A simple Online Fair Division problem*

Anna Bogomolnaia[♦♣], **Hervé Moulin**[♦♣], **Fedor Sandomirskiy**[★♣]

♦: University of Glasgow, UK
♣: the Higher School of Economics, St Petersburg, Russia
★: the Technion, Haifa, Israel

Abstract

A fixed set of n agents share a random object: the distribution μ of the profile of utilities is IID across periods, but arbitrary across agents. We consider a class of online division rules that learn the realized utility profile, and only know from μ the individual expected utilities. They have no record from past realized utilities, and do not know either if and how many new objects will appear in the future. We call such rules *prior-independent.*

A rule is fair if each agent, ex ante, expects at least $1/n$-th of his utility for the object if it is a good, at most $1/n$-th of his disutility for it if it is a bad. Among fair prior-independent rules to divide goods (bads) we uncover those collecting the largest (lowest) total expected (dis)utility. There is exactly one fair rule for bads that is optimal in this sense. But for goods the set of optimal fair rules is one dimensional. Both in the worst case and in the asymptotic sense, our optimal rules perform much better than the natural Proportional rule (for goods or for bads), and not much worse than the optimal fair *prior-dependent* rule that knows the full distribution μ in addition to realized utilities.

1 Online and Fair division rules

A rule to allocate resources in an intertemporal context is *online*[1] if the allocation taking place in each period relies only on the information available at this particular time and possibly on the record of past allocations, but not on any precise expectations about the future periods. Critically, it is not known how long the process will continue, if at all.

We compare here the performance of a family of online fair division rules that are both history- and prior-independent (as explained below), with that of the best rule that an omniscient manager – with full knowledge of the past and the best available statistical information about the future – can design. We find that the gap in performance is small, which is a strong argument in favor of

*Comments by seminar participants at the University of Liverpool, Zurich Technical University, the University of Lancaster, the Weizmann Institute, Tel-Aviv University, and the Technion in Israel, the PSE and the MSE in Paris, University of Rochester, and HSE St.Petersburg are gratefully acknowledged. Remarks of William Thomson and Yossi Azar were especially helpful. The numerical simulations reported in Sections 8 are due to Yekaterina Rzhewskaya, PhD student at the HSE St Petersburg.
Support from the Basic Research Program of the National Research University Higher School of Economics is gratefully acknowledged. Sandomirskiy is partially supported by Grant 19-01-00762 of the Russian Foundation for Basic Research.

[1] The term “online” originates in the computer science literature: online algorithms deal with sequences of requests that emerge dynamically and are not known in advance ([13]).

the prior-independent rules, much easier to implement as they eschew the potentially difficult and/or costly acquisition and processing of complex information.

We consider the fair division of one or more random objects. When an object arrives in a given period it must be divided (physically, or by means of a lottery if it is indivisible) between a fixed set of agents. The (random) profiles of individual utilities are probabilistically independent across periods, but typically not across agents. If the object is a desirable "good" the design goal is to maximize total relative utility while ensuring that each participant receive a *Fair Share* of the resources: that is, an agent's expected utility is no less than $\frac{1}{n}$-th of his expected utility for the full object.[2] If the object is a non disposable "bad" (chore) we wish symmetrically to minimize total relative disutility while ensuring that no one gets to cover more than a fair share of the chore.

Think of a dispatcher distributing randomly arriving customers across the fleet of taxis, or more generally, to a fixed set of specialists serving the customers one at a time. For efficiency she wishes to assign customers where they generate the most benefit, while fairness requires that each member of the cooperative be guaranteed a significant share of total business. Online allocation rules may well be the only feasible rules here. Symmetrically, the online distribution of workloads is compelling if we assign emergency care patients between hospitals, refugees between shelters and so on. Here the concern is to minimize costs while making sure no one receives an unduly large burden of tasks. See more examples in the literature review.

The *relative (dis)utility* of agent i is the ratio of i's realized (dis)utility to his expected (dis)utility with respect to the distribution common to every arrival of a new object. We assume that upon the arrival of each object, the corresponding profile of relative (dis)utilities is revealed, it is the input of our online rules. In other words the rule learns how lucky or unlucky each agent would be to receive the object that just appeared. There are no incentives considerations in our model, we do not allow strategic misreports by the agents. This is realistic in the above examples where the interaction between dispatcher and receivers of the objects is one of mutual trust.

We similarly measure the efficiency performance of a rule by the expected sum of relative (dis)utilities: interpersonal comparisons of absolute (dis)utilities have no normative meaning, but comparing relative (dis)utilities does.

If the online rule only learns absolute (dis)utilities, but is unable to calibrate them with respect to their expected value, the only way to ensure Fair Share in expectation is to divide the object equally. Contrast this with the options available when relative (dis)utilities are known. Say the object is a good, worth more than its average value to Ann, and less to Bob: we can give a bigger share to Ann than to Bob and still guarantee that, in expectation (at the ex ante stage), Bob gets at least one half of his average value for the good. We need of course to return the favor to Bob if and when his relative utility is higher than average while Ann's is below average. And this discriminating policy clearly pushes up the expected total relative utility of our two agents.

Here we develop the full possibilities opened up by this simple insight. We define a *history-and-prior-independent online* rule as a one period division rule based on nothing more than the realized profile of relative (dis)utilities. Its expected fairness and efficiency performances in any period are the same as in the stationary dynamic context where new objects may or may not arrive tomorrow, and if they do the probability distribution of (dis)utility profiles are independent across periods. By contrast, *a prior-dependent* rule has full knowledge of the probability distribution of the profile of (dis)utilities in each period. Both types of rules seek to optimize total relative utility under the Fair

[2] This mild and uncontroversial test of fairness goes back to the earliest modern discussion of fair division by Steinhaus ([26]).

Share constraint.

Since vectors of (dis)utilities are independent and identically distributed across periods, a prior-dependent rule cannot gain from conditioning the current allocation on the history. Thus a one-period division problem is enough to compare history-and-prior-independent online rules (for brevity: prior-independent rules) to the best prior-dependent rule.

2 Our results

Equal split, irrespective of (dis)utilities, relative or absolute, is the simplest way to ensure Fair Share **(FS)**. It is easy to guarantee FS and achieve a much higher total relative (dis)utility than Equal Split does. The simplest method is the *Proportional* rule, where the probability that an agent gets the good (resp. bad) is proportional (resp. inversely proportional) to her relative utility (resp. disutility): Proposition 1.

Many other prior-independent rules guarantee FS, and some of them have a better efficiency performance than the Proportional one. Our main results, Theorems 1 and 2 in Sections 5 and 6, describe the *optimal* ones. Optimality here is the following (strong) ex post statement: if a prior-independent division rule φ guaranteeing FS is not optimal, there is at least one optimal rule φ^* such that, in every realization of the profile of relative utilities (for a good), φ collects (weakly) less total relative utility than φ^*; and the inequality is sometimes strict. These inequalities are of course reversed for a bad.

The set of optimal prior-independent rules to divide goods is infinite and one-dimensional; it is given in closed form by Theorem 1. We call these rules *Top Heavy* because they put as much weight on the agents with high relative utility as permitted by the Fair Share constraint. For two-agent problems, and only those, there is a *single* Top Heavy rule as follows: if the relative utilities of the agents for the good are x_1, x_2 and $x_1 \leq x_2$, the shares (y_1, y_2) are $(0, 1)$ if $\frac{x_1}{x_2} \leq \frac{1}{2}$ and $(1 - \frac{x_2}{2x_1}, \frac{x_2}{2x_1})$ if $\frac{1}{2} \leq \frac{x_1}{x_2} \leq 1$.

When we divide bads, we speak of a *Bottom Heavy* rule. For two-agent problems, it is the mirror image of the Top Heavy rule above: with relative disutilities $x_1 \leq x_2$, the shares are $(1, 0)$ if $\frac{x_1}{x_2} \leq \frac{1}{2}$ and $(\frac{x_2}{2x_1}, 1 - \frac{x_2}{2x_1})$ if $\frac{1}{2} \leq \frac{x_1}{x_2} \leq 1$. But for problems with three or more agents, surprisingly, there is a *single* optimal Bottom Heavy rule to divide bads, given in closed form by Theorem 2.[3] This rule collects, ex post, less relative disutility than *any* other prior-independent rule meeting FS.

By contrast, the most efficient prior-dependent division rule meeting FS cannot in general be represented in closed form because it processes more complex information through a straightforward linear program.

The rest of our results compare the efficiency performance of our Top (resp. Bottom) Heavy rules to that of the optimal prior-dependent rules ensuring Fair Share, and show that the gap is often very small. Two types of results sustain this claim.

First, in Section 7, we use the relevant "worst case" performance index to compare rules. For brevity call the expected total relative utility collected by a rule for goods, whether prior-independent or prior-dependent, its *relative gain*; in the case of bads the *relative loss* is similarly the expected total relative disutility collected by the rule.

[3]Unexpected differences between the fair division of goods and that of bads have already emerged in the deterministic fair division of several commodities: see [11] and [12].

Fix a prior-independent rule φ to divide goods and meeting Fair Share. Its *Price of Independence* (PoI) is the largest possible ratio of the relative gain of an prior-dependent rule ξ meeting FS, to that of φ. If the prior-independent rule φ, or prior-dependent rule ξ, meets FS, its *Price of Fairness* (PoF) is the largest ratio of the optimal relative gain unconstrained by FS,[4] to the relative gain of φ or ξ.

If we divide bads, the PoI of the prior-independent rule φ is the largest possible ratio of the relative loss of φ to that of a prior-dependent rule ξ meeting FS. And the PoF of φ or ξ is the largest ratio of their relative loss to the smallest feasible relative loss unconstrained by FS.

We observe first that the two indices PoI and PoF coincide for any prior-independent rule, for goods or for bads: Lemma 1.

Proposition 4 looks at the division of goods. Between two agents ($n = 2$), the PoI=PoF of the Proportional rule is 121%, that of the Top Heavy rule is 109%, and the PoF of the best prior-dependent rule is 108%. As n grows, the PoF of the best prior-dependent rule grows as $\frac{\sqrt{n}}{2}$, see [14]; this is also the growth rate of the PoF for the Proportional rule and for some of the Top Heavy rules (for other TH rules, the rate is faster). This rate is much slower than n, which is the PoI=PoF of the Equal Split rule, the worst possible among all prior-independent rules meeting FS.

In Proposition 5 we divide bads. Between two agents, the PoI=PoF of Equal Split is unbounded, that of the Proportional rule is 2, but for the optimal Top Heavy rule it is 112.5%, exactly like the PoF of the best prior-dependent rule. As n grows, the PoI=PoF of the Proportional rule is n, the PoI=PoF the optimal Bottom Heavy rule increases as $\frac{n}{4}$, just like the PoF of the best prior-dependent rule.

In Section 8 we discuss asymptotic probabilistic results, assuming that individual (dis)utilities are statistically independent and drawn from familiar distributions: uniform, exponential, etc.

The utilitarian performance of the best Top Heavy rules, and of the single optimal Bottom Heavy rule proves to be strong. These rules collect, independently of n, a substantial fraction of the maximal relative gain (or minimal relative loss) bounded away from zero. For example, when n is large the maximal gain is only 132% more than the gain captured by the Top Heavy rule for the uniform distribution and and 288% for the exponential.

We conclude that the collection of detailed statistical information in our simple fair division problem brings little utilitarian gain, when compared with the performance of the prior-independent rules we discovered: the best prior-dependent rule typically garners not much more utilitarian surplus than the best fair prior-independent rule. This conclusion translates to history-dependent[5] rules as well. Such a rule can improve upon history-and-prior-independent rules but cannot outperform the best prior-dependent rule in IID environment. Therefore dependence on history gives only a tiny improvement over our history-independent Top Heavy (for goods) and Bottom Heavy (for bads) rules.

3 Literature review

Problems of dynamic (online) resource-allocation have attracted some attention in the computer science and economic communities, but the literature is still sparse.

An early and influential reference on the efficiency aspect of online resource allocation is [23] in the matching context; follow-up work include [18], and [17]. The fairness aspect was touched only recently.

[4]Obtained by dividing any object only between the agents with the highest utility.

[5]A history-dependent prior-independent rule may learn something of the underlying distribution from the past allocations. Optimal rules in this full-fledged online context appear to be quite complicated.

In the food bank problem of [2], like in our setting, objects are arriving online and are allocated to a fixed population of agents, under a fairness constraint and toward the efficiency objective. Thanks to the simplifying assumption that agents have dichotomous preferences, i.e., they either "like" or "dislike" the kind of food delivered right now, the space of possible rules is finite-dimensional and they find some with appealing properties. In the algorithmic paper [6] agents have additive valuations over indivisible objects and fairness is achieved without lotteries by a complicated derandomization technique; the resulting algorithm provides no welfare guarantees.

The study of a "dual" setting where the resources to allocate are known in advance but agents arrive online was initiated by the model of online fair cake-cutting of [29]. The division of a single unit of durable resource (unlike here) between agents randomly arriving and departing is discussed in [19] and [20], where the goal is to maintain approximate fairness while disrupting the allocation of as few agents as possible in each period. When allocating several computational resources in a cloud among different clients [21] it is natural to assume agents have Leontief preferences (they need CPU and memory in a given proportion): online algorithms ensuring fairness for such preferences are constructed by [24]. Then [10] discusses the case of additive preferences.

