (h \circ \nu) = (f \circ h) \circ \nu = f \circ \nu = k$ implies that $k \circ \tilde{h} = k$ and $\tilde{h} \in G(k)$. Thus for any $h \in G(f)$ there is $\tilde{h} \in G(k)$ lying over h with respect to $\nu: \mathcal{N}_0 \to \mathcal{N}'$.

Let us show that every $\tilde{g} \in G(k)$ lies over some $h \in G(f)$ with respect to ν. As $G(\nu) \subset G(k)$ and $G(\nu)$ lies over $\mathrm{id}_{\mathcal{N}'}$, it is sufficient to consider $\tilde{g} \notin G(\nu)$. Fix a regular point $x_0 \in \mathcal{N}$ and a point $z_0 \in k^{-1}(x_0)$. Then $z_1 = \tilde{g}(z_0) \neq z_0$. Denote $y_0 = \nu(z_0)$, $y_1 = \nu(z_1)$. As $\tilde{g} \notin G(\nu)$, $y_0 \neq y_1$. Since $f: \mathcal{N}' \to \mathcal{N}$ is a regular covering, the group $G(f)$ acts transitively on the fiber $f^{-1}(x_0)$. Therefore, there is an element $h \in G(f)$ such that $h(y_0) = y_1$. As shown above, for $h \in G(f)$, there is $\tilde{h} \in G(k)$, lying over h with respect to ν. Remark that $\tilde{h} \circ \gamma$ and $\gamma \circ \tilde{h}$ lie over h for all $\gamma \in G(\nu)$. Let $\tilde{h}(z_0) = z_2$. Since $\tilde{h}$ lies over h with respect to ν, we have $\nu(z_2) = (\nu \circ \tilde{h})(z_0) = (h \circ \nu)(z_0) = h(y_0) = y_1$. Hence, $z_1, z_2 \in \nu^{-1}(y_1)$, and there exists $g_0 \in G(\nu)$, satisfying $g_0(z_2) = z_1$. Thus, $g_0 \circ \tilde{h}(z_0) = z_1$ and $\tilde{g}(z_0) = z_1$, where $g_0 \circ \tilde{h}, \tilde{g} \in G(k)$. Since the deck transformation group $G(k)$ acts simply transitively on the fiber $k^{-1}(x_0)$ over the regular point x_0, it is necessary that $\tilde{g} = g_0 \circ \tilde{h}$, hence $\tilde{g}$ lies over h.

Define a map $\rho \colon G(k) \to G(f)$ by $\rho(\tilde{g}) := h$, where $\tilde{g}$ lies over h with respect to ν. It is easy to check that ρ is a group homomorphism. As shown above, the map ρ is surjective. Since $\ker \rho = \{\tilde{g} \in G(k) \mid \nu \circ \tilde{g} = \nu\} = G(\nu)$, the group $G(\nu)$ is a normal subgroup of $G(k)$, and the group $G(f)$ is isomorphic to the quotient group $G(k)/G(\nu) = \pi_1^{orb}(\mathcal{N})/\pi_1^{orb}(\mathcal{N}')$.

Prove otherwise. Assume now that $G(\nu)$ is a normal subgroup of $G(k)$. For any $g \in G(k)$, define a map $h \colon \mathcal{N}' \to \mathcal{N}'$ by the equality $h(x) = (\nu \circ g)(\tilde{x})$, $x \in \mathcal{N}$, where $\tilde{x} \in \nu^{-1}(x)$ is an arbitrary point. Let us show that the definition is correct. Take another point $\tilde{x}' \in \nu^{-1}(x)$. Since the fundamental group $G(\nu)$ acts transitively on the fiber $\nu^{-1}(x)$ of the universal cover $\nu \colon \mathcal{N}_0 \to \mathcal{N}'$, there is $\gamma \in G(\nu)$ such that $\tilde{x}' = \gamma(\tilde{x})$. As $G(\nu)$ is a normal subgroup of $G(k)$, there exists $\gamma' \in G(\nu)$, satisfying the equality $g \circ \gamma = \gamma' \circ g$. The chain of equalities $\nu \circ g(\tilde{x}') = \nu \circ (g \circ \gamma)(\tilde{x}) = \nu \circ (\gamma' \circ g)(\tilde{x}) = (\nu \circ \gamma') \circ g(\tilde{x}) = \nu \circ g(\tilde{x})$ implies that the map $h \colon \mathcal{N}' \to \mathcal{N}'$ is independent of the choice of $\tilde{x} \in \nu^{-1}(x)$, i.e. h is well defined. It is not difficult to show that $h \in \mathit{Diff}(\mathcal{N}')$. By definition of h, we have $h \circ \nu(\tilde{x}) = \nu \circ g(\tilde{x})$ $\forall \tilde{x} \in \mathcal{N}_0$, i.e. g lies over h with respect to ν.

Therefore, we have a map $\mu \colon G(k) \to \mathit{Diff}(\mathcal{N}')$ given by $\mu(g) := h$, where g lies over h with respect to ν. Hence $\mu(g) = \rho(g)$, and $G(f) = \rho(G(k))$. Let us show that the group $G(f)$ acts transitively on every fiber $f^{-1}(z)$, $z \in \mathcal{N}$. Let $z_1, z_2 \in f^{-1}(z)$ and $y_1 \in \nu^{-1}(z_1)$, $y_2 \in \nu^{-1}(z_2)$. Since $y_1, y_2 \in k^{-1}(z)$, there exists a deck transformation $g \in G(k)$ such that $g(y_1) = y_2$. Put $h = \rho(g) \in G(f)$, hence $h(z_1) = z_2$. Therefore, the group $G(f)$ acts transitively on every fiber $f^{-1}(z)$, $z \in \mathcal{N}$, and $f : \mathcal{N}' \to \mathcal{N}$ is a regular covering. $\square$

2.3. The Euler–Satake characteristic of orbifolds and its properties

Let $\mathcal{N}$ be a compact n-dimensional orbifold and let $\mathcal{N}^{ef}$ be the effective orbifold associated to $\mathcal{N}$. It is known that $\mathcal{N}$ (as well as $\mathcal{N}^{ef}$) admits a good finite triangulation; i.e. a finite triangulation such that the orbifold groups at the points in the interior of each simplex σ are the same [7,15]. Let $\mathcal{K}$ be such a triangulation of $\mathcal{N}$. The *Euler–Satake characteristic* of $\mathcal{N}$ is defined by the formula [17,19]

$$\chi^{ES}(\mathcal{N}) = \sum_{\sigma \in \mathcal{K}} (-1)^{\dim \sigma} \frac{1}{|\Gamma_\sigma|}, \tag{1}$$

where Γ_σ is the orbifold group at any point in the interior of σ.