In economics, the impact of changing the set of agents entitled to a share of a fixed bundle of resources has been discussed extensively since [27] in the static and deterministic version of fair division.

Matching markets are another popular area where the arrival of agents (e. g., job-seekers and firms) can be modeled as "online": [1],[4], [5], [3]. See also the dynamic kidney exchange model in [28], where both agents and objects arrive online but preferences are only on one side of the market. This literature is concerned about the welfare implications of strategic behavior, of congestion (waiting is costly but leads to a better match), and of various signaling policies. For a fixed population of prioritized agents and online objects' arrival, these questions were also studied by [9].

Our approach to online fair division is methodologically close to the design of prior-independent [16] and prior-free auctions [22] and the applications of robust-optimization to contract theory [15]. There, as here, in contrast to the classical Bayesian approach where the designer knows the prior distribution, either no information about prior is available at all or it is known that the prior belongs to a certain wide class of distributions. Hence the optimal worst-case behavior becomes the design objective.

Measuring the trade-off between fairness and the utilitarian objective by the Price of Fairness (PoF) was suggested by [14] and [7] in the context of offline cake cutting and bargaining, respectively. A similar idea of comparing online and offline rules (the latter with full information about the future and the past) by the worst-case ratio of collected welfare is known as competitive analysis [13].

4 The model

Definitions 1 to 4 apply to the division of a good or a bad.

Definition 1

A fair division problem $\mathcal{P} = (N, \mu, X)$ *is described by the fixed set* N *of* n *agents, the probability distribution* $\mu \in \Delta(\mathbb{R}_+^N)$*, and the random variable* X *in* $\mathbb{R}_+^N$ *with distribution* μ*. We always assume that the expectations* $\mathbb{E}_\mu(X_i)$ *is bounded and positive for each* i*.*

We interpret $X_i, i \in N$, as agent i's random utility or disutility for the object realized at a certain period. We impose no additional restriction on the probability space or the distribution of X:

(dis)utilities X_i may be arbitrarily correlated across agents. We write $X_i^r = \frac{1}{\mathbb{E}_\mu(X_i)} X_i$ for agent i's relative utility or disutility.

Definition 2
A prior-independent[6] *division rule is a measurable mapping from* $\varphi : \mathbb{R}_+^N \to \Delta(N)$*, symmetric in* N*.*[7] *Given a realization* $x^r \in \mathbb{R}_+^N$ *of the relative (dis)utility profile* X^r*, agent* i *gets the share* $\varphi_i(x^r)$ *of the object.*
A prior-dependent division rule is a collection of measurable mappings $\xi^{\mathcal{P}} : \mathbb{R}_+^N \to \Delta(N)$*, one for each problem* $\mathcal{P}$*; it is symmetric in* N*. Given* $\mathcal{P}$ *and the realisation* $x^r \in \mathbb{R}_+^N$ *agent* i *gets the share* $\xi_i^{\mathcal{P}}(x^r)$*.*

Here "dividing the object" can be interpreted either literally if the object is divisible, or as assigning probabilistic shares, or time shares.

The fairness constraint of our division rules, whether prior-independent or prior-dependent, sets a lower (resp. upper) bound on every agent's expected utility (resp. disutility).

Definition 3
The division rule rule φ *or* ξ *guarantees Fair Share (FS) if every agent's expected (dis)utility is at least (at most)* $\frac{1}{n}$*-th of his expected (dis)utility for the entire object. If the object is a good this means, for each agent* i

$$\mathbb{E}_\mu(\varphi_i(X^r) \cdot X_i) \geq \frac{1}{n}\mathbb{E}_\mu(X_i) \;\; ; \;\; \mathbb{E}_\mu(\xi_i^{\mathcal{P}}(X^r) \cdot X_i) \geq \frac{1}{n}\mathbb{E}_\mu(X_i) \text{ for each } \mathcal{P} \tag{1}$$

These inequalities are reversed if we divide a bad.

Our design goal, conditional upon meeting Fair Share, is to maximize the expected relative utilitarian welfare $\sum_{i \in N} \frac{\mathbb{E}_\mu(\varphi_i(X^r) \cdot X_i)}{\mathbb{E}_\mu(X_i)}$ in case of a good, or to minimize this quantity in case of a bad (and similarly for a prior-dependent rule). For a prior-dependent rule, this is a straightforward optimization problem related to previous literature about cake division, and with an essentially unique solution (see [14]).

Definition 4 *We call the problem* $\mathcal{P}$ *normalized if* $\mathbb{E}_\mu(X_i) = 1$ *for all* $i \in N$*.*

In the rest of the paper it is very convenient and causes no confusion to restrict attention to such problems, where absolute and relative (dis)utilities coincide. All proofs are given for normalised problems and extend automatically to general problems by replacing everywhere X_i by $X_i^r = \frac{1}{\mathbb{E}_\mu(X_i)} X_i$.

If $S \subseteq N$ we use repeatedly the notation $z_S = \sum_{j \in S} z_j$, and e^S for the vector $e_i^S = 1$ if $i \in S$, $= 0$ if $i \notin S$. Finally $x \gg y$ means $x_i > y_i$ for all i.

Two benchmark examples The *Equal Split* rule, $\varphi^{es}(x) = \frac{1}{n} e^N$ for all x, is the simplest prior-independent rule of all, and it implements Fair Shares. To see that its utilitarian performance is poor, we compare it to the natural Proportional rule:

Proposition 1
i) The Proportional rule for a good is

$$\varphi_i^{pro}(x) = \frac{x_i}{x_N} \text{ for a good if } x \neq 0 \text{ and } \varphi^{pro}(0) = \frac{1}{n} e^N$$

[6] If X is observable while X^r is not, then, for a prior-independent rule, we must additionally know $\mathbb{E}X_i, i \in N$. In this case, by calling a rule prior-independent, we abuse terminology since we still need a collection of simple statistics of the prior.

[7] A permutation of the agents permutes their shares accordingly.

The Proportional rule for a bad is

$$\varphi_i^{pro}(x) = \frac{\frac{1}{x_i}}{\sum_{j\in N}\frac{1}{x_j}} \text{ if } x \gg 0$$

$$\varphi^{pro}(x) = \frac{1}{|S|}e^S \text{ if } x_i = 0 \text{ for } i \in S \text{ and } x_j > 0 \text{ for } j \notin S \text{ ; } \varphi^{pro}(0) = \frac{1}{n}e^N$$

In both cases it guarantees Fair Shares.

ii) For goods, there are utility profiles x where $\frac{\sum_{i\in N}\varphi_i^{pro}(x)\cdot x_i}{\sum_{i\in N}\varphi_i^{es}(x)\cdot x_i} = n$. For bads, there are disutility profiles where $\frac{\sum_{i\in N}\varphi_i^{es}(x)\cdot x_i}{\sum_{i\in N}\varphi_i^{pro}(x)\cdot x_i}$ is arbitrarily large.

Proof. *Statement i) for a good.* Suppose $\mathcal{P}$ is normalised and apply the Cauchy-Schwartz inequality to the two variables $\frac{X_i^2}{X_N}$ and X_N:

$$\mathbb{E}_\mu\left(\frac{X_i^2}{X_N}\right)\cdot\mathbb{E}_\mu(X_N) \geq (\mathbb{E}_\mu X_i)^2$$

Now the left most expectation is simply $\mathbb{E}_\mu(\varphi_i^{pro}(X)\cdot X_i)$, agent i's expected utility, while by the normalisation the other two terms are respectively n and 1.

Statement i) for a bad. Agent i's expected utility under φ^{pro} is now

$$\mathbb{E}_\mu\left(\varphi_i^{pro}(X)\cdot X_i\right) = \mathbb{E}_\mu\left(\frac{1}{\sum_{j\in N}\frac{1}{X_j}}\right) = \frac{1}{n}\mathbb{E}_\mu(\widetilde{X})$$

where $\widetilde{X}$ is the harmonic mean of the X_i-s. The conclusion follows from the inequality $\widetilde{X} \leq \frac{1}{n}X_N$ between harmonic and arithmetic means.

Statement ii). It is enough to take for a good the utility profile $x = e^1$; and for a bad the disutility profile $x = \varepsilon e^1 + e^{N\diagdown 1}$, where ε is arbitrarily small. ■

Remark: The rules assigning probabilities to agents in proportion (or inverse proportion) to some strictly higher power q of their relative (dis)utilities improve efficiency in our utilitarian sense, but they fail FS.[8]

5 Goods: the family of optimal prior-independent rules

Our first main result (Theorem 1 below) compares prior-independent rules for goods meeting Fair Share in terms of their relative utilitarian performance. We use the following binary relation.

Definition 5: *Fix two prior-independent rules φ^1 and φ^2 for dividing a good. We say that φ^1 dominates φ^2 if it always collects, ex post (for every realization of the relative utilities) at least as much utilitarian surplus, and sometimes strictly more*

$$\sum_{i\in N}\varphi_i^2(x)\cdot x_i \leq \sum_{i\in N}\varphi_i^1(x)\cdot x_i \text{ for all } x\in\mathbb{R}_+^N \text{ , with a strict inequality for some } x \tag{2}$$

[8]Say we divide a good and μ picks, for each $i\geq 2$, the vector $x^i = e^1 + (n-1)e^i$ with probability $\frac{1}{n-1}$. Then the expected utility of agent 1 is $\frac{1}{1+(n-1)^q}$, below $\frac{1}{n}$ for $n\geq 3$. The proof for a bad is similar.

The key step toward Theorem 1 characterizes the restriction imposed by Fair Share on any prior-independent rule φ. Given a vector x in $\mathbb{R}_+^N$, we write its arithmetic average as $\overline{x} = \frac{1}{n}x_N$.

Proposition 2

The prior-independent rule φ dividing a good satisfies Fair Share if and only if there exists a number $\theta, 0 \leq \theta \leq 1$, such that

$$\varphi_i(x) \geq \max\left\{\frac{1}{n} + \frac{\theta}{n-1}(1 - \frac{\overline{x}}{x_i}), 0\right\} \text{ for all } i \in N \text{ and } x \in \mathbb{R}_+^N \tag{3}$$

(where we use $\frac{1}{0} = +\infty$)

Proof

Statement **if:** Assume the division rule φ for a good satisfies (3), that implies

$$\varphi_i(x) \cdot x_i \geq \frac{1}{n}x_i + \frac{\theta}{n-1}(x_i - \overline{x}) \text{ for all } x$$

For an arbitrary normalised problem $\mathcal{P}$ (Definition 4) we have $\mathbb{E}_\mu(X_i - \overline{X}) = 0$ so the first inequality in (1) follows. If $\mathcal{P}$ is not normalised the random variables X_i^r define a normalised problem and we are done.

Statement **only if:** Assume the rule φ meets Fair Share and define the real valued function $f(x) = \varphi_1(x) \cdot x_1$. Symmetry of φ implies $f(e^N) = \frac{1}{n}$. Consider a convex combination in $\mathbb{R}_+^N$, with an arbitrary number of terms, such that $\sum_{k=1}^K \mu_k y^k = e^N$. The problem $\mathcal{P}$ in which $X = y^k$ with probability μ_k is normalised and FS implies

$$\sum_{k=1}^K \mu_k f(y^k) \geq \frac{1}{n} = f(e^N)$$

The convexification g of f at x is $g(x) = \inf\{\sum_{k=1}^K \mu_k f(y^k)\}$, over all convex combinations such that $\sum_{k=1}^K \mu_k y^k = x$, see [25].

The inequality above says $g(e^N) \geq f(e^N)$ and the opposite inequality is true by definition of g, so $g(e^N) = f(e^N)$. Because g is convex and finite at e^N there exists a vector $\alpha \in \mathbb{R}^N$ supporting its graph at $(e^N, g(e^N))$, i. e. such that for all $x \in \mathbb{R}_+^N$:

$$g(x) \geq g(e^N) + \alpha \cdot (x - e^N) \Longrightarrow f(x) = \varphi_1(x) \cdot x_1 \geq \frac{1}{n} + \alpha \cdot (x - e^N)$$

Apply the inequality above to $x = \lambda e^N$ for any $\lambda > 0$. By symmetry of φ we get

$$\frac{1}{n}\lambda \geq \frac{1}{n} + (\lambda - 1)\alpha \cdot e^N \text{ for any } \lambda > 0$$

implying $\alpha \cdot e^N = \frac{1}{n}$, therefore $\varphi_1(x) \cdot x_1 \geq \alpha \cdot x$ for all x

Again symmetry of φ implies that we can take $\alpha_j = \alpha_i$ for all $i, j \geq 2$. Indeed if x' obtains from x by permuting coordinates i, j we have

$$\varphi_1(x) \cdot x_1 = \varphi_1(x') \cdot x_1 \geq \frac{1}{2}(\alpha \cdot x + \alpha \cdot x') = \widetilde{\alpha} \cdot x$$

where $\widetilde{\alpha}_i = \widetilde{\alpha}_j$ and $\widetilde{\alpha} \cdot e^N = \frac{1}{n}$ is preserved.