Let us introduce another, equivalent definition of the Euler–Satake characteristic. Let X be a vector field on $\mathcal{N}$, x be a singularity of X, and $(\tilde{U}, \Gamma_U, \varphi_U)$ be a chart with the coordinate neighborhood $U \ni x$. Denote by X_U the lift of X to $\tilde{U}$ under the map $\varphi_U \colon \tilde{U} \to U \subset \mathcal{N}$. Then the vector field X_U has a singularity at $\tilde{x} \in (\varphi_U)^{-1}(x)$. Let $I_{\tilde{x}}(X_U)$ denote the index of singularity of X_U at $\tilde{x}$ in the usual sense. As known from [17, Section 3], the number $I_x(X) = \frac{1}{|\Gamma_x|} I_{\tilde{x}}(X_U)$ is independent of the choices of $(\tilde{U}, \Gamma_U, \varphi_U)$ and $\tilde{x}$. The number $I_x(X)$ is called *the index of singularity* of X at x. Note that $I_x(X)$ is not necessarily integer.

In [17, Theorem 3], I. Satake proved the following generalization of the Poincaré–Hopf theorem: if X is a vector field with singularities $x_1, \dots, x_k$ on a compact orbifold $\mathcal{N}$, then

$$\chi^{ES}(\mathcal{N}) = \sum_{i=1}^{k} I_{x_i}(X), \tag{2}$$

This formula may serve as an equivalent definition of the Euler–Satake characteristic of $\mathcal{N}$.

One more, equivalent approach to the Euler–Satake characteristic for an orbifold $\mathcal{N}$ was presented by H. Ding [5]. He introduced the concept of Euler characteristic of Fukaya–Ono type $\chi^{FO}(\mathcal{N})$ and proved that, for a compact orbifold $\mathcal{N}$, this characteristic $\chi^{FO}(\mathcal{N})$ coincides with the Euler–Satake characteristic $\chi^{ES}(\mathcal{N})$.

By [7, Section 2.2], if $\mathcal{N}$ is a connected ineffective orbifold, then

$$\chi^{ES}(\mathcal{N}) = \frac{1}{|K^{ef}_{\mathcal{N}}|} \chi^{ES}(\mathcal{N}^{ef}), \tag{3}$$

where $K^{ef}_{\mathcal{N}}$ is the inefficiency group of $\mathcal{N}$.

I. Satake showed that the Euler–Satake characteristic of an odd dimensional compact orbifold is zero [17, Theorem 4].

The Euler–Satake characteristic of an orbifold has the additivity and multiplicativity properties [7, Section 2.2]. Let $\mathcal{K}$ be a good triangulation of an orbifold $\mathcal{N}$. Suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ are closed subsets of $\mathcal{N}$ such that $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{N}$, and the sets $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_1 \cap \mathcal{N}_2$ correspond to subcomplexes of $\mathcal{K}$; then

$$\chi^{ES}(\mathcal{N}_1 \cup \mathcal{N}_2) = \chi^{ES}(\mathcal{N}_1) + \chi^{ES}(\mathcal{N}_2) - \chi^{ES}(\mathcal{N}_1 \cap \mathcal{N}_2).$$

The Euler–Satake characteristic of the product $\mathcal{N}_1 \times \mathcal{N}_2$ of compact orbifolds $\mathcal{N}_1$ and $\mathcal{N}_2$ is equal to the product of their Euler–Satake characteristics, i.e.

$$\chi^{ES}(\mathcal{N}_1 \times \mathcal{N}_2) = \chi^{ES}(\mathcal{N}_1) \cdot \chi^{ES}(\mathcal{N}_2).$$

The following statement is given in [7, Lemma 2.2] without a proof. It is stated in [19, Proposition 13.3.4] in Section 13.3 "Two-dimensional orbifolds" with a short proof. In view of importance of the statement for the sequel, we give its new proof here.

Proposition 2. *Let $f:\mathcal{N}'\to\mathcal{N}$ be a regular covering of a compact orbifold $\mathcal{N}$ whose deck transformations group G is finite. Then*

$$\chi^{ES}(\mathcal{N})=\frac{1}{|G|}\chi^{ES}(\mathcal{N}'). \tag{4}$$

Proof. Consider the case when $\mathcal{N}$ is an effective orbifold. Then $\mathcal{N}'$ is also effective. Let X be a vector field on $\mathcal{N}$ with finitely many singular points $\{x_1,\dots,x_k\}$. Denote X' to be the lift of X to $\mathcal{N}'$ under the covering f and $f^{-1}(x_i)=\{x_{i1},\dots,x_{im_i}\}$. Then each $x_{ij}\in f^{-1}(x_i)$ is a singular point of X'. Since the points $x_{i1},\dots,x_{im_i}$ belong to the same orbit of G, their isotropy groups are pairwise isomorphic, and the number m_i is equal to $|G/G_{x_{i1}}|=|G|/|G_{x_{i1}}|$. Moreover, since the notion of the index of singularity is local, $I_{x_{i1}}(X')=\cdots=I_{x_{im_i}}(X')$, where $I_{x_{ij}}(X')$ is the index of singularity of X' at x_{ij}, $i=1,\dots,k$, $j=1,\dots,m_i$.

By the definition of a covering orbifold, for points $x_i\in\mathcal{N}$ and $x_{i1}\in\mathcal{N}'$, there are charts $(\tilde U,\Gamma_U,\varphi_U)$ and $(\tilde U',\Gamma_{U'},\varphi_{U'})$ with the following properties: (a) $\tilde U=\tilde U'$ and $\Gamma_{U'}$ is a subgroup of the group Γ_U; (b) $x_i\in U=\varphi_U(\tilde U)$ and $x_{i1}\in U'=\varphi_{U'}(\tilde U')$; (c) $f|_{U'}\circ\varphi_{U'}=\varphi_U$. As above, let Γ_{x_i} and $\Gamma_{x_{i1}}$ denote the orbifold groups at x_i and x_{i1} respectively. Without loss of generality, we assume that $\Gamma_{x_{i1}}\subset\Gamma_{x_i}$. It is easy to see that the isotropy group $G_{x_{i1}}$ is isomorphic to the quotient group $\Gamma_{x_i}/\Gamma_{x_{i1}}$. Therefore, $m_i=|G|/|G_{x_{i1}}|=\frac{|G||\Gamma_{x_{i1}}|}{|\Gamma_{x_i}|}$.

Let X_U be the lift of X to $\tilde U$. Then X_U is also the lift of X' to $\tilde U'=\tilde U$. Therefore, $I_{x_{i1}}(X')=\frac{1}{|\Gamma_{x_{i1}}|}I_{\tilde x_i}(X_U)$, where $\tilde x_i\in(\varphi_U)^{-1}(x_i)$. Thus, using (2), we get the following chain of equalities:

$$\chi^{ES}(\mathcal{N}')=\sum_{i=1}^{k}\sum_{j=1}^{m_i}I_{x_{ij}}(X')=\sum_{i=1}^{k}m_iI_{x_{i1}}(X')=\sum_{i=1}^{k}\frac{|G||\Gamma_{x_{i1}}|}{|\Gamma_{x_i}|}\cdot\frac{1}{|\Gamma_{x_{i1}}|}I_{\tilde x_i}(X_U)=$$
$$=|G|\sum_{i=1}^{k}\frac{1}{|\Gamma_{x_i}|}I_{\tilde x_i}(X_U)=|G|\sum_{i=1}^{k}I_{x_i}(X)=|G|\chi^{ES}(\mathcal{N}).$$

Suppose that $\mathcal{N}$ and $\mathcal{N}'$ are ineffective. Since $f:\mathcal{N}'\to\mathcal{N}$ is a covering, it is necessary $K^{ef}_{\mathcal{N}}\cong K^{ef}_{\mathcal{N}'}$. Therefore, by (3), the formula (4) remains valid in this case. □

Corollary 1. *Let G be a finite group acting on a compact manifold M and let $\mathcal{N}=M/G$ be a very good orbifold. Then*

$$\chi^{ES}(\mathcal{N})=\frac{1}{|G|}\chi(M),$$

where $\chi(M)$ is the topological Euler characteristic of the manifold M.

2.4. The topological and orbifold Euler characteristics

Let $\mathcal{N}$ be a compact orbifold. The Euler characteristic of the underlying topological space of $\mathcal{N}$ is called the *topological Euler characteristic* of $\mathcal{N}$ and is denoted by $\chi(\mathcal{N})$. Note that $\chi(\mathcal{N})$ is always an integer. If $\mathcal{N}$ is a manifold, then $\chi^{ES}(\mathcal{N})$ coincides with $\chi(\mathcal{N})$.

Let G be a finite group acting on a compact manifold X. Then the compact quotient space $\mathcal{N}=X/G$ admits an orbifold structure. Recall that, in this case, the orbifold $\mathcal{N}$ is called very good. As known (see, for example, [9]), the topological Euler characteristic of a very good orbifold $\mathcal{N}$ can be calculated by the following formula

$$\chi(\mathcal{N})=\frac{1}{|G|}\sum_{g\in G}\chi(X^g), \tag{5}$$

where X^g is the fixed point set of $g\in G$. It is well-known that X^g is a smooth submanifold of X which is disconnected in general.

L. Dixon, J. A. Harvey, C. Vafa, E. Witten [6] introduced the orbifold Euler characteristic $\chi^{orb}(\mathcal{N})$ for a very good compact orbifold $\mathcal{N}=X/G$. Denote by $[g]$ the conjugacy class of an element $g\in G$. Let $C(g)$ be the centralizer of g in G. As known [9], the orbifold Euler characteristic $\chi^{orb}(\mathcal{N})$ of $\mathcal{N}$ can be calculated by the formula

$$\chi^{orb}(\mathcal{N})=\sum_{[g]}\chi(X^g/C(g)), \tag{6}$$

where $\chi(X^g/C(g))$ is the topological Euler characteristic of the quotient $X^g/C(g)$.

If the action of G on X is free, then $\chi(\mathcal{N})=\chi^{orb}(\mathcal{N})$. If G is an abelian group, then the conjugacy class $[g]$ consists of the element g, $C(g)=G$ and the formula (6) can be rewritten in the following form

$$\chi^{orb}(\mathcal{N})=\sum_{g\in G}\chi(X^g/G). \tag{7}$$

In contrast with the Euler–Satake characteristic, the orbifold Euler characteristic is always an integer.

Remark 3. Generalized Euler characteristics for orbifolds are defined in [7,8].

3. Equivalence of Chern's conjecture to its analog for affine orbifolds

3.1. A linear connection as a Cartan geometry

Let us first recall the definition of a Cartan geometry [20]. Let G be a Lie group and H be a closed subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. Let N be a smooth (not necessarily connected) manifold. *A Cartan geometry* on N of type (G, H) (or $\mathfrak{g}/\mathfrak{h}$) is a principal right H-bundle $P(N, H)$ with the projection $p : P \to N$ and a $\mathfrak{g}$-valued 1-form β on P, satisfying the following conditions:

(c_1) the map $\beta_w : T_wP \to \mathfrak{g}$ is an isomorphism of vector spaces for every $w \in P$ (nondegeneracy of β);
(c_2) $R_h^*\beta = \mathrm{Ad}_G(h^{-1})\beta$ for all $h \in H$, where $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation of G on $\mathfrak{g}$ (equivariance of β);
(c_3) $\beta(A^*) = A$ for any $A \in \mathfrak{h}$, where A^* is the fundamental vector field associated with A.

This Cartan geometry is denoted by $\xi = (P(N, H), \beta)$.

Let us consider the homogeneous space G/H and let G act on G/H by left translations. If there exists an $\mathrm{Ad}_G(H)$-invariant vector subspace V of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus V$, then G/H is called *reductive*. A Cartan geometry $\xi = (P(N, H), \beta)$ of type (G, H), where G/H is a reductive homogeneous space, is called *a reductive Cartan geometry*.

A $\mathfrak{g}$-valued 2-form $\Omega = d\beta + [\beta, \beta]$ is said to be the *curvature* of a Cartan geometry $\xi = (P(N, H), \beta)$. If $\Omega = 0$, the Cartan geometry ξ is called *flat*.

Cartan geometries form a category $\mathfrak{Car}$, where morphisms of two Cartan geometries $\xi = (P(M, H), \beta)$ and $\xi' = (P'(M', H), \beta')$ of the same type (G, H) are principle H-bundles morphisms $\Gamma : P(M, H) \to P'(M', H)$, satisfying the condition $\Gamma^*\beta' = \beta$.

Denote by $Aff(A^n)$ the group of all affine transformations of the n-dimensional affine space A^n. Elements of $Aff(A^n)$ can be represented in the form $\langle B, b\rangle$, where $B \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. The group operation in $Aff(A^n)$ is defined by the formula $\langle B, b\rangle\langle C, c\rangle = \langle BC, Bc + b\rangle \ \forall\langle B, b\rangle, \langle C, c\rangle \in Aff(A^n)$. Therefore, $Aff(A^n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$ is the semidirect product of the subgroup $GL(n, \mathbb{R})$ and the normal subgroup $\mathbb{R}^n$.

Further, we put $G = Aff(A^n)$ and $H = GL(n, \mathbb{R})$. The Lie algebra $\mathfrak{g}$ of G is the semidirect sum of the Lie algebra $\mathfrak{h}$ of $GL(n, \mathbb{R})$ and the Lie algebra $\mathfrak{p}$ of $\mathbb{R}^n$. By the $Ad_G(H)$-invariance of $\mathfrak{p}$, G/H is reductive. Consider the frame bundle $\mathcal{T}(M, H)$ over an n-dimensional manifold M. Recall that an H-connection in this bundle is an H-invariant n-dimensional distribution Q on $\mathcal{T}(M, H)$. We emphasize that there is a one-to-one correspondence between linear connections ∇ on M and H-connections Q in $\mathcal{T}(M, H)$. Note that, in the notation introduced above, a linear connection ∇ on M can be considered as a reductive Cartan geometry $\xi = (\mathcal{T}(M, H), \beta)$ of type G/H with $\beta = \omega + \theta$, where ω is the $\mathfrak{h}$-valued 1-form of the connection Q associated with ∇ and θ is the $\mathfrak{p}$-valued soldering 1-form of Q.