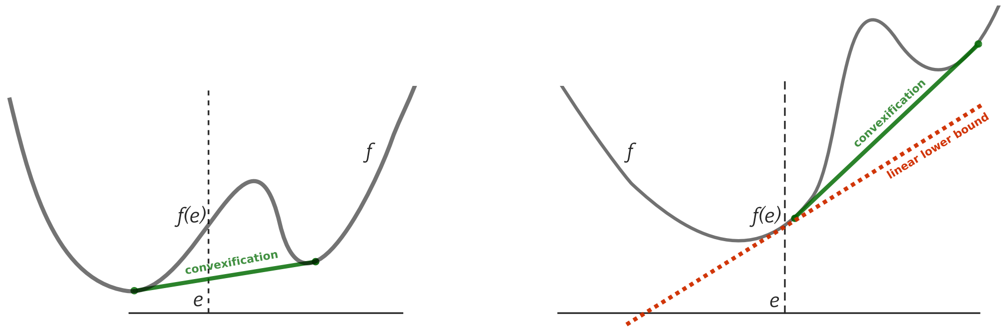

Figure 1: The geometric intuition behind the proof of Proposition 2. Right figure: the convexification of a function f coincides with f at $x = e$ if the graph of f is supported by a linear function. Left figure illustrates necessity of this condition.

Set $\beta = \alpha_i$ for all $i \geq 2$ and note that $\beta \leq 0$ because of the inequality $x_1 \geq \varphi_1(x) \cdot x_1 \geq \alpha_1 x_1 + \beta x_{N \diagdown 1}$. Combining this with $\alpha \cdot e^N = \frac{1}{n}$ we see that there is a non negative constant γ such that

$$\varphi_1(x) \cdot x_1 \geq \alpha \cdot x = \frac{1}{n} x_1 + \gamma((n-1)x_1 - x_{N \diagdown 1})$$

Changing the parameter γ to $\delta = n\gamma$ this gives

$$\Longrightarrow \varphi_i(x) \geq \frac{1}{n} + \delta \left(1 - \frac{\overline{x}}{x_i}\right) \text{ for all } i \in N \text{ and } x \in \mathbb{R}_+^N$$

It remains to find the bounds on γ derived from the fact that $\varphi(x)$ is in $\Delta(N)$. For all $x \gg 0$ the inequalities above (and $\varphi(x) \geq 0$) imply

$$\sum_{i \in N} \max \left\{ \frac{1}{n} + \delta \left(1 - \frac{\overline{x}}{x_i}\right), 0 \right\} \leq 1 \text{ for all } x \in \mathbb{R}_+^N \tag{4}$$

which is equivalent to the following property:

$$\text{for all } S \subseteq N: \sum_{i \in S} \frac{1}{n} + \delta \left(1 - \frac{\overline{x}}{x_i}\right) = |S| \left(\frac{1}{n} + \delta\right) - \delta \overline{x} \left(\sum_{i \in S} \frac{1}{x_i}\right) \leq 1 \text{ for all } x \in \mathbb{R}_+^N$$

The infimum of $\overline{x}(\sum_{i \in S} \frac{1}{x_i})$ is $\frac{|S|^2}{n}$, achieved for any x parallel to e^S, therefore

$$|S| \left(\frac{1}{n} + \delta\right) \leq 1 + \delta \frac{|S|^2}{n} \Longleftrightarrow \left(1 - \frac{|S|}{n}\right)(\delta |S| - 1) \leq 0$$

and we conclude that $\delta \leq \frac{1}{n-1}$. This gives the desired inequality (3) by setting $\theta = (n-1)\delta$.■

Armed with Proposition 2, it is now easy to identify the undominated prior-independent division rules (Definition 5) meeting FS for goods.

We fix $\theta, 0 < \theta \leq 1$, and use the corresponding inequalities (3) to define the canonical *Top Heavy* rule φ^θ for goods. For any $x \in \mathbb{R}_+^N$ we write $x^* = (x^{*1}, \cdots, x^{*n})$ for the order statistics of x,[9] and

[9]The vector with the same set of coordinates as x, rearranged in increasing order.

$\tau(x) = \{i \in N | x_i = \max_{j \in N} x_j \Longleftrightarrow x_i = x^{*n}\}$ for the set of agents with largest utility. Then we set

$$\varphi_j^\theta(x) = \max\left\{\frac{1}{n} + \frac{\theta}{n-1}(1 - \frac{\overline{x}}{x_j}), 0\right\} \text{ for all } j \in N \diagdown \tau(x)$$

$$\varphi_i^\theta(x) = \frac{1}{|\tau(x)|}\left(1 - \sum_{j \in N \diagdown \tau(x)} \varphi_j^\theta(x)\right) \text{ for all } i \in \tau(x) \tag{5}$$

Inequality (4) guarantees that the highest share above is non negative. It also implies that the i-sequence of shares $\varphi_i^\theta(x)$ is co-monotonic with that of utilities x_i.[10]

The rule φ^θ converges to Equal Split when θ goes to zero, but Equal Split is clearly dominated by *any* rule φ^θ for $\theta > 0$. This is why we excluded 0 from the range of θ.

Note that all rules φ^θ are discontinuous at any x where at least two agents have the highest utility ($x^{*(n-1)} = x^{*n}$).

Theorem 1 *for goods*
i) If $n = 2$ *the Top Heavy rule* φ^1 *dominates every other prior-independent rule meeting Fair Share.*
ii) If $n \geq 3$*: every prior-independent rule meeting Fair Share is dominated by, or equal to, one Top Heavy rule* φ^θ*,*$0 < \theta \leq 1$*; the Top Heavy rules themselves are undominated.*
iii) The proportional rule is dominated by the Top Heavy rule $\varphi^{\frac{n-1}{n}}$*, but not by any other rule* φ^θ*.*

For two agent problems the rule φ^1 has a simple expression. By symmetry it is enough to define it when $x_1 \leq x_2$:

$$\varphi^1(x) = (0,1) \text{ if } \frac{x_1}{x_2} \leq \frac{1}{2} \;;\; = \left(1 - \frac{x_2}{2x_1}, \frac{x_2}{2x_1}\right) \text{ if } \frac{1}{2} \leq \frac{x_1}{x_2} \leq 1 \tag{6}$$

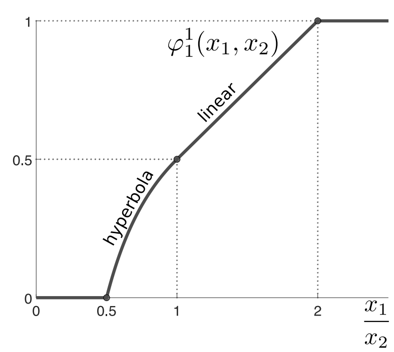

Figure 2: The amount of the good received by the first agent under the TH rule φ^1 for two agents as a function of the ratio $t = \frac{x_1}{x_2}$. If the ratio is below $\frac{1}{2}$ or above 2, the TH rule coincides with the utilitarian one, which gives the whole good to the agent with higher value. If the relative values are closer, both agents receive a non-zero amount of the good: $\varphi_1 = 1 - \frac{1}{2t}$ on $\left[\frac{1}{2}, 1\right]$ and $\varphi_1 = \frac{1}{2}t$ on $[1, 2]$.

[10] This is clear if we compare the shares of two agents i, k outside $\tau(x)$; if $i \notin \tau(x)$ and $k \in \tau(x)$ inequality $\varphi_i^\theta(x) \leq \varphi_k^\theta(x)$ is

$$|\tau(x)|\varphi_i^\theta(x) + \sum_{j \in N \diagdown \tau(x)} \varphi_j^\theta(x) \leq 1$$

which follows from $\varphi_i^\theta(x) = \max\{\frac{1}{n} + \delta(1 - \frac{Ex}{x_i}), 0\}$ and (4).

Proof of Theorem 1

Statement i) Fix $\theta < \theta'$ and $x_1 \leq x_2$. , the definition (5) implies $\varphi_1^\theta(x) \geq \varphi_1^{\theta'}(x)$ because the coefficient of θ in $\varphi_1^\theta(x)$ is $\frac{1}{2}(1-\frac{x_2}{x_1}) \leq 0$, therefore $\varphi_2^\theta(x) \leq \varphi_2^{\theta'}(x)$ and inequality (2) follows; both can be strict as well. Note that this argument does not extend to the case $n \geq 3$ because if agent i's utility is neither the smallest nor the largest, the sign of the coefficient of θ in $\varphi_i^\theta(x)$ is ambiguous.

Thus φ^1 dominates φ^θ for $\theta < 1$. The fact that it also dominates other prior-independent rules meeting FS follows from the proof of Statement ii).

Statement ii) Fix a prior-independent rule φ satisfying FS. There is a $\theta, 0 \leq \theta \leq 1$, s. t. the inequalities (3) hold for all i and x (Proposition 2). If $\theta = 0$, our rule is Equal Split, which we already noticed is dominated by each rule φ^θ. If $\theta > 0$, these inequalities imply $\varphi_i(x) \geq \varphi_i^\theta(x)$ for all x and all $i \notin \tau(x)$. Hence $(\varphi_i(x)-\varphi_i^\theta(x))x_i \leq (\varphi_i(x)-\varphi_i^\theta(x))x^{*n}$ for all $i \notin \tau(x)$. Summing up these inequalities and adding $\sum_{j\in\tau(x)}(\varphi_i(x) - \varphi_i^\theta(x))x_j$ on both sides gives the desired weak inequalities in (2). If none of the inequalities in (2) is strict, we deduce $\varphi_i(x) = \varphi_i^\theta(x)$ for all x and all $i \notin \tau(x)$ s.t. $x_i > 0$. If there is some i s. t. $x_i = 0$ and $\varphi_i(x) > 0$ (while $\varphi_i^\theta(x) = 0$) then $\varphi(x)$ has less weight to distribute on $\tau(x)$ than φ^θ, contradicting our assumption. Because φ is symmetric, we conclude $\varphi(x) = \varphi^\theta(x)$.

We check now that no TH rule φ^θ dominates another TH rule $\varphi^{\theta'}$. Assume $0 < \theta < \theta'$ and consider first the profile $x_i = \frac{3}{4}$ if $i \neq n$, $x_n = 1 + \frac{n-1}{4}$. Then $\overline{x} = 1$ and all coordinates of $\varphi_i^\theta(x)$ and $\varphi_i^{\theta'}(x)$ are strictly positive. Compute $\varphi_i^\theta(x) - \varphi_i^{\theta'}(x) = \frac{\theta'-\theta}{3(n-1)} > 0$ for all $i \neq n$, so that $\varphi^{\theta'}$ collects more surplus at x than φ^θ.

To show an instance of the reverse comparison, we choose

$$x_1 = \frac{\theta}{3} \ ; \ x_i = 1 + \frac{\frac{3}{4} - \frac{\theta}{3}}{n-2} \text{ for } 2 \leq i \leq n-2 \ ; \ x_n = \frac{5}{4}$$

Thus $\overline{x} = 1$ and $\overline{x} < x_i < x_n$ for $2 \leq i \leq n-2$. This implies $\varphi_1^\theta(x) = \varphi_1^{\theta'}(x) = 0$, $\varphi_i^\theta(x) < \varphi_i^{\theta'}(x)$, and $\varphi_n^\theta(x) > \varphi_i^{\theta'}(x)$.

Statement iii) In the previous proof we showed the the rule φ is dominated by φ^θ if it satisfies inequalities (3). Thus the rule φ^{pro} is dominated by the TH rule $\varphi^{\frac{n-1}{n}}$ if for all $x \in \mathbb{R}_+^N$ we have

$$\frac{x_1^2}{x_N} \geq \frac{1}{n} + \frac{1}{n}\left(1 - \frac{\overline{x}}{x_i}\right) \Longleftrightarrow \frac{x_1^2}{x_N} + \frac{x_N}{n^2} \geq \frac{2}{n}x_1$$

and the latter inequality is easily checked.

To check for instance that φ^{pro} is not dominated by the TH rule φ^1, we pick a profile $x = (0, a, a, \cdots, a, b)$ in $\mathbb{R}_+^N$ such that $a < b$ and

$$\varphi_2^{pro}(x) = \frac{a}{(n-2)a+b} < \varphi_2^1(x) = \frac{1}{n} + \frac{1}{n-1}\left(1 - \frac{(n-2)a+b}{na}\right)$$

for instance if $a = 2, b = 3$. Then φ^{pro} puts less weight than φ^1 on each $x_i, 2 \leq i \leq n-1$, and more on $x_n = b$. ■

6 Bads: the unique optimal prior-independent rule

We adapt the relation of domination between two prior-independent rules φ^1 and φ^2 for dividing a bad, by simply reversing the inequality (2): φ^1 dominates φ^2 iff

$$\sum_{i\in N}\varphi_i^2(x)\cdot x_i \geq \sum_{i\in N}\varphi_i^1(x)\cdot x_i \text{ for all } x\in\mathbb{R}_+^N \text{ , with a strict inequality for some } x \tag{7}$$

Next we adapt the characterization of prior-independent rules meeting Fair Share.

Proposition 3
The prior-independent rule φ dividing a bad satisfies Fair Share if and only if there exists a number $\theta, 0\leq\theta\leq 1$, such that

$$\varphi_i(x)\leq \min\left\{\frac{1}{n}+\frac{\theta}{n-1}\left(\frac{\overline{x}}{x_i}-1\right),1\right\} \text{ for all } i\in N \text{ and } x\in\mathbb{R}_+^N \tag{8}$$

(where we use $\frac{1}{0}=+\infty$)

Our proof of Proposition 3 resembles the one for Proposition 1 about goods and can be found in Appendix A.