3.2. Affine orbifolds

Let $\mathcal{N}$ be an n-dimensional orbifold with atlas $\mathcal{A} = \{(\tilde{U}, \Gamma_U, \varphi_U)\}$. Suppose that for every chart $(\tilde{U}, \Gamma_U, \varphi_U) \in \mathcal{A}$, the set $\tilde{U}$ is an open subset of the affine space A^n and the group Γ_U acts on $\tilde{U}$ by restrictions of transformations from the affine group $Aff(A^n)$. This action may be ineffective. If every embedding $\varphi_{VU}: \tilde{U} \to \tilde{V}$, satisfying $\varphi_V \circ \varphi_{VU} = \varphi_U$, is a restriction of an affine transformation of A^n, then $\mathcal{N}$ is called an *affine orbifold*.

3.3. The holonomy group of an affine orbifold

In the sequel, we assume that the orbifolds under consideration are effective unless otherwise stated. We present another approach to the concept of affine orbifold. The definition of a fiber bundle over an orbifold is known (see for example [1,2]). The total space of a fiber bundle over an orbifold $\mathcal{N}$ is an orbifold $\mathcal{P}$ and its projection $\pi : \mathcal{P} \to \mathcal{N}$ is a submersion in the category $\mathfrak{Orb}$. Here we will only work with principal bundles whose total spaces are manifolds.

We consider H-spaces, where a Lie group H acts on the right.

Let a Lie group H act properly and almost freely on a manifold P. Then there exists an orbifold structure on the quotient space P/H such that the quotient map $f : P \to P/H$ is a submersion in the orbifold category $\mathfrak{Orb}$. This structure is unique up to isomorphisms in $\mathfrak{Orb}$. Let $h : P/H \to \mathcal{N}$ be an isomorphism of orbifolds. Then we say that we have a *good principal H-bundle P over the orbifold $\mathcal{N}$ with the projection* $\pi = h \circ f : P \to \mathcal{N}$ and use the notation $P(\mathcal{N}, H)$.

The frame bundle $\mathcal{T}(\mathcal{N}, H)$ over an n-dimensional orbifold $\mathcal{N}$ is an example of a good principal H-bundle over $\mathcal{N}$, where $H = GL(n, \mathbb{R})$. By analogy with the manifold case, an H-connection Q in $\mathcal{T}(\mathcal{N}, H)$ is an H-invariant n-dimensional distribution Q on $\mathcal{T}$. We emphasize the correspondence between linear connections ∇ on $\mathcal{N}$ and H-connections Q in the frame bundle $\mathcal{T}(\mathcal{N}, H)$.

Consider an affine geometry on an orbifold $\mathcal{N}$ as a Cartan geometry $\xi = (\mathcal{T}(\mathcal{N}, H), \beta)$ defined in Section 3.1, where β is a nondegenerate $\mathfrak{g}$-valued equivariant 1-form on $\mathcal{T}$. A diffeomorphism h of $\mathcal{T}$ is an automorphism of the affine orbifold $\mathcal{N}$ if $h^*\beta = \beta$ and $h \circ R_a = R_a \circ h \ \ \forall a \in H$.

Note that an n-dimensional orbifold $\mathcal{N}$ is affine if and only if there exists a flat torsion free linear connection ∇ on $\mathcal{N}$. The vanishing both of the torsion and of the curvature of the affine geometry on $\mathcal{N}$ is equivalent to the vanishing of the curvature of the associated Cartan geometry: $\Omega = 0$. Using properties of the curvature of a Cartan geometry [20, Section 1.5], it is easy to check the equality

$$\Omega_u(X_u, Y_u) = [\beta_u(X_u), \beta_u(Y_u)] - \beta_u([X, Y]_u), \quad X, Y \in \mathfrak{X}(\mathcal{T}),\ u \in \mathcal{T}. \tag{8}$$

In other words, Ω may be interpreted as a measure of the difference between the Lie algebra brackets and the brackets of the corresponding vector fields on $\mathcal{T}$.

Fix a basis E_i, $i = 1, \ldots, n$, of the vector space $\mathfrak{p} \cong \mathbb{R}^n$. The vector fields $\mathcal{E}_i$, where $\mathcal{E}_i|_u = (\beta_u)^{-1}(E_i)$, $u \in \mathcal{T}$, belong to $\mathfrak{X}_Q(\mathcal{T})$ and $\{\mathcal{E}_i|_u\}$ is a basis of Q_u. Since $\Omega = 0$ and $[A, B] = 0$ for all $A, B \in \mathfrak{p}$, by (8), it is necessary $[\mathcal{E}_i, \mathcal{E}_j]_u = \beta_u^{-1}([E_i, E_j]) = 0$ for each $i, j \in \{1, \ldots, n\}$ and for every $u \in \mathcal{T}$. This implies that $[X, Y] \in \mathfrak{X}_Q(\mathcal{T})$ for every vector fields $X, Y \in \mathfrak{X}_Q(\mathcal{T})$. By the Frobenius theorem, the distribution Q is integrable. Thus, $(\mathcal{N}, \nabla)$ is an affine orbifold if and only if the H-connection Q on $\mathcal{T}$ defined by the linear connection ∇ is an integrable distribution, i.e. there exists an n-dimensional foliation $(\mathcal{T}, F)$ such that $TF = Q$. This foliation is called *horizontal*. The H-invariance of $Q = TF$ implies the H-invariance of the foliation $(\mathcal{T}, F)$.

Definition 3. Take any $x \in \mathcal{N}$ and $u \in \pi^{-1}(x)$. Let $L = L(u)$ be the leaf of $(\mathcal{T}, F)$ through u. The subgroup $\Phi(u) = \{a \in H \mid R_a(L) = L\}$ of the group $H = GL(n, \mathbb{R})$ is called the *holonomy group* at u of the affine orbifold $\mathcal{N}$.

If $u' = ub$, $b \in H$, then $\Phi(u') = b^{-1}\Phi(u)b$. Thus, the holonomy group of $\mathcal{N}$ is defined up to conjugacy in H.

Remark 4. In the case when $\mathcal{N}$ is an affine manifold, Definition 3 is equivalent to the usual definition of the holonomy group ([13, Chap II, Section 4]).

Proposition 3. *Let $\mathcal{N}$ be an n-dimensional affine orbifold which is not necessarily compact. Let $\pi : \mathcal{T} \to \mathcal{N}$ be the projection of the frame bundle $\mathcal{T}(\mathcal{N}, H)$. Then:*

(i) *the affine connection on $\mathcal{N}$ defines an integrable H-invariant distribution Q tangent to the foliation $(\mathcal{T}, F)$;*

(ii) *for every leaf $L = L(u)$ of $(\mathcal{T}, F)$, $u \in \mathcal{T}$, the restriction $\pi|_L : L \to \mathcal{N}$ is a regular covering whose deck transformation group is the holonomy group $\Phi(u)$.*

Proof. The statement (i) was proved above in this section.