For each $\theta, 0<\theta\leq 1$, we can now use inequality (8) to construct, as in the previous section, the canonical *Bottom Heavy* rule φ^θ placing as much weight on the smallest disutilities as permitted by (8). The construction relies on the same order statistics x^*, but is slightly more involved. We write $\sigma(x;t)=\{i\in N|x_i=x^{*t}\}$ (so $\sigma(x;n)=\tau(x)$) and use the convention $\sigma(x;0)=\varnothing$. Note that the minimum of $\sum_N\frac{\overline{x}}{x_i}$ over $\mathbb{R}_+^N$ is n, and is achieved by any x parallel to e^N, and only by those: for such a vector, symmetry imposes $\varphi^\theta(x)=\frac{1}{n}e^N$. For all other vectors x the inequality $\sum_N\frac{\overline{x}}{x_i}>n$ implies $\sum_{i\in N}(\frac{1}{n}+\frac{\theta}{n-1}(\frac{\overline{x}}{x_i}-1))>1$.

There is a unique $\widetilde{t}, 0\leq\widetilde{t}\leq n-1$ s. t.

$$\sum_{i:x_i\leq x^{*\widetilde{t}}}\left(\frac{1}{n}+\frac{\theta}{n-1}\left(\frac{\overline{x}}{x_i}-1\right)\right)\leq 1<\sum_{i:x_i\leq x^{*(\widetilde{t}+1)}}\left(\frac{1}{n}+\frac{\theta}{n-1}\left(\frac{\overline{x}}{x_i}-1\right)\right) \tag{9}$$

Indeed the left inequality holds by our convention at $\widetilde{t}=0$, and the right one holds at $\widetilde{t}=n-1$ as we just saw. A simple induction argument proves the existence of $\widetilde{t}$. Uniqueness comes from the fact that each term $(\frac{1}{n}+\frac{\theta}{n-1}(\frac{\overline{x}}{x_i}-1))$ is non negative (by Proposition 3, or directly from $\frac{\overline{x}}{x_i}\geq\frac{1}{n}$).

We define now the *Bottom Heavy* rule $\varphi^\theta(x)$:

$$\varphi_i^\theta(x)=\frac{1}{n}+\frac{\theta}{n-1}\left(\frac{\overline{x}}{x_i}-1\right) \quad \text{for all } \ i \text{ s. t. } x_i\leq x^{*\widetilde{t}} \tag{10}$$

$$\varphi_j^\theta(x)=\frac{1}{|\sigma(x;\widetilde{t}+1)|}\left(1-\sum_{i:x_i\leq x^{*\widetilde{t}}}\varphi_i^\theta(x)\right) \quad \text{for all } \ j \text{ s. t. } x_j\in\sigma(x;\widetilde{t}+1) \tag{11}$$

$$\varphi_j^\theta(x)=0 \ \text{ if } \ x_j>x^{*(\widetilde{t}+1)} \tag{12}$$

We note, for future reference, a few facts about φ^θ.

The sequence of shares $\varphi_i^\theta(x)$ is anti-monotonic to the sequence of disutilities x_i.

If $\widetilde{t}=0$ the only agents with a positive share are those in $\sigma(x;1)$, who have the smallest disutility, so φ^θ selects an optimal utilitarian allocation.

Finally $\varphi_j^\theta(x) < \frac{1}{n}+\frac{\theta}{n-1}(\frac{\overline{x}}{x_j}-1)$ for each agent j in $\sigma(x;\widetilde{t}+1)$. And the mapping φ^θ is discontinuous at x if and only if $|\sigma(x;\widetilde{t}+1)|>1$.

Theorem 2 *for bads*
For any $n\geq 2$, the Bottom Heavy rule φ^1 dominates every other prior-independent rule for bads meeting Fair Share.

The proof of Theorem 2 proves more difficult than in the case of goods, see Appendix A.

If $n=2$, the dominant BH rule φ^1 for bads is the mirror image of the dominant TH rule φ^1 (6):

$$\varphi^1(x)=(1,0) \text{ if } \frac{x_1}{x_2}\leq\frac{1}{2} \text{ ; } =\left(\frac{x_2}{2x_1},1-\frac{x_2}{2x_1}\right) \text{ if } \frac{1}{2}\leq\frac{x_1}{x_2}\leq 1$$

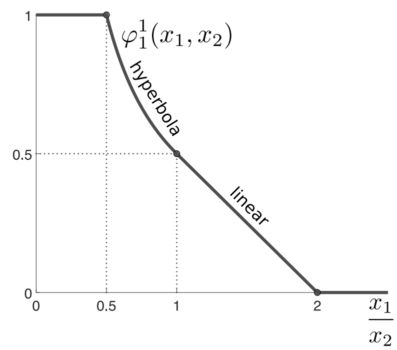

Figure 3: The share of the first agent under the BH rule φ^1 for two agents as a function of $\frac{x_1}{x_2}$.

If $n\geq 3$, the definition of φ^1 in (10) takes a simpler form because $\frac{1}{n}+\frac{1}{n-1}(\frac{\overline{x}}{x_i}-1)=\frac{1}{n(n-1)}\frac{x_{N\diagdown i}}{x_i}$. If $x\in\mathbb{R}_+^N$ is not parallel to e^N, there is a unique $\widetilde{t}, 0\leq\widetilde{t}\leq n-1$ s. t.

$$\frac{1}{n(n-1)}\sum_{i:x_i\leq x^{*\widetilde{t}}}\frac{x_{N\diagdown i}}{x_i}\leq 1<\frac{1}{n(n-1)}\sum_{i:x_i\leq x^{*(\widetilde{t}+1)}}\frac{x_{N\diagdown i}}{x_i}$$

Then we set

$$\varphi_i^1(x)=\frac{1}{n(n-1)}\frac{x_{N\diagdown i}}{x_i} \text{ if } x_i\leq x^{*\widetilde{t}}$$

and $\varphi_j^1(x)$ for j such that $x_j>x^{*\widetilde{t}}$ is given by (11) and (12) as above.

7 Worst case performances

Notation. We write $\Phi(FS)$ for the set of prior-independent rules φ meeting Fair Share; Ξ for the set of prior-dependent rules ξ; and $\Xi(FS)$ for that of prior-dependent rules meeting FS. See Definitions 2 and 3. Next Π_n is the set of normalised problems with n agents (Definition 4). Finally the relative gain (or loss) of the prior-independent and prior-dependent rules φ and ξ at problem $\mathcal{P}$ are

$$\pi(\varphi,\mathcal{P})=\mathbb{E}_\mu\left(\sum_{i\in N}\varphi_i(X)\cdot X_i\right)\text{ ; }\pi(\xi,\mathcal{P})=\mathbb{E}_\mu\left(\sum_{i\in N}\xi_i^{\mathcal{P}}(X)\cdot X_i\right).$$

Definition 6
The n-Price of Independence (PoI_n) of the rule $\varphi \in \Phi(FS)$ is

$$\text{goods: } PoI_n(\varphi) = \sup_{\xi \in \Xi(FS)} \sup_{\mathcal{P} \in \Pi_n} \frac{\pi(\xi, \mathcal{P})}{\pi(\varphi, \mathcal{P})}$$

$$\text{bads: } PoI_n(\varphi) = \sup_{\xi \in \Xi(FS)} \sup_{\mathcal{P} \in \Pi_n} \frac{\pi(\varphi, \mathcal{P})}{\pi(\xi, \mathcal{P})}$$

For a good, the utilitarian performance of the rule $\psi \in \Phi(FS) \cup \Xi(FS)$ at a problem $\mathcal{P}$ is the ratio of the the optimal unconstrained relative gain to the gain collected by ψ. For a bad it is the ratio of the relative loss generated by ψ to the optimal relative loss:

$$\text{goods: } R(\psi, \mathcal{P}) = \frac{\mathbb{E}_\mu \max_i X_i}{\pi(\psi, \mathcal{P})} \quad \text{bads: } \quad R(\psi, \mathcal{P}) = \frac{\pi(\psi, \mathcal{P})}{\mathbb{E}_\mu \min_i X_i}.$$

The n-Price of Fairness (PoF_n) of $\psi \in \Phi(FS) \cup \Xi(FS)$ is its worst-case utilitarian performance

$$PoF_n(\psi) = \inf_{\mathcal{P} \in \Pi_n} R(\psi, \mathcal{P}) \geq 1.$$

It should be clear that the restriction to normalised problems in this Definition is without loss of generality.

Lemma 1 *If the prior-independent rule $\varphi \in \Phi(FS)$ divides goods, we have*

$$PoI_n(\varphi) = PoF_n(\varphi) = \sup_{x \in \mathbb{R}_+^N} \frac{\max_i x_i}{\sum_{i \in N} \varphi_i(x) \cdot x_i}$$

If $\varphi \in \Phi(FS)$ divides bads, we have

$$PoI_n(\varphi) = PoF_n(\varphi) = \sup_{x \in \mathbb{R}_+^N} \frac{\sum_{i \in N} \varphi_i(x) \cdot x_i}{\min_i x_i}$$

Proposition 4 *for goods*
i) The PoI_n of any rule $\varphi \in \Phi(FS)$ is at most n; the PoI of Equal Split is exactly n.
ii) The PoI_n of the Proportional rule is $\frac{\sqrt{n}+1}{2}$; for instance 121% for $n = 2$.
iii) The PoI_n of the Top Heavy rule φ^θ is decreasing in θ. Moreover:

$$PoI_n(\varphi^1) = \frac{n}{2\sqrt{n} - 1} = \frac{\sqrt{n}}{2} + \frac{1}{4} + O\left(\frac{1}{\sqrt{n}}\right)$$

$$PoI_n(\varphi^\theta) = \frac{n}{2\sqrt{(n-1+\theta)\theta} + 1 - 2\theta} \geq PoI_n(\varphi^1)$$

For instance $PoI_2(\varphi^1) \simeq 109\%$ for $n = 2$.
iv) The smallest feasible PoF_n of a prior-dependent rule in $\Xi(FS)$ is such that

$$\frac{n}{2\sqrt{n} - 1} \geq \inf_{\xi \in \Xi(FS)} PoF_n(\xi) \geq \frac{n}{2\sqrt{n} - \frac{1}{2}} = \frac{\sqrt{n}}{2} + \frac{1}{8} + O\left(\frac{1}{\sqrt{n}}\right)$$

For $n=2$ it is 108%*.*

Statements *iii*) and *iv*), together with Lemma 1, make clear that the PoF$_n$ of the TH rule φ^1 is essentially the best PoF$_n$ of any fair prior-dependent rule.

Proposition 5 *for bads*
i) The PoI_n of Equal Split is unbounded (for any fixed n); that of the Proportional rule is n;
ii) The PoI_n of the Bottom Heavy prior-independent rule φ^1 is such that

$$\frac{n}{4}+\frac{5}{4} \geq PoI_n(\varphi^1) \geq \frac{n}{4}+\frac{1}{2}+\frac{1}{4n}$$

It is 109% *for $n=2$.*
iii) The smallest feasible PoF_n of a prior-dependent rule in $\Xi(FS)$ is

$$\inf_{\xi\in\Xi(FS)} PoF_n(\xi) = \frac{n}{4}+\frac{1}{2}+\frac{1}{4n}$$

For $n=2$ it is 108%*.*

Again, the last two statements and Lemma 1 imply that the PoF$_n$ of the BH rule φ^1 is essentially the best PoF$_n$ of any fair prior-dependent rule.

All three results (Lemma 1 and Propositions 5,6) are proved in Appendix B.

8 Asymptotic performance for standard distributions

We evaluate the utilitarian performance of the TH, the BH, and the Proportional rules in the benchmark setting, where the number of agents is large and their values are given by independent identically distributed (IID) random variables.

Fix a distribution $\nu \in \Delta(\mathbb{R}_+)$ with unit mean and assume that the vector $X=(X_i)_{i=1,..,n}$ of values is distributed according to $\mu=\otimes_{i=1}^n \nu$, i.e., the values are independent random variables with distribution ν. The corresponding problem $\mathcal{P}_n(\nu)$ is both normalised and symmetric.

In Appendix C we derive the somewhat cumbersome general formulas describing the utilitarian performance $R(\varphi,\mathcal{P}_n(\nu))$ for these three rules when n is large. Here we discuss examples and corollaries of the general results.

8.1 Goods

8.1.1 Bounded support: ν is the uniform distribution on $[0,1]$.

In this case the TH rule φ^1 and the Proportional rule φ^{pro} have similar utilitarian performances.

For $n=2$ the TH almost achieves the optimal welfare level; the Proportional rule is 10% behind: simple computations show that $R(\varphi^1,\mathcal{P}_2(\text{uni}[0,1])) = \frac{8}{5+4\ln 2} \approx 1.03$ and $R(\varphi^{pro},\mathcal{P}_2(\text{uni}[0,1])) = \frac{2}{\ln 2 - 1} \approx 1.13$. Compare these numbers with the worst-case guarantees from Proposition 4: $PoF_2(\varphi^{pro}) = \frac{\sqrt{2}+1}{2} \approx 1.21$ and $PoF_2(\varphi^1) = \frac{2}{2\sqrt{2}-1} \approx 1.09$. We see that the Proportional rule captures less welfare for the uniform distribution that the TH rule for *any* distribution.

For $n\to\infty$, Proposition 6 from Appendix C and Lemma 3 below imply that the utilitarian performances of our two rules converge as follows

$$R(\varphi^1,\mathcal{P}_\infty(\text{uni}[0,1])) = \frac{1}{\frac{1}{16}+\ln 2} \approx 1.32 \text{ and } R(\varphi^{pro},\mathcal{P}_\infty(\text{uni}[0,1])) = 1.5$$

This result is in a sharp contrast with the worst-case behavior (Section 7): there are problems $\mathcal{P}$ with n agents such that the TH rule collects only a $2/\sqrt{n}$ fraction of the optimal relative gain. Our next result generalizes this observation.

8.1.2 The TH rule keeps a positive fraction of the optimal relative gain.

This holds in general, not just in the above example. Fix a distribution ν with mean 1 and with average absolute deviation $D(\nu) = \int |x-1| d\nu(x)$.

Lemma 2: *If ν has mean 1 and a finite moment $\int_{\mathbb{R}_+} x^{\beta} d\nu(x) < 0$ for some $\beta > 2$, then the utilitarian performance of the TH rule converges to a limit value which satisfies the following upper bound*[11]

$$R(\varphi^1, \mathcal{P}_\infty(\nu)) \leq \frac{2}{D} + \frac{4}{D^2} \tag{13}$$

If in addition ν has unbounded support, then

$$R(\varphi^1, \mathcal{P}_\infty(\nu)) \geq \frac{1}{D} \tag{14}$$

Proof in Appendix C.

For instance if ν is the exponential distribution we find

$$R(\varphi^1, \mathcal{P}_\infty(\exp)) = \frac{1}{1 - 2e^{-\frac{1}{2}} - \mathrm{Ei}(-1/2)} \approx 2.88$$

where Ei stands for a special function, the exponential integral. Contrast this with the situation for the Proportional rule.

Lemma 3: *Under the assumptions of Lemma 2*

$$R(\varphi^{pro}, \mathcal{P}_n(\nu)) = \frac{\mathbb{E}_\mu \max_i X_i}{\mathbb{E}_\nu (X_1)^2}(1 + o(1)), \quad \text{as } n \to \infty.$$

(where $a_n = o(1)$ means that $a_n \to 0$, as $n \to \infty$)

Indeed, by the law of large numbers

$$\pi(\varphi^{pro}, \mathcal{P}_n(\nu)) = \mathbb{E}_\mu \sum_{i \in N} X_i \varphi_i^{pro}(X) = n \cdot \mathbb{E}_\mu \frac{(X_1)^2}{\sum_{i \in N} X_i} \to \mathbb{E}_\mu \frac{(X_1)^2}{\mathbb{E}_\mu X_1} = \mathbb{E}_\nu (X_1)^2.$$

Lemma 3 implies that $R(\varphi^{pro}, \mathcal{P}_\infty(\nu))$ tends to $+\infty$ if ν has unbounded support, because $\mathbb{E}_\mu \max_i X_i$ tends to infinity. For instance $R(\varphi^{pro}, \mathcal{P}_n(\exp)) = \frac{\ln n}{2}(1 + o(1))$.

Of course this limit is positive and finite if the support of ν is bounded.

8.2 Bads

When a bad is divided, the performance of the BH and the Proportional rule is determined by the behaviour of the distribution at the leftmost point of the support. Both rules generate a bounded multiple of the optimal relative loss when 0 does not belong to the support of ν and the TH rule does also well when ν has a non-zero density at 0. However both rules have poor performance if the support touches 0 but ν has not enough "weight" near 0. Here we give three examples to illustrate the general asymptotic results in Appendix C.

[11] Note that D is at most 2 so this upper is never below 2.

8.2.1 The support does not touch zero: ν is uniform on $[\frac{1}{2}, \frac{3}{2}]$.

By Proposition 7 in Appendix C, the utilitarian performances of the BH and the Proportional rules converge to limit values which are pretty close to each other:

$$R(\varphi^1, \mathcal{P}_\infty(\text{uni}[\frac{1}{2}, \frac{3}{2}])) = e - 1 \approx 1.72 \text{ and } R(\varphi^{pro}, \mathcal{P}_\infty(\text{uni}[\frac{1}{2}, \frac{3}{2}])) = \frac{2}{\ln 3} \approx 1.82$$

8.2.2 The support touches zero but there is not enough weight around it: ν has density $\frac{3}{4}x(2-x)$ on $[0, 2]$.

For this distribution the optimal relative loss tends to zero while the losses of the BH and the Proportional rules remain positive. Proposition 7 shows that the utilitarian performances of both rules tend to infinity at the speed of $\sqrt{n}$ while their ratio converges to $\frac{1}{\sqrt{3}} \approx 0.58$:

$$R(\varphi^1, \mathcal{P}_n(\nu)) = \frac{2}{3\sqrt{\pi}}\sqrt{n}(1 + o(1)) = R(\varphi^{pro}, \mathcal{P}_n(\nu))\frac{1}{\sqrt{3}}(1 + o(1))$$

8.2.3 The distribution has non-zero density at 0 (e.g., ν is uniform on $[0, 2]$).

Then the BH rule outperforms the Proportional one in the limit.
Lemma 4: *Assume the distribution ν has a continuous density f on an interval $[0, a]$ and $f(0) > 0$. Then $R(\varphi^1, \mathcal{P}_n(\nu))$ converges to a finite positive limit as n becomes large, whereas $R_n(\varphi^{pro}, \mathcal{P}_n(\nu)) = \Omega\left(\frac{n}{\ln(n)}\right)$ as $n \to \infty$.*[12]

A similar result for the case when the density is infinite at $x = 0$ is the object of Lemma 5 in Appendix C.

The statement about the BH rule follows from the asymptotic result for the order statistic: the expected values of X^{*k} for small numbers k are equal to $\frac{k}{f(0)\cdot n}(1 + o(1))$ as $n \to \infty$.[13] Therefore, on average only a bounded number of agents with smallest X_i receive a non-zero portion of a bad, which implies that the utilitarian performance is bounded away from infinity.

For the Proportional rule we have $\pi(\varphi^{pro}, \mathcal{P}_n(\nu)) = n \cdot \mathbb{E}\frac{1}{\sum_k \frac{1}{X^{*k}}}$. For large n we can estimate the denominator from below by the harmonic series; taking into account that $\mathbb{E}X^{*1} = \frac{1}{f(0)\cdot n}(1 + o(1))$ we get the desired asymptotic formula.

9 Concluding comments

Envy Freeness An alternative, much more demanding interpretation of fairness in our model is (ex ante) *Envy-Freeness*, which means, in the case of goods:

$$\mathbb{E}_\mu(\varphi_i(X^r) \cdot X_i) \geq \mathbb{E}_\mu(\varphi_j(X^r) \cdot X_i) \text{ for all } i, j \text{ and } \mathcal{P}$$

The critical Proposition 2 can be adapted as follows. Set $g(x) = (\varphi_1(x) - \varphi_2(x)) \cdot x_1$ so that Envy Freeness means $\mathbb{E}_\mu(g(Y)) \geq g(e^N) = 0$ whenever $\mathbb{E}_\mu(Y) = e^N$, and deduce in the same way that

[12] Recall that $a_n = \Omega(b_n)$ if there exist n_0 and $C > 0$ such that $|a_n| \geq C|b_n|$ for all $n \geq n_0$.

[13] The order statistic X^{*k} has the same distribution as $F^{-1}(Y^{*k})$, where F is the distribution function of ν and Y_i, $i \in N$, are IID random variables uniformly distributed on $[0, 1]$. By symmetry, $\mathbb{E}Y^{*k} = \frac{k}{n+1}$.

there is a vector $\beta \in \mathbb{R}^n$ such that $(\varphi_1(x) - \varphi_2(x)) \cdot x_1 \geq \beta \cdot (x - e^N)$ for all x. Symmetry of φ and $\varphi(x) \in \Delta(N)$ imply promptly the existence of $\theta \geq 0$ such that, for any x with weakly increasing coordinate:

$$\theta \left(1 - \frac{x_{i-1}}{x_i}\right) \leq \varphi_i(x) - \varphi_{i-1}(x) \leq \theta \left(\frac{x_i}{x_{i-1}} - 1\right) \text{ for all } i = 1, \cdots, n.$$

Applying this when x_i is a geometric sequence with a large exponent gives $\theta \leq \frac{2}{n(n-1)}$, and by choosing $\theta^* = \frac{2}{n(n-1)}$ and defining φ appropriately, we guarantee a worst case utilitarian performance of the order of $O(\frac{1}{n})$, comparable to the Price of Envy Freeness for a prior dependent rule: [14].

Similarly for bads we find that, if the coordinates of x are weakly increasing, an Envy Free rule φ is such that

$$\theta \left(1 - \frac{x_{i-1}}{x_i}\right) \leq \varphi_{i-1}(x) - \varphi_i(x) \leq \theta \left(\frac{x_i}{x_{i-1}} - 1\right) \text{ for all } i = 1, \cdots, n$$

where again the parameter θ is at most $\frac{2}{n(n-1)}$. But this time the utilitarian performance of such a rule is fairly poor as one can see with $\theta^* = \frac{2}{n(n-1)}$ and the disutility profile $x_i = 2^{i-1}$ for all i. The most efficient profile of share is then $\varphi_i(x) = (n-i)\theta^*$ and the ratio $\frac{1}{x_1}(\sum_1^n \varphi_i(x) x_i)$ is then in the order of $\frac{2^n}{n^2}$!

Asymmetric rules If the agents are endowed with unequal ownership rights on the objects, captured by the shares $\lambda \in \Delta(N)$, it is natural to adapt Fair Share as follows (for goods): $\mathbb{E}_\mu(\varphi_i(X^r) \cdot X_i) \geq \lambda_i \mathbb{E}_\mu(X_i)$ for all i. We can again adapt the argument in Proposition 2 to characterize this constraint by the existence, for each i, of a linear form lower bounding the function $x \to \varphi_i(x) \cdot x_i$. But these linear forms move in a space of high dimension and the characterization of the undominated fair rules is much more difficult.

References

[1] M Akbarpour, S Li, S Oveis Gharan. *Dynamic Matching Market Design*. arXiv.org arXiv:1402.3643. (2014).

[2] M Aleksandrov, H Aziz, S Gaspers, T Walsh. *Online Fair Division: Analysing a Food Bank Problem*. Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence IJCAI 2015, 2540-2546.(2015).

[3] N Arnosti, R Johari, K Yash. *Managing Congestion in Matching Markets*. Available at SSRN: https://ssrn.com/abstract=2427960 or http://dx.doi.org/10.2139/ssrn.2427960. (May 7, 2018).

[4] I Ashlagi, M Braverman, Y Kanoria, P. Shi. *Clearing matching markets efficiently: informative signals and match recommendations*. available on SSRN, (August 2018).

[5] I Ashlagi, M Burq, P Jaillet, A Saberi. *Maximizing Efficiency in Dynamic Matching Markets*. arXiv:1803.01285 [cs.DS] https://arxiv.org/abs/1803.01285.(2018).

[6] G Benade, AM Kazachkov, AD Procaccia, .A Psomas. *How to Make Envy Vanish Over Time*. EC-18: Proceedings of the 19th ACM Conference on Economics and Computation, (June 2018).

[7] D Bertsimas, VF Farias, N Trichakis. *The Price of Fairness*. Operations Research, 59(1), 17-31. (2011).

[8] D Bertsimas, VF Farias, N Trichakis. *On the efficiency-fairness trade-off.* Management Science 58.12, 2234-2250.(2012).

[9] F Bloch, D Cantala. *Dynamic Assignment of Objects to Queuing Agents*. American Economic Journal: Microeconomics, 9 (1): 88-122. (2017).

[10] L Bo, L Wenyang, L Yingka. *Dynamic Fair Division Problem with General Valuations.* arXiv:1802.05294v2 [cs.GT] (18 Apr 2018).

[11] A Bogomolnaia, H Moulin, F Sandomirskiy, E Yanovskaya. 2017. *Competitive division of a mixed manna*. Econometrica. Vol. 85. No. 6. P. 1847-1871.(2017).

[12] A Bogomolnaia, H Moulin, F Sandomirskiy, E Yanovskaya. 2018. *Dividing bads under additive utilities*. Social Choice and Welfare (2018): 1-23.

[13] A Borodin, R El-Yaniv. Online computation and competitive analysis. Cambridge University Press. (2005)

[14] I Caragiannis, C Kaklamanis, P Kanellopoulos, M Kyropoulou. *The Efficiency of Fair Division*. Proceedings of the 5th International Conference on Web and Internet Economics (WINE), 475-482.(2009).

[15] G Carroll. *Robustness and linear contracts*. American Economic Review 105,2,536-6. (2015).

[16] N Devanur, J Hartline, A Karlin, T Nguyen. Prior-independent multi-parameter rule design. In International Workshop on Internet and Network Economics, pp. 122-133. Springer, Berlin, Heidelberg, 2011.

[17] N Devanur, K Jain, B Sivan,CA Wilens. *Near Optimal Online Algorithms and Fast Approximation Algorithms for Resource Allocation Problems*. Journal of the ACM (JACM) 66.1 (2019): 7.