(ii) Let us describe the local structure of the principal H-bundle $\pi : \mathcal{T} \to \mathcal{N}$. Take any $x \in \mathcal{N}$ and a linearized orbifold chart $(\tilde{U}, \Gamma_U, \varphi_U)$ at x. In this case, Γ_U can be considered as a subgroup of H. Hence, Γ_U acts freely on $\tilde{U} \times H$ by the formula $\gamma(y, a) := (\gamma(y), \gamma^{-1} \cdot a)$, $(y, a) \in \tilde{U} \times H$. The quotient manifold $(\tilde{U} \times H)/\Gamma_U$ as well as the projection $r_U : (\tilde{U} \times H)/\Gamma_U \to \tilde{U}/\Gamma_U$ are defined. Denote by $\Gamma_U.(y, a)$ the Γ_U-orbit of $(y, a) \in \tilde{U} \times H$. The equality $\Gamma_U.(y, a) \cdot b := \Gamma_U.(y, a \cdot b)$, $b \in H$ defines a right action of H on $(\tilde{U} \times H)/\Gamma_U$. Since Γ_U acts freely on H, we have a locally trivial bundle with the projection $(\tilde{U} \times H)/\Gamma_U \to H/\Gamma_U$ and with the standard fiber $\tilde{U} \cong \mathbb{R}^n$. The fibers of this bundle form a simple foliation, which is called *horizontal*.

Thus, we have the following commutative diagram

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "p_U"] \arrow[d, "\pi|_{\pi^{-1}(U)}"] & (\tilde{U} \times H)/\Gamma_U \arrow[d, "r_U"] \\
U & \tilde{U}/\Gamma_U, \arrow[l, "q_U"]
\end{tikzcd}$$

where $p_U : \pi^{-1}(U) \to (\tilde{U} \times H)/\Gamma_U$ is an isomorphism both of the induced principal H-bundles and of the horizontal foliations. Let us identify $\pi^{-1}(U)$ with $(\tilde{U} \times H)/\Gamma_U$ via p_U.

Let $u \in \pi^{-1}(x)$ and $L = L(u)$ be the leaf of $(\mathcal{T}, F)$. Therefore, $(\pi|_L)^{-1}(U) = L \cap \pi^{-1}(U) = \coprod V_i$, where $V_i \cong \mathbb{R}^n$ are local leaves of the horizontal foliation, and there exists a diffeomorphism $s_i : V_i \to \tilde{U}$, satisfying $\pi|_{V_i} = \varphi_U \circ s_i$. Therefore, $\pi|_{V_i} : V_i \to U \subset \pi(L)$ is a diffeomorphism. This means that the restriction $\pi|_L : L \to \pi(L)$ is a covering onto an open subset $\pi(L)$ in $\mathcal{N}$.

Let us show that $\pi(L) = \mathcal{N}$. Suppose that there is $x' \in \mathcal{N} \setminus \pi(L)$. Take $u' \in \pi^{-1}(x')$. Let L' be the leaf of $(\mathcal{T}, F)$ through u'. As proved above, $\pi(L')$ is open in $\mathcal{N}$. Assume that there exists $x_0 \in \pi(L) \cap \pi(L')$. Then there are $y \in \pi^{-1}(x_0) \cap L$ and $y' \in \pi^{-1}(x_0) \cap L'$. Since H acts transitively on every fiber of $\pi : \mathcal{T} \to \mathcal{N}$, there exists $b \in H$ such that $R_b(y') = y$. The H-invariance of $(\mathcal{T}, F)$ implies $R_b(L') = L$. Therefore $\pi(L') = \pi(L)$. This contradicts with the choice of x', hence $\pi(L) \cap \pi(L') = \emptyset$. Thus, for any $x' \in \mathcal{N} \setminus \pi(L)$, there is an open neighborhood $\pi(L')$ belonging to $\mathcal{N} \setminus \pi(L)$, hence $\pi(L)$ is a closed subset in $\mathcal{N}$. Since $\pi(L)$ is an open-closed subset in $\mathcal{N}$, the connectivity of $\mathcal{N}$ implies $\pi(L) = \mathcal{N}$.

Therefore, $\pi|_L : L \to \mathcal{N}$ is a covering. The transitivity of the H-action on every fiber $\pi^{-1}(x)$, $x \in \mathcal{N}$, implies the transitivity of the action of the holonomy group $\Phi(u)$ on every fiber $(\pi|_L)^{-1}(x)$. This means that $\Phi(u)$ is the deck transformation group of this covering. Thus $\mathcal{N} = L/\Phi(u)$, hence $\pi|_L : L \to \mathcal{N}$ is a regular covering. □

3.4. The structure of affine orbifolds

We call an affine orbifold $\mathcal{N}$ very good, if there exist an affine manifold M and a finite group Ψ of automorphisms of M such that $\mathcal{N} = M/\Psi$.

Proposition 4. *Every compact n-dimensional affine orbifold is very good.*

Proof. Let $\mathcal{N}$ be a compact n-dimensional affine orbifold. By Proposition 3, there exists a regular covering $f: L \to \mathcal{N}$, where L is a manifold and the deck transformation group $\Phi = \Phi(u)$ is a subgroup of $H = GL(n, \mathbb{R})$. Therefore, Φ is a matrix group.

If $\nu: L^0 \to L$ is a universal cover for L, then $k = f \circ \nu: L^0 \to \mathcal{N}$ is a universal cover for $\mathcal{N}$. Since $f: L \to \mathcal{N}$ is a regular covering, by Proposition 1, the deck transformation group Φ is isomorphic to the quotient group $\pi_1^{orb}(\mathcal{N})/\pi_1^{orb}(L)$. Recall ([10, p. 132]) that the fundamental group of a compact orbifold is finitely generated. Therefore, the group Φ is finitely generated as the quotient group of the finitely generated group $\pi_1^{orb}(\mathcal{N})$ by $\pi_1^{orb}(L)$.

According to a fundamental result of A. Selberg [18], a finitely generated linear group over a field of characteristic zero is virtually torsion-free. Therefore, the finitely generated matrix group Φ has a normal subgroup Φ_0 without finite subgroups such that the quotient group $\Psi = \Phi/\Phi_0$ is finite.