[18] J Feldman, A Mehta, V Mirrokni, S Muthukrishnan. *Online stochastic matching: Beating 1-1/e*. Foundations of Computer Science, 2009. FOCS'09. 50th Annual IEEE Symposium (2009 Oct 25) (pp. 117-126). IEEE. https://arxiv.org/pdf/0905.4100.

[19] E Friedman, C-A Psomas, S Vardi. *Dynamic fair division with minimal disruptions*. EC-15: Proceedings of the 16th ACM Conference on Economics and Computation, 697-713. (2015).

[20] E Friedman, C-A Psomas, S Vardi.*Controlled dynamic fair division.* EC-17: Proceedings of the 18th ACM Conference on Economics and Computation, 461–478. (2017).

[21] F Hao, M Kodialam, TV Lakshman. *Online Allocation of Virtual Machines in a Distributed Cloud.* IEEE/ACM Transactions on Networking, 25,1, 238-249, (Feb 2017).

[22] J Hartline, A Goldberg, A Wright. *Competitive Auctions and Digital Goods*. SODA 2001, Proceedings of the 12th ACM-Siam Symposium on Discrete Algorithms. (2001).

[23] RM Karp, UV Vazirani, and VV Vazirani. *An optimal algorithm for on-line bipartite matching.* Proceedings of the ACM STOCS90 Conference, pp. 352,358, (1990).

[24] I Kash, AD Procaccia, N Shah. *No Agent Left Behind: Dynamic Fair Division of Multiple Resources*. Journal of Artificial Intelligence Research 51 (2014) 579-603.

[25] R Laraki. On the regularity of the convexification operator on a compact set. Journal of Convex Analysis 11, no. 1 (2004): 209-234.

[26] H Steinhaus. *The Problem of Fair Division*, Econometrica, 16, 101–104.(1948).

[27] W Thomson. *The fair division of a fixed supply among a growing population.* Mathematics of Operations Research, 8(3):319-326.(1983).

[28] MU Unver. *Dynamic kidney exchange.* The Review of Economic Studies, 77(1):372-414.(2010).

[29] T Walsh. *Online cake cutting.* International Conference on Algorithmic DecisionTheory. Springer, Berlin, Heidelberg. (2011).

A Proofs for Section 6

A.1 Proof of Proposition 3

Statement **if:** The proof is the same as in Proposition 2 for goods, upon reversing inequalities.

Statement **only if:** Again the proof mimics that of the only if statement in Proposition 2. Fix a prior-independent rule φ meeting FS and define $f(x) = \varphi_1(x) \cdot x_1$; by symmetry $f(e^N) = \frac{1}{n}$. For any convex coefficients $\mu \in \Delta(K)$ and convex combination $\sum_{k=1}^{K} \mu_k y^k = e^N$ in $\mathbb{R}_+^N$, we apply FS to the normalised problem in which $X = y^k$ with probability μ_k and obtain $\sum_{k=1}^{K} \mu_k f(y^k) \leq f(e^N)$. Therefore the concavification g of f coincides with f at e^N, and there is some $\alpha \in \mathbb{R}^N$ supporting its graph at $(e^N, g(e^N))$, which means

$$\varphi_1(x) \cdot x_1 \leq \alpha \cdot (x - e^N) + \frac{1}{n} \text{ for all } x \in \mathbb{R}_+^N$$

The same symmetry arguments show that α takes the form $\alpha = (\alpha_1, \beta, \beta, \cdots, \beta)$ and $\alpha \cdot e^N = \frac{1}{n}$. This time the inequality $0 \leq \varphi_1(x) \cdot x_1 \leq \alpha_1 x_1 + \beta x_{N \diagdown 1}$ implies $\alpha \geq 0$. Setting $\delta = n\beta$ and rearranging we get finally:

$$\varphi_i(x) \leq \frac{1}{n} + \delta \left(\frac{\overline{x}}{x_i} - 1 \right) \text{ for all } i \in N \text{ and } x \in \mathbb{R}_+^N$$

Because $\frac{\overline{x}}{x_i} \geq \frac{1}{n}$ the inequality $\varphi_i(x) \geq 0$ holds everywhere if it holds at $x = e^i$, and there it implies the bound $\delta \leq \frac{1}{n-1}$. Then the change of parameters $\theta = (n-1)\delta$ implies the desired inequality (8). Finally we note that the inequality $\sum_N \varphi_i(x) = 1 \leq \sum_N (\frac{1}{n} + \frac{\theta}{n-1}(\frac{\overline{x}}{x_i} - 1))$ holds automatically for any x and δ, because $\min_{x \in \mathbb{R}_+^N} \sum_N \frac{\overline{x}}{x_i} = n$. ■

A.2 Proof of Theorem 2

Step 1 We prove first that if the prior-independent rule φ meets inequalities (8) for some $\theta, 0 \leq \theta \leq 1$, then φ^θ dominates φ or equals φ. In Step 2 we show that φ^1 dominates φ^θ if $\theta < 1$.

First for $\theta = 0$, inequalities (8) imply that φ itself is Equal Split i. e., φ^0. From now on we assume $\theta > 0$.

Along the ray through e^N the rules φ and φ^θ coincide by Symmetry. Now we fix $x \in \mathbb{R}_+^N$ not parallel to e^N and let $\widetilde{t}$ be defined by (9). From (8) we get $\varphi_i(x) \leq \varphi_i^\theta(x)$ for all i s. t. $x_i \leq x^{*\widetilde{t}}$, hence

$$\sum_{i:x_i \leq x^{*\widetilde{t}}} (\varphi_i(x) - \varphi_i^\theta(x))x_i \geq \sum_{i:x_i \leq x^{*\widetilde{t}}} (\varphi_i(x) - \varphi_i^\theta(x))x^{*(\widetilde{t}+1)} \tag{15}$$

Next we have $\sum_{i:x_i \geq x^{*(\widetilde{t}+1)}} \varphi_i^\theta(x)x_i = \sum_{i:x_i \geq x^{*(\widetilde{t}+1)}} \varphi_i^\theta(x)x^{*(\widetilde{t}+1)}$ because $\varphi_i^\theta(x) = 0$ if $x_i > x^{*(\widetilde{t}+1)}$. Thus

$$\sum_{i:x_i \geq x^{*(\widetilde{t}+1)}} (\varphi_i(x) - \varphi_i^\theta(x))x_i \geq \sum_{i:x_i \geq x^{*(\widetilde{t}+1)}} (\varphi_i(x) - \varphi_i^\theta(x))x^{*(\widetilde{t}+1)} \tag{16}$$

Summing up these two inequalities gives the corresponding weak inequality (7).

Assume finally that all inequalities (7) are equalities. If at least one x_i is zero, (8) implies that $\varphi(x)$ does not put any weight outside $\sigma(x, 1)$, so $\varphi(x) = \varphi^\theta(x)$. If each x_i is strictly positive our assumption implies that (15) is an equality; but (9) implies $x^{*\widetilde{t}} < x^{*(\widetilde{t}+1)}$ therefore $\varphi_i(x) = \varphi_i^\theta(x)$ as long as $x_i \leq x^{*\widetilde{t}}$. Now (16) cannot be an equality if $\varphi(x)$ puts any weight on agents with disutilities larger than $x^{*(\widetilde{t}+1)}$, and we conclude $\varphi(x) = \varphi^\theta(x)$ by symmetry of φ.

Step 2. We show that φ^{θ^+} dominates φ^{θ^-} if $\theta^+ > \theta^- > 0$. We write these rules φ^+ and φ^- for simplicity, and fix $x \in \mathbb{R}_+^N$. So for $\varepsilon = +, -$, inequalities (9) define the integer t^ε for $\varphi^\varepsilon(x)$.

We use the notation

$$\delta_i = \frac{1}{n-1}\left(\frac{\overline{x}}{x_i} - 1\right) \text{ and } \psi_i^\varepsilon = \frac{1}{n} + \theta^\varepsilon \delta_i$$

We prove inequality (7) between φ^+ and φ^- for a vector x with no two equal coordinates. This will be enough because each mapping φ^θ is only discontinuous at x if $|\sigma(x, \widetilde{t}+1)| > 1$, and the total disutility $\sum_N \varphi_i^\theta(x)x_i$ is continuous at such points.

Finally we label the coordinates of x increasingly, so that $x_i = x^{*i}$ for all i, and the definition of $\varphi^\varepsilon(x)$ is notationally simpler: $\varphi_i^\varepsilon(x) = \psi_i^\varepsilon > 0$ for $1 \leq i \leq t^\varepsilon$; $0 \leq \varphi_{t^\varepsilon+1}^\varepsilon(x) < \psi_{t^\varepsilon+1}^\varepsilon$; $\varphi_j^\varepsilon(x) = 0$ for $j > t^\varepsilon + 1$.

We claim first $t^+ \leq t^-$, and if $t^+ = t^- = t$ then $\lambda = \frac{\varphi_{t^++1}^+(x)}{\psi_{t^++1}^+} < \mu = \frac{\varphi_{t^-+1}^-(x)}{\psi_{t^-+1}^-}$ where $0 \leq \lambda, \mu < 1$. To prove this we compute

$$1 = \sum_1^{t^+} \psi_i^+ + \lambda\psi_{t^++1}^+ = \frac{t^+ + \lambda}{n} + \theta^+(\delta_{\{1,\cdots,t^+\}} + \lambda\delta_{t^++1})$$

As $\frac{t^++\lambda}{n} < 1$, this implies $\delta_{\{1,\cdots,t^+\}} + \lambda\delta_{t^++1} > 0$, therefore

$$1 > \frac{t^+ + \lambda}{n} + \theta^-(\delta_{\{1,\cdots,t^+\}} + \lambda\delta_{t^++1})$$

But by repeating the computation above for $\varphi^-(x)$ we get

$$1 = \frac{t^- + \mu}{n} + \theta^-(\delta_{\{1,\cdots,t^-\}} + \mu\delta_{t^-+1})$$

We see that $t^- < t^+$ brings a contradiction between the last two statements. And if $t^- = t^+ = t$ they imply $\lambda\psi^-_{t+1} < \mu\psi^-_{t+1}$ so $\lambda < \mu$ because $\psi^-_i > 0$ for all i. The claim is proved.

Now we evaluate the difference Δ in total disutility collected by our two rules:

$$\Delta = \sum_N (\varphi^+_i(x) - \varphi^-_i(x))x_i$$

$$= \sum_1^{t^+}(\psi^+_i - \psi^-_i)x_i + (\lambda\psi^+_{t^++1} - \psi^-_{t^++1})x_{t^++1} - \sum_{t^++2}^{t^-} \psi^-_i x_i - \mu\psi^-_{t^-+1}x_{t^-+1}$$

where we have assumed $t^+ < t^-$; if instead $t^+ = t^- = t$ the last three terms of the sum reduce to $(\lambda\psi^+_{t+1} - \mu\psi^-_{t+1})x_{t^++1}$. As x_i increases in i we have

$$\Delta \leq \sum_1^{t^+}(\psi^+_i - \psi^-_i)x_i + \lambda\psi^+_{t^++1}x_{t^++1} - (\psi^-_{\{t^++1,\ldots,t^-\}} + \mu\psi^-_{t^-+1})x_{t^++1}$$

and from $\varphi^+_N(x) = \varphi^-_N(x)$ we get $\psi^-_{\{t^++1,\ldots,t^-\}} + \mu\psi^-_{t^-+1} = \sum_1^{t^+}(\psi^+_i - \psi^-_i) + \lambda\psi^+_{t^++1}$. Rearranging the right hand term in the inequality above, and going back to the definition of ψ^ε_i this gives

$$\Delta \leq \sum_1^{t^+}(\psi^+_i - \psi^-_i)(x_i - x_{t^++1}) = (\theta^+ - \theta^-)\sum_1^{t^+}\delta_i(x_i - x_{t^++1})$$

We show finally that the right hand term above is strictly negative, as desired.

The sequence δ_i is (strictly) decreasing and initially positive. As $\delta_{\{1,\cdots,t^+\}} + \lambda\delta_{t^++1} > 0$, we have $\delta_{\{1,\cdots,t^+\}} > 0$. The sequence $\gamma_i = x_{t^++1} - x_i$ is positive and (strictly) decreasing. These facts imply that $\sum_1^{t^+}\delta_i\gamma_i$ strictly positive. Let δ_{i^*} be the first strictly negative term in the sequence δ_i: we have $\sum_1^{i^*-1}\delta_i\gamma_i \geq \sum_1^{i^*-1}\delta_i\gamma_{i^*}$ as all terms are non negative and γ_i decreases; also $\sum_{i^*}^{t^+}\delta_i\gamma_i > \sum_{i^*}^{t^+}\delta_i\gamma_{i^*}$ as $\delta_i < 0$ and $\gamma_i < \gamma_{i^*}$. Thus $-\Delta = \sum_1^{t^+}\delta_i\gamma_i > \delta_{\{1,\cdots,t^+\}}\gamma_{i^*}$.