It is easy to see that the affine structure on $\mathcal{N}$ induces an affine structure on L such that the deck transformation group Φ becomes a group of affine automorphisms of L. Since $\mathcal{N} = L/\Phi$, the group Φ acts on L properly discontinuously. Hence, the torsion-free subgroup Φ_0 of Φ acts freely and properly discontinuously on L. Therefore, the group $\Psi = \Phi/\Phi_0$ acts on the affine quotient manifold $M = L/\Phi_0$ as an automorphism group, and $\mathcal{N} = M/\Psi$. The quotient maps $s: L \to L/\Phi_0 = M$ and $q: M \to M/\Psi = \mathcal{N}$ satisfy $f = q \circ s$. Thus, $\mathcal{N} = M/\Psi$, where M is an n-dimensional affine manifold and Ψ is a finite group of its automorphisms. □

3.5. The proof of Theorem 1

Let $\mathcal{N}$ be an effective n-dimensional compact affine orbifold. By Proposition 4, $\mathcal{N}$ is very good and there exist an n-dimensional affine manifold M and a finite automorphism group Ψ of M such that $\mathcal{N} = M/\Psi$. By Proposition 2, the Euler–Satake characteristic $\chi^{ES}(\mathcal{N})$ of $\mathcal{N}$ satisfies the equality

$$\chi^{ES}(\mathcal{N}) = \frac{1}{|\Psi|}\chi(M), \tag{9}$$

where $\chi(M)$ is the Euler characteristic of M.

If $\mathcal{N}$ is ineffective and has the inefficiency group $K_{\mathcal{N}}^{ef}$, then there exists the associated effective orbifold $\mathcal{N}^{ef}$. In this case, by the formula (3), we obtain the following equality

$$\chi^{ES}(\mathcal{N}) = \frac{1}{|\Psi||K_{\mathcal{N}}^{ef}|}\chi(M). \tag{10}$$

The equalities (9) and (10) imply the equivalence of Chern's conjecture with its analog for the Euler–Satake characteristic of compact affine orbifolds stated above. □

4. Proofs of Theorems 2–4

Since the Euler–Satake characteristics of a compact affine orbifold $\mathcal{N}$ and the associated effective orbifold $\mathcal{N}^{ef}$ satisfy the equality $\chi^{ES}(\mathcal{N}) = \frac{1}{|K_{\mathcal{N}}^{ef}|}\chi^{ES}(\mathcal{N}^{ef})$, it is sufficient to prove Theorems 2–4 for effective orbifolds. Therefore, without loss of generality, we will further assume that all orbifolds under consideration are effective.

4.1. Special affine orbifolds

First, we prove the following lemma. As above, we put $H = GL(n, \mathbb{R})$.

Lemma 2. *Let M be an n-dimensional compact affine manifold admitting a finite automorphism group Ψ effectively acting on M. Let $\mathcal{N} = M/\Psi$ and let $q: M \to M/\Psi = \mathcal{N}$ be the quotient map. Denote by $\widehat{\xi} = (P(M, H), \alpha)$ and $\xi = (\mathcal{T}(\mathcal{N}, H), \beta)$ the Cartan geometries defined by the affine structures on M and $\mathcal{N}$ respectively. Then:*

(i) The induced automorphism group $\widehat{\Psi} \cong \Psi$ of ξ acts on P such that $\mathcal{T} = P/\widehat{\Psi}$ and the quotient map $\widehat{q}: P \to P/\widehat{\Psi} = \mathcal{T}$ satisfies the conditions:

$$R_a \circ \widehat{q} = \widehat{q} \circ R_a \quad \forall a \in H; \quad \widehat{q}_*(\widehat{Q}) = Q,$$

where $\hat{q}_*$ is the differential of $\hat{q}$, $\widehat{Q}$ and Q are the H-connections given by the affine structures on M and $\mathcal{N}$ respectively.

(ii) The holonomy groups $\widehat{\Phi}(\widehat{u})$, $\widehat{u} \in P$, and $\Phi(u)$, $u = \widehat{q}(\widehat{u}) \in \mathcal{T}$, satisfy the inclusions

$$\widehat{\Phi}(\widehat{u}) \subset \Phi(u) \subset H. \tag{11}$$

Proof. Denote by $p : P \to M$ the projection of the frame bundle $P(M, H)$. If $\widehat{u} = \{X_1, \dots, X_n\}$ is a frame of T_xM, then $\widehat{u} \in P$ and $p(\widehat{u}) = x$. Every element $\psi \in \Psi$ defines a diffeomorphism of P by the formula $\widehat{\psi}(\widehat{u}) = (\psi_{*x}X_1, \dots, \psi_{*x}X_n)$, where ψ_{*x} is the differential of ψ at x. As known, $\widehat{\psi}$ is an isomorphism of the frame bundle, $\widehat{\psi} \circ R_a = R_a \circ \widehat{\psi}$ for every $a \in H$. By ([13, Chap VI, Proposition 1.3]), $\widehat{\psi}$ satisfies the condition $\widehat{\psi}^*\alpha = \alpha$, where $\hat{\psi}^*$ is the codifferential of $\hat{\psi}$. Thus, $\widehat{\psi}$ is an automorphism of the affine structure considered as a Cartan geometry. Thus, ψ generates an automorphism group $\widehat{\Psi} = \langle \widehat{\psi} \rangle$ isomorphic to Ψ, which lies over Ψ with respect to $p : P \to M$. The equality $\widehat{q}_*(\widehat{Q}) = Q$ follows from the condition $\widehat{\psi}^*\alpha = \alpha$. Thus. the statement (i) is proved.

Denote by $(P, \widehat{F})$ and $(\mathcal{T}, F)$ the foliations such that $T\widehat{F} = \widehat{Q}$ and $TF = Q$. By the property $\widehat{q}_*(\widehat{Q}) = Q$ of the projection $\widehat{q} : P \to \mathcal{T}$, it is necessary $\widehat{q}(\widehat{L}) = L$, where $\widehat{L} = \widehat{L}(\widehat{u})$ and $L = L(u)$, $u = \widehat{q}(\widehat{u}) \in \mathcal{T}$ are the leaves of the foliations $(P, \widehat{F})$ and $(\mathcal{T}, F)$ respectively.

Consider the holonomy group $\widehat{\Phi}(\widehat{u})$ of the H-connection $\widehat{Q}$ at $\widehat{u}$. Let $u = \widehat{q}(\widehat{u})$ and let $\Phi(u)$ be the holonomy group of the H-connection Q at u. By the definition of the holonomy group $\widehat{\Phi}(\widehat{u})$, for any $a \in \widehat{\Phi}(\widehat{u})$, the equality $R_a(\widehat{L}) = \widehat{L}$ holds. Therefore, we have the following chain of equalities: $R_a(L) = R_a(\widehat{q}(\widehat{L})) = (R_a \circ \widehat{q})\widehat{L} = (\widehat{q} \circ R_a)\widehat{L} = \widehat{q}(R_a(\widehat{L})) = \widehat{q}(\widehat{L}) = L$. This means that $a \in \Phi(u)$, hence, the inclusions (11) hold true. □