B Proofs for section 7

B.1 Proof of Lemma 1

Step 1. For goods. The inequality $PoI_n(\varphi) \leq PoF_n(\varphi)$ is clear. Next, for any $\mathcal{P} \in \Pi_n$, not necessarily symmetric, there exists some $x \in \mathbb{R}^N_+$ such that

$$\frac{\mathbb{E}_\mu(\max_i X_i)}{\pi(\varphi, \mathcal{P})} \leq \frac{\max_i x_i}{\sum_{i\in N}\varphi_i(x)\cdot x_i}$$

This proves $PoF_n(\varphi) \leq \sup_{x\in\mathbb{R}^N_+}\frac{\max_i x_i}{\sum_{i\in N}\varphi_i(x)\cdot x_i}$. Now we pick an arbitrary $x \in \mathbb{R}^N_+$ and check the inequality $\frac{\max_i x_i}{\sum_{i\in N}\varphi_i(x)\cdot x_i} \leq PoI_n(\varphi)$, thus completing the proof.

Consider the symmetric problem $\mathcal{P} \in \Pi_n$ that selects each of the $n!$ permutations of $\frac{1}{x}x$ with equal probability $\frac{1}{n!}$. By Symmetry of φ we have $\pi(\varphi, \mathcal{P}) = \sum_{i \in N} \varphi_i(x) \cdot x_i$. It will be enough to construct a rule $\xi \in \Xi(FS)$ such that $\pi(\xi, \mathcal{P}) = \max_i x_i$, because $\frac{\pi(\xi,\mathcal{P})}{\pi(\varphi,\mathcal{P})} \leq PoI_n(\varphi)$. To this end we note that the utilitarian rule ${}^{ut}\xi$, that for each $\mathcal{P} \in \Pi_n$ divides the good equally between all agents with highest utility, violates FS in general but not if the problem $\mathcal{P}$ is symmetric (the distribution μ is symmetric in all variables x_i).[14] Thus we can pick to choose ξ equal to ${}^{ut}\xi$ for symmetric problems, and meeting FS elsewhere.

The similar argument for bads is omitted.

B.2 Proof of Proposition 4

Statement i) Pick $\varphi \in \Phi(FS)$ and $\mathcal{P} \in \Pi_n$. The FS property implies

$$\pi(\varphi, \mathcal{P}) = \sum_{i \in N} \mathbb{E}_\mu(\varphi_i(X) \cdot X_i) \geq \frac{1}{n} \sum_{i \in N} \mathbb{E}_\mu X_i \geq \frac{1}{n} \mathbb{E}_\mu(\max_i X_i)$$

and the first claim follows. If φ is the Equal Split rule, the first inequality above is an equality, and the second one is an equality if the random variable X is uniform over the coordinate profiles e^i.

Statement ii) By Lemma 1 we must evaluate $\sup_{x \in \mathbb{R}^N_+ \setminus 0} \frac{\sum_{i \in N} x_i}{\sum_{i \in N} x_i^2} \max_i x_i$. By rescaling x we can assume $x_1 = 1 = \max_{i \geq 2} x_i$, then we must show

$$\sup \frac{1 + \sum_2^n x_i}{1 + \sum_2^n x_i^2} = \frac{\sqrt{n} + 1}{2}$$

where the supremum is on all $x_2, \cdots, x_n \in [0, 1]$. We omit the straightforward argument.

Statement iii) We fix $\theta, 0 < \theta \leq 1$, set $N = \{1, \cdots, n\}$ and rewrite inequalities (3) as

$$\varphi_i^\theta(x) \geq \max\left\{ \left(\frac{1}{n} + \frac{\theta}{n-1}\right) - \frac{\theta}{n(n-1)} \frac{x_N}{x_i}, 0 \right\} \text{ for all } i \text{ and } x \in \mathbb{R}^N_+$$

By Lemma 1 we must evaluate the smallest feasible value of $\frac{1}{x^{*n}}\{\sum_{i=1}^n \varphi_i^\theta(x) \cdot x_i\}$ in $\mathbb{R}^N_+$. As noted already, this function is continuous in x (even though φ^θ itself is not at those profiles where several agents have the highest utility) so it will be enough to compute the infimum of this ratio for profiles x such that $x_i < x_n$ for all $i \leq n-1$.

We first compute the desired lower bound when $(\frac{1}{n} + \frac{\theta}{n-1}) - \frac{\theta}{n(n-1)} \frac{x_N}{x_i} \geq 0$ for all i, so that all agents $i \leq n-1$ get exactly this share and agent n gets

$$\varphi_n^\theta(x) = 1 - \sum_{i=1}^{n-1} \varphi_i^\theta(x) = \frac{1}{n} - \theta + \frac{\theta}{n(n-1)} \left(\left(\sum_{i=1}^{n-1} \frac{1}{x_i} \right) x_n + n - 1 + \sum_{\{i,j\} \subset \{1,\cdots,n-1\}} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} \right) \right)$$

In the right hand side, if we fix the sum $\sum_{i=1}^{n-1} x_i$ the first sum is minimal when all utilities are equal; the second sum is also minimal and equal to $(n-1)(n-2)$ when utilities are equal. It is also

[14]
$$\mathbb{E}_\mu(X_1) \leq \mathbb{E}_\mu(\max_i X_i) = \pi({}^{ut}\xi, \mathcal{P}) = \sum_i \mathbb{E}_\mu({}^{ut}\xi_i^{\mathcal{P}}(X) \cdot X_i) = n \mathbb{E}_\mu({}^{ut}\xi_1^{\mathcal{P}}(X) \cdot X_1)$$

clear that for $i,j \leq n-1$ the sum $\varphi_i^\theta(x) \cdot x_i + \varphi_j^\theta(x) \cdot x_j$ is constant when we equalize x_i and x_j while keeping their sum constant. Thus we can assume that $x_i = y$ for $1 \leq i \leq n-1$, so that the share of agent n is

$$\varphi_n^\theta(x) = \frac{1}{n} - \theta + \frac{\theta}{n}\left(\frac{x_n}{y} + n - 1\right) = \frac{1}{n}(1-\theta) + \frac{\theta}{n}\frac{x_n}{y}$$

Then we compute

$$\frac{1}{x_n}\left(\sum_{i=1}^{n} \varphi_i^\theta(x) \cdot x_i\right) = \varphi_n^\theta(x) + (n-1)\frac{\varphi_1^\theta(x)}{x_n} = \frac{1}{n}\left((1-2\theta) + \theta\frac{x_n}{y} + (n-1+\theta)\frac{y}{x_n}\right)$$

and the minimum in x_n, y of this expression is achieved for $\frac{x_n}{y} = (\frac{(n-1+\theta)}{\theta})^{\frac{1}{2}}$ (which is larger than 1 as needed) and its value is

$$\frac{1}{n}\left((1-2\theta) + 2\sqrt{(n-1+\theta)\theta}\right)$$

as stated. Clearly it decreases in θ.

It remains to consider the case where for some $i^* \leq n-1$ we have, for all $i \leq i^*-1$ and all $j \geq i^*$:

$$\left(\frac{1}{n} + \frac{\theta}{n-1}\right) - \frac{\theta}{n(n-1)}\frac{x_N}{x_i} < 0 \leq \left(\frac{1}{n} + \frac{\theta}{n-1}\right) - \frac{\theta}{n(n-1)}\frac{x_N}{x_j}$$

Observe that if we decrease x_i to zero for all $i \leq i^*-1$, without changing other coordinates, the share of each agent $j, i^* \leq j \leq n-1$, increases (strictly if some x_i is positive), while that of agent n decreases, therefore the ratio $\frac{1}{x^{*n}}\{\sum_{i=1}^n \varphi_i^\theta(x) \cdot x_i\}$ decreases. Thus it is enough to assume $x_i = 0$ for all $i \leq i^*-1$. Computing the share of agent n and the total utility $\sum_{i=1}^n \varphi_i^\theta(x) \cdot x_i$ is then more tedious but very similar, and the argument that we can assume $x_i = y$ for $i^* \leq i \leq n-1$ is unchanged. In turn we find

$$\varphi_n^\theta(x) = \frac{i^*}{n}\left(1 - \frac{n-i^*}{n-1}\theta\right) + \frac{n-i^*}{n(n-1)}\theta\frac{x_n}{y}$$

$$\frac{1}{x_n}\left(\sum_{i=1}^{n} \varphi_i^\theta(x) \cdot x_i\right) = \frac{i^*}{n} - \frac{(n-i^*)(i^*+1)}{n(n-1)}\theta + \frac{n-i^*}{n(n-1)}\left(\theta\frac{x_n}{y} + (n-1+i^*\theta)\frac{y}{x_n}\right)$$

of which the minimum in x_n, y is

$$\frac{i^*}{n} - \frac{(n-i^*)}{n(n-1)}\left((i^*+1)\theta - 2\sqrt{(n-1+i^*\theta)\theta}\right)$$

and this quantity increases in i^* because $(i^*+1)\theta - 2\sqrt{(n-1+i^*\theta)\theta}$ does. Therefore the wost case is for $i^* = 1$, and we are done.

Statement iv) Clearly $\inf_{\xi \in \Xi(FS)} PoF_n(\xi) \leq \inf_{\varphi \in \Phi(FS)} PoF_n(\varphi) \leq PoF_n(\varphi^1)$, so the inequality $\inf_{\xi \in \Xi(FS)} PoF_n(\xi) \leq \frac{n}{2\sqrt{n}-1}$ follows from Lemma 1 and statement *iii*).

Next we fix $n, p, 1 \le p \le n-1$ and consider the problem $\mathcal{P}(n,p) \in \Pi_n$ with n agents and p equiprobable states:

$$\begin{array}{cccc} & state & \omega_1 & \cdots & \omega_p \\ & proba & 1/p & \cdots & 1/p \\ & X_1 & p & 0 & 0 \\ & \cdots & 0 & p & 0 \\ \text{utilities:} & X_p & 0 & 0 & p \\ & X_{p+1} & 1 & 1 & 1 \\ & \cdots & 1 & 1 & 1 \\ & X_n & 1 & 1 & 1 \end{array}$$

Let N_1 be the set of the p "single-minded" agents and N_2 that of the other $n-p$ "indifferent" agents. Fix an arbitrary prior-dependent rule $\xi \in \Xi(FS)$ and write $\mathbb{E}_\mu(Y_i) = \mathbb{E}_\mu(\xi_i^{\mathcal{P}(n,p)}(X) \cdot X_i)$ the expected utility of agent i.

We call λ_k the total share ξ gives to N_2 at state ω_k. Then $\mathbb{E}_\mu(Y_{N_2}) = \frac{1}{p}\sum_{k=1}^p \lambda_k$ and Fair Share implies $\sum_{k=1}^p \lambda_k \ge \frac{p(n-p)}{n}$. If ξ gives the remaining shares to single minded agent k in state ω_k, then $\mathbb{E}_\mu(Y_{N_1}) = \frac{1}{p}\sum_{k=1}^p (1-\lambda_k)p = p - \sum_{k=1}^p \lambda_k$. This is the best ξ can do for the utilitarian objective. Compute

$$\mathbb{E}_\mu(Y_N) = \left(p - \sum_{k=1}^p \lambda_k\right) + \left(\frac{1}{p}\sum_{k=1}^p \lambda_k\right) = p - \frac{p-1}{p}\sum_{k=1}^p \lambda_k$$
$$\le p - \frac{(p-1)(n-p)}{n} = \frac{p^2}{n} - \frac{p}{n} + 1$$
$$\Longrightarrow \left(\frac{\mathbb{E}_\mu(\max_i X_i)}{\mathbb{E}_\mu(Y_N)}\right)^{-1} = \frac{\mathbb{E}_\mu(Y_N)}{p} \le \frac{p}{n} + \frac{1}{p} - \frac{1}{n}$$

The minimum of $\frac{p}{n} + \frac{1}{p} - \frac{1}{n}$ over real numbers is achieved for $p = \sqrt{n}$, and is worth $\frac{2}{\sqrt{n}} - \frac{1}{n} = (PoI_n(\varphi^1))^{-1}$. As p is integer and $p \to f(p) = \frac{p}{n} + \frac{1}{p}$ is convex, the minimum over integers is at most $\alpha = \max\{f(\sqrt{n}+\frac{1}{2}), f(\sqrt{n}-\frac{1}{2})\}$. Routine computations show $\alpha \le \frac{2}{\sqrt{n}} + \frac{1}{2n}$ therefore $[\frac{\mathbb{E}_\mu(\max_i X_i)}{\mathbb{E}_\mu(Y_N)}]^{-1} \le \frac{2\sqrt{n}-\frac{1}{2}}{n}$ and the proof is complete.

B.3 Proof of Proposition 5

Statement i) If φ is the Equal Split rule, then $\frac{1}{\min_i x_i}(\sum_{i\in N}\varphi_i(x)\cdot x_i) = \frac{x_N}{n\cdot\min_i x_i}$ for all $x \in \mathbb{R}_+^N$, and this ratio is clearly unbounded, so the claim follows by Lemma 1.

Recall from Proposition 1 that the prior-independent rule φ^{pro} is efficient at any profile $x \in \mathbb{R}_+^N$ with at least one zero coordinate. For $x \gg 0$ we have $\frac{1}{\min_i x_i}(\sum_{i\in N}\varphi_i^{pro}(x)\cdot x_i) = \frac{1}{\min_i x_i}\frac{n}{\sum_{i\in N}\frac{1}{x_i}} = \frac{\widetilde{x}}{\min_i x_i}$ where $\widetilde{x}$ is the harmonic mean of the x_i. The inequality $\widetilde{x} \le n\min_i x_i$ is always true, and becomes an equality when $x_1 = \min_i x_i$, and all other coordinates are equal and go to infinity. So the $PoI(\varphi^{pro})$ is indeed n.