Proof of Theorem 2. Let $\mathcal{N}$ be an effective compact n-dimensional affine orbifold. Assume that $\mathcal{N}$ is special, i.e. the holonomy group of $\mathcal{N}$ belongs to the special linear subgroup $SL(n, \mathbb{R})$. By Proposition 4, there exist a compact n-dimensional affine manifold M and a finite group Ψ of its automorphisms such that $\mathcal{N} = M/\Psi$. Therefore, by Proposition 2, $\chi^{ES}(\mathcal{N}) = \frac{1}{|\Psi|}\chi(M)$. By Lemma 2, using the notations introduced above, the holonomy groups $\widehat{\Phi}(\widehat{u})$ and $\Phi(u)$ of M and $\mathcal{N}$ are related by the inclusions $\widehat{\Phi}(\widehat{u}) \subset \Phi(u)$. Since $\Phi(u) \subset SL(n, \mathbb{R})$, we have $\widehat{\Phi}(\widehat{u}) \subset SL(n, \mathbb{R})$. Therefore, the affine manifold M is also special. As proved by B. Klingler [12], the Euler characteristic of any special compact affine manifold is zero. Therefore $\chi^{ES}(\mathcal{N}) = \frac{1}{|\Psi|}\chi(M) = 0$. □

4.2. The application to flat pseudo-Riemannian orbifolds

Proofs of Theorem 3. Assume that $(\mathcal{N}, g)$ is a compact n-dimensional pseudo-Riemannian orbifold, and the Levi-Civita connection ∇ of g is flat. Therefore, $(\mathcal{N}, g)$ is a compact affine orbifold.

Let (r, s) be the signature of the pseudo-Riemannian metric g. Since $(\mathcal{N}, g)$ is a pseudo-Riemannian orbifold, its frame bundle has a reduction $\mathcal{R}(\mathcal{N}, O(r, s))$ to the pseudo-orthogonal subgroup $O(r, s)$, $r+s = n$, of $H = GL(n, \mathbb{R})$, and ∇ defines an $O(r, s)$-connection on $\mathcal{R}$. Denote by $\Phi(u)$, $u \in \mathcal{R}$, the holonomy group of $(\mathcal{N}, \nabla)$. Then $\Phi(u)$ is a subgroup of $O(r, s)$. It is well known that the determinant of any matrix from $O(r, s)$ is equal to ± 1.

First, assume $(\mathcal{N}, g)$ is oriented. Then the determinant of every matrix from the holonomy group $\Phi(u)$, $u \in \mathcal{R}$, is equal to 1. Therefore, $\mathcal{N}$ is special. By Theorem 2, $\chi^{ES}(\mathcal{N}) = 0$.

Let $(\mathcal{N}, g)$ be non-oriented. There is a 2-fold regular covering $f : \mathcal{N}' \to \mathcal{N}$ such that $\mathcal{N}'$ is an oriented orbifold. Let g' be the induced pseudo-Riemannian metric on $\mathcal{N}'$. Then the deck transformation group isomorphic to $\mathbb{Z}_2$ becomes an isometry group of $(\mathcal{N}', g')$. By Proposition 2, it is necessary $\chi^{ES}(\mathcal{N}) = \frac{1}{2}\chi^{ES}(\mathcal{N}')$. As proved above, $\chi^{ES}(\mathcal{N}') = 0$, hence $\chi^{ES}(\mathcal{N}) = 0$. □

4.3. Complete affine orbifolds

Definition 4. Let $\mathcal{N}$ be an n-dimensional affine orbifold. It is called *complete*, if its universal covering space is the affine space A^n.

Proof of Theorem 4. Let $\mathcal{N}$ be an effective n-dimensional compact complete affine orbifold. By Proposition 4, $\mathcal{N}$ is very good. Thus, there are a compact affine manifold M and a finite group Ψ of automorphisms of M such that $\mathcal{N} = M/\Psi$ and the quotient map satisfies the relation $\nu = q \circ k$, where $k : A^n \to M$, $\nu : A^n \to \mathcal{N}$ are universal covers and $q : M \to \mathcal{N} = M/\Psi$ is the quotient map. Completeness of A^n implies completeness of M. By [14], the Euler characteristic of the complete compact affine manifold M is zero, i.e. $\chi(M) = 0$. By Corollary 1, $\chi^{ES}(\mathcal{N}) = \frac{1}{|\Psi|}\chi(M)$, hence $\chi^{ES}(\mathcal{N}) = 0$. □

5. Examples

5.1. Affine orbifolds which are neither special nor complete

Example 1. Fix $\lambda \in (0, 1)$ and a natural number $n \geq 3$. Consider the homothety $\varphi : A^n \setminus \{0\} \to A^n \setminus \{0\}$, $\varphi(x) = \lambda x$, $x \in A^n \setminus \{0\}$. The group $\Phi \cong \mathbb{Z}$ generated by φ acts on $A^n \setminus \{0\}$ freely and properly discontinuously. Therefore, the quotient map $\nu : A^n \setminus \{0\} \to (A^n \setminus \{0\})/\Phi = M$ is a regular covering and the compact quotient space $M = (A^n \setminus \{0\})/\Phi$ admits a structure of n-dimensional affine manifold. Since $A^n \setminus \{0\}$ is incomplete, the affine manifold M is also incomplete. The homothety φ does not preserve the volume form of $A^n \setminus \{0\}$, hence M is not special. Remark that M is diffeomorphic to the product $S^{n-1} \times S^1$, therefore $\chi(M) = \chi(S^{n-1} \times S^1) = \chi(S^{n-1}) \cdot \chi(S^1) = 0$, because $\chi(S^1) = 0$.

Let $\tilde{\gamma}: A^n\backslash\{0\} \to A^n\backslash\{0\}$ be the rotation of $A^n\backslash\{0\}$ given by

$$\tilde{\gamma}(x_1, x_2, x_3, \dots, x_n) = (x_1 \cos\frac{2\pi}{k} - x_2 \sin\frac{2\pi}{k}, x_1 \sin\frac{2\pi}{k} + x_2 \cos\frac{2\pi}{k}, x_3, \dots, x_n),$$

where $(x_1, x_2, x_3, \dots, x_n) \in A^n\backslash\{0\}$ and $k \in \mathbb{N}$, $k \geq 2$. Since $\tilde{\gamma} \circ \varphi = \varphi \circ \tilde{\gamma}$, the affine automorphism $\tilde{\gamma}$ defines an automorphism γ of the affine manifold M such that $\gamma \circ \nu = \nu \circ \tilde{\gamma}$. Denote by Γ_k the group generated by γ. Remark that $\Gamma_k \cong \mathbb{Z}_k$. Since Γ_k is an affine automorphism group of M, the compact quotient space $\mathcal{N}_k = M/\Gamma_k$ admits a structure of n-dimensional affine orbifold.

For $n = 3$, the singular points of $\mathcal{N}_k$ form two 1-dimensional strata diffeomorphic to the circle S^1.

For $n \geq 4$, the orbifold $\mathcal{N}_k$ has the following stratification $\Delta = \{\Delta_n, \Delta_{n-2}\}$, where the stratum Δ_{n-2} is diffeomorphic to $S^{n-3} \times S^1$.

Since $\chi(M) = 0$, Corollary 1 implies $\chi^{ES}(\mathcal{N}_k) = \chi(M)/|\Gamma_k| = \frac{1}{k}\chi(M) = 0$. Since M is neither special nor complete, $\mathcal{N}_k$ is also neither special nor complete, however $\chi^{ES}(\mathcal{N}_k) = 0$.

Example 1 shows that the analog of Chern's conjecture stated above is still open.

5.2. Special affine orbifolds

Further, we construct some special affine orbifolds $\mathcal{N}$ and compute their Euler–Satake characteristic $\chi^{ES}(\mathcal{N})$, topological Euler characteristic $\chi(\mathcal{N})$ and orbifold Euler characteristic $\chi^{orb}(\mathcal{N})$.

Example 2 (*n-Dimensional Pillow*)**.** Consider the n-dimensional affine torus $X = T^n$ as the quotient space $A^n/\mathbb{Z}^n$ of the affine space A^n by the integer lattice $\mathbb{Z}^n$. Denote by $[x_1, x_2, \dots, x_n]$ the image of $(x_1, x_2, \dots, x_n) \in A^n$ under the quotient map $A^n \to A^n/\mathbb{Z}^n = T^n$. Let $G \cong \mathbb{Z}_2$ be the group generated by the involution

$$g: T^n \to T^n: [x_1, x_2, \dots, x_n] \mapsto [-x_1, -x_2, \dots, -x_n].$$

The compact quotient space $\mathcal{N} = X/G$ admits a structure of n-dimensional affine orbifold.

Let X^g be the fixed point set of g. Obviously, X^g consists of the 2^n points $[x_1, x_2, \dots, x_n] \in T^n$, where $x_i \in \{0,\ 1/2\}$, $i = 1, \dots, n$. Let us denote them by A_i, $i = 1, \dots, 2^n$. Therefore, the set of singular points of $\mathcal{N}$ consists of the 2^n points $q(A_i)$, where $q: X \to \mathcal{N}$ is the quotient map.

By Corollary 1, $\chi^{ES}(\mathcal{N}) = \frac{1}{|\mathbb{Z}_2|}\chi(T^n) = 0$, because $\chi(T^n) = 0$.

Using the fact that

$$|G| = |\mathbb{Z}_2| = 2, \quad \chi(X^e) = \chi(T^n) = 0, \quad \chi(X^g) = \chi(\bigsqcup_{i=1}^{2^n} A_i) = \sum_{i=1}^{2^n} \chi(A_i) = 2^n$$

and applying the formula (5), we get

$$\chi(\mathcal{N}) = \frac{1}{|G|}(\chi(X^e) + \chi(X^g)) = \frac{2^n}{2} = 2^{n-1}. \tag{12}$$

By (7), we get

$$\chi^{orb}(\mathcal{N}) = \chi(X^e/G) + \chi(X^g/G) = \chi(\mathcal{N}) + \chi(X^g) = 2^{n-1} + 2^n = 3 \cdot 2^{n-1}. \tag{13}$$

Thus, for the n-dimensional orbifold $\mathcal{N}$ constructed above, we obtain

$$\chi^{ES}(\mathcal{N}) = 0, \quad \chi(\mathcal{N}) = 2^{n-1}, \quad \chi^{orb}(\mathcal{N}) = 3 \cdot 2^{n-1}.$$

In particular, for $n = 2$, the orbifold $\mathcal{N}$ is the 2-dimensional pillow with the underlying topological space homeomorphic to the 2-dimensional sphere S^2, and $\chi^{ES}(\mathcal{N}) = 0$, $\chi(\mathcal{N}) = 2$, $\chi^{orb}(\mathcal{N}) = 6$.

For $n = 4$, the orbifold $\mathcal{N}$ is the Kummer surface [1], and $\chi^{ES}(\mathcal{N}) = 0$, $\chi(\mathcal{N}) = 8$, $\chi^{orb}(\mathcal{N}) = 24$.

Example 3. Let $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional affine torus T^2 constructed in Example 2 for the case $n = 2$. Take two reflections of the torus T^2 :

$$\tau_1([x_1, x_2]) = [1 - x_1, x_2], \quad \tau_2([x_1, x_2]) = [x_1, 1 - x_2] \quad \forall [x_1, x_2] \in T^2.$$

Let the group G be generated by τ_1, τ_2. Then $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The compact quotient space $\mathcal{N} = X/G$ admits a structure of 2-dimensional affine orbifold which can be considered as a square on the plane. Four vertices of the square are the 0-dimensional strata of $\mathcal{N}$, four sides (without the vertices) of the square are the 1-dimensional strata of $\mathcal{N}$.

By Corollary 1, we have $\chi^{ES}(\mathcal{N}) = \frac{1}{|\mathbb{Z}_2 \times \mathbb{Z}_2|}\chi(T^2) = 0$.

The fixed point set X^{τ_i} of the reflection τ_i is the disjoint sum of two circles $S^1 \sqcup S^1$, $i = 1, 2$. Hence, $\chi(X^{\tau_i}) = \chi(S^1 \sqcup S^1) = 2\chi(S^1) = 0$. The composition $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1$ has four fixed points $X^{\tau_1 \circ \tau_2} = \sqcup_{i=1}^4 A_i$. Therefore, $\chi(X^{\tau_1 \circ \tau_2}) = \chi(\sqcup_{i=1}^4 A_i) = 4\chi(A_i) = 4$. Using $|G| = |\mathbb{Z}_2 \times \mathbb{Z}_2| = 4$, $\chi(X^e) = \chi(T^2) = 0$ and the formula (5), we get

$$\chi(\mathcal{N}) = \frac{1}{|G|}(\chi(X^e) + \chi(X^{\tau_1}) + \chi(X^{\tau_2}) + \chi(X^{\tau_1 \circ \tau_2})) = 1.$$

Since the restriction $\tau_k|_{X^{\tau_l}}$ fixes four points $\sqcup_{i=1}^4 A_i = X^{\tau_1 \circ \tau_2}$, $k \neq l$, we have $\chi(X^{\tau_k}/G) = \chi(X^{\tau_k}/\mathbb{Z}_2) = \frac{1}{2}(\chi(X^{\tau_k}) + \chi(X^{\tau_1 \circ \tau_2})) = 2$, $k = 1, 2$. Note that $X^e/G = \mathcal{N}$, $X^{\tau_1 \circ \tau_2}/G = X^{\tau_1 \circ \tau_2}$. By (7), we obtain

$$\chi^{orb}(\mathcal{N}) = \chi(X^e/G) + \chi(X^{\tau_1}/G) + \chi(X^{\tau_2}/G) + \chi(X^{\tau_1 \circ \tau_2}/G) = 1 + 2 + 2 + 4 = 9.$$

Thus, we have $\chi^{ES}(\mathcal{N}) = 0$, $\chi(\mathcal{N}) = 1$, $\chi^{orb}(\mathcal{N}) = 9$.

Examples 2 and 3 show that the direct analog of Chern's conjecture for the topological and orbifold Euler characteristics does not exist.

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