Statement ii) The lower bound follows from the lower bound on $\inf_{\xi\in\Xi(FS)} PoF_n(\xi)$ (statement *iii* proven below) and from $PoI_n(\varphi^1) = PoF_n(\varphi^1) \ge \inf_{\xi\in\Xi(FS)} PoF_n(\xi)$.

To prove the upper bound $PoF_n(\varphi^1) \leq \frac{n}{4} + \frac{5}{4}$, we fix an arbitrary profile x and majorize $\frac{1}{\min_i x_i}(\sum_{i\in N} \varphi_i^1(x) \cdot x_i)$. Because φ^1 is homogeneous of degree zero and symmetric, and φ^1 is efficient if $x_1 = 0$, we can without loss assume $x_1 = 1$ and x_i increases weakly in i. We must bound $U_N(x) = \sum_{i\in N} \varphi_i^1(x) \cdot x_i$.

By definition of φ^1 there exists an index $s+1$ such that

$$\frac{1}{n(n-1)} \sum_{i=1}^{s+1} \frac{x_{N\setminus i}}{x_i} \leq 1 < \frac{1}{n(n-1)} \sum_{i=1}^{s+2} \frac{x_{N\setminus i}}{x_i}$$

and $\varphi_i^1(x) = \frac{1}{n(n-1)} \frac{x_{N\setminus i}}{x_i}$ for $i \leq s+1$.

We set $\Delta = n(n-1) - \sum_{i=1}^{s+1} \frac{x_{N\setminus i}}{x_i}$, $\Delta \geq 0$, and develop $U_N(x)$ as follows

$$n(n-1)U_N(x) = \sum_{i=1}^{s+1} x_{N\setminus i} + \Delta x_{s+2} = s\sum_{i=1}^{s+1} x_i + (s+1) \sum_{j=s+2}^{n} x_j + \Delta x_{s+2}$$

Say we replace each $x_i, 2 \leq i \leq s+1$ by their average $y = \frac{1}{s}\sum_{i=2}^{s+1}$, ceteris paribus: this will decrease the total weight given by φ^1 to these coordinates, which is $\frac{x_N}{n(n-1)}(\sum_2^{s+1} \frac{1}{x_i})$, and increase the weight to coordinates x_{s+2} and beyond. Therefore this move increases $U_N(x)$, so we can assume that these s coordinates are all equal to y. We also set $\sum_{j=s+2}^{n} x_j = w$. Now we try to bound

$$n(n-1)U_N(x) = s(1+sy) + (s+1)w + \Delta x_{s+2}$$

under the constraints

$$\Delta = n(n-1) + s + 1 - (1+sy+w)(1+\frac{s}{y}) \geq 0 \ ; \ 0 \leq \Delta x_{s+2} \leq 1 + sy + w \ ; \ w \geq (n-s-1)y$$

where the second inequality comes from the fact that $\Delta \leq \frac{x_{N\setminus(s+2)}}{x_{s+2}}$ and the third one from the fact that the coordinates of x increase weakly. These inequalities imply

$$n(n-1)U_N(x) \leq (s+1)(1+sy) + (s+2)w$$

$$(1+sy+w)\left(1+\frac{s}{y}\right) \leq n(n-1)+s+1 \Longrightarrow \left(1+\frac{s}{y}\right)w \leq n(n-1) - s\left(y+\frac{1}{y}\right) + s - s^2$$

$$\Longrightarrow w \leq (n(n-1)+s)\frac{y}{y+s} - sy$$

Combining $w \geq (n-s-1)y$ and the upper bound above gives

$$(n-s-1)y \leq (n(n-1)+s)\frac{y}{y+s} - sy \Longrightarrow y + s \leq n + \frac{s}{n-1} \leq n+1$$

Next we combine the upper bound on $n(n-1)U_N(x)$ with that on w:

$$n(n-1)U_N(x) \leq (s+1)(1+sy) + (s+2)(n(n-1)+s)\frac{y}{y+s} - (s+2)sy$$

$$= s + 1 - sy + (s+2)(n(n-1)+s)\frac{y}{y+s}$$

We now majorize the upper bound above in the two real variables s, y such that $y+s \leq n+1$. Observe first this bound increases in y because its derivative has the sign of $\frac{(s+2)(n(n-1)+s)}{(y+s)^2} - 1$ and $\frac{(s+2)(n(n-1)+s)}{(y+s)^2} \geq \frac{3(n^2-n+1)}{(n+1)^2}$. Thus we can take $y+s = n-1$ and use the inequality $\frac{s+2}{n+1} \leq 1$ to deduce the bound

$$n(n-1)U_N(x) \leq s+1+\frac{n(n-1)(s+2)y}{n+1}+sy\left(\frac{s+2}{n+1}-1\right) \leq n+\frac{n(n-1)}{n+1}(s+2)(n+1-s)$$

The maximum in s of $(s+2)(n+1-s)$ is $\frac{(n+3)^2}{4}$ for $s = \frac{n-1}{2}$, therefore

$$\Longrightarrow U_N(x) \leq \frac{1}{n-1}+\frac{(n+3)^2}{4(n+1)} = \frac{n}{4}+\frac{5}{4}-\frac{2}{n^2-1}$$

completing the proof of statement ii).

Statement iii)

Step 1. lower bound on $\inf_{\xi \in \Xi(FS)} PoF_n(\xi)$. Consider the normalised problem $\mathcal{P}$ with two equally probable states ω, ω', and the corresponding profiles of disutilities

$$x_1 = \frac{4}{n+1}, x_i = 2 \text{ for } 2 \leq i \leq n \ ; \ x_1' = 2\frac{n-1}{n+1}, x_i' = 0 \text{ for } 2 \leq i \leq n$$

Without the FS constraint total disutility is minimized by giving to agent 1 the whole bad in state ω, and no share at all in state ω', so that $\mathbb{E}_\mu(\min_i X_i) = \frac{2}{n+1}$. The FS constraint caps the share of agent 1 at $\frac{n+1}{2n}$ in state ω so at least $\frac{n-1}{2n}$ goes to the other agents and expected total disutility is at least $\frac{1}{n}+\frac{1}{2}\frac{n-1}{2n}2 = \frac{n+1}{2n}$. Therefore for any $\xi \in \Xi(FS)$ we have

$$\frac{\pi(\xi, \mathcal{P})}{\mathbb{E}_\mu(\min_i X_i)} \geq \frac{(n+1)^2}{4n} = \frac{n}{4}+\frac{1}{2}+\frac{1}{4n}$$

Step 2. upper bound on $\inf_{\xi \in \Xi(FS)} PoF_n(\xi)$. We omit for brevity the proof that the above lower bound is achieved by the rule in $\Xi(FS)$ with the smallest utilitarian disutility at each $\mathcal{P} \in \Pi_n$.[15] In any event the upper bound on $PoF_n(\varphi^1)$ in statement ii) applies to $\inf_{\xi \in \Xi(FS)} PoF_n(\xi)$ as well.

C Asymptotic results and missing proofs for Section 8

C.1 Goods

Proposition 6 *Fix a distribution ν of X_i with $E_\nu X_1 = 1$ and $E_\nu (X_1)^\beta < \infty$ for some $\beta > 2$. Consider a problem $\mathcal{P}_n(\nu)$ with n agents and $\mu = \otimes_{i=1}^n \nu$. Then the utilitarian performance of the TH rule φ^θ, $\theta \in (0,1]$, satisfies*

$$R(\varphi^\theta, \mathcal{P}_n(\nu)) = \frac{1}{1-\mathbb{E}_\nu\left(1+\theta-\frac{\theta}{X_1}\right)_+ + \frac{\mathbb{E}_\nu(X_1(1+\theta)-\theta)_+}{\mathbb{E}_\mu \max_i X_i}}\left(1++O\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{\beta}}}\right)\right), \tag{17}$$

for large number of agents[16] *n. Here $(y)_+$ denotes* $\max\{y, 0\}$.

Note that the only dependence on n in formula (17) is through the expected value of $X^{*n} = \max_{i=1,..,n} X_i$ and the error-term.

[15] It is actually the PoF of a general normalized bargaining set; see the discussion in [7], [8].

[16] $a_n = O(b_n)$ if there exist n_0 and $C > 0$ such that $|a_n| \leq C|b_n|$ for all $n \geq n_0$.

C.1.1 Proof of Proposition 6

To simplify heavy formulas we assume that $\theta = 1$ (proof for other values of θ follows the same logic).

By the definition of the TH rule φ^1 we can represent the relative gain as

$$\sum_i X_i \varphi_i^1(X) = \sum_{i=1}^n X_i \left(\frac{2}{n} - \frac{X_N - X_i}{n(n-1)X_i}\right)_+ + X^{*n}\left(1 - \sum_{i=1}^n \left(\frac{2}{n} - \frac{X_N - X_i}{n(n-1)X_i}\right)_+\right)$$
$$= A + X^{*n} - B.$$

Consider the contribution of A first. Since all X_i have the same distribution $\mathbb{E}_\mu A = \mathbb{E}_\mu \left(2X_1 - \frac{\sum_{j\neq 1} X_j}{n-1}\right)_+$. Let us show that $\Delta_0 = \mathbb{E}_\mu A - \mathbb{E}_\nu (2X_1 - 1)_+$ is small. The function $(\,\cdot\,)_+$ is Lipschitz with constant one, thus by the Cauchy inequality and independence of X_j

$$|\Delta_0| \le \mathbb{E}_\mu \left|1 - \frac{\sum_{j\neq 1} X_j}{n-1}\right| = \frac{1}{n-1}\mathbb{E}_\mu \left|\sum_{j\neq 1}(X_j - 1)\right|$$

$$\le \frac{1}{n-1}\sqrt{\mathbb{E}_\mu \left(\sum_{j\neq 1}(X_j - 1)\right)^2} = \frac{\sqrt{\mathbb{V}_\nu X_1}}{\sqrt{n-1}} = O\left(\frac{1}{\sqrt{n}}\right)$$

if the variance $\mathbb{V}_\nu$ of X_1 is finite.

Now we will check that $\mathbb{E}_\mu B$ is close to $\mathbb{E}_\mu X^{*n} \cdot \mathbb{E}_\nu (2 - 1/X_1)_+$ (as if X^* is independent of X_i and $\sum X_j$ approximately equals its expectation). This is done in two steps:

Step 1: proving that $\mathbb{E}_\mu B$ does not change much if we put $(2-1/X_1)_+$ instead of $(2 - \sum_{j\neq 1} X_j/(n-1)X_1)_+$

Step 2: showing that the random variables X^{*n} and $(2 - 1/X_1)_+$ can be decoupled; the expected value of the product is close to the product of expectations.

Step 1. Proving that $\sum_{j\neq 1} X_j/(n-1)$ can be replaced by its expectation:

Since X_j are IID we have

$$\mathbb{E}_\mu B = \mathbb{E}_\mu X^{*n}\left(2 - \frac{\sum_{j\neq 1} X_j}{(n-1)X_1}\right)_+ = \mathbb{E} X^{*n}\left(2 - \frac{1}{X_1}\right)_+ + \Delta_1,$$

where

$$\Delta_1 = \mathbb{E}_\mu X^*(n)\left(\left(2 - \frac{\sum_{j\neq 1} X_j}{(n-1)X_1}\right)_+ - \left(2 - \frac{1}{X_1}\right)_+\right) = \mathbb{E}_\mu X^{*n} h(X)$$

Consider two cases depending on how far is the sum $\sum_{j\neq 1} X_j$ from its expected value. Let Q be the event that $\left|\frac{\sum_j X_j}{n-1} - 1\right| > \frac{1}{2}$. Then the probability of $\mathbb{P}_\mu(Q)$ is at most $\frac{8\mathbb{V}_\nu X_1}{n-1}$ by Markov inequality. Let us represent Δ_1 as $\mathbb{E}_\mu X^{*n} h(X) 1_Q + \mathbb{E}_\mu X^{*n} h(X) 1_{\bar{Q}}$. For the first term we use the estimate $h \le 2$ and then apply Holder inequality:

$$\mathbb{E}_\mu X^{*n}|h(X)|1_Q \le 2\mathbb{E}_\mu X^{*n} 1_Q \le \left(\mathbb{E}_\mu |X^{*n}|^{q'}\right)^{\frac{1}{q'}} (P_\mu(Q))^{\frac{1}{q}}.$$

To bound the second term consider the following inequality for $y, z \le 2$: $||y|_+ - |z|_+| \le (1_{y\ge 0} + 1_{z\ge 0})|y-z|$. Applying it to h we get:

$$|h(x)| \le \left(1_{\left\{\frac{1}{x_1} \le \frac{2(n-1)}{\sum_{j\neq 1} x_j}\right\}} + 1_{\left\{\frac{1}{x_1}\le 2\right\}}\right)\left|\frac{\sum_{j\neq 1} x_j}{(n-1)x_1} - \frac{1}{x_1}\right|$$

For $x \in \bar{Q}$ the function h is non-zero only if $\frac{1}{x_1} \le \frac{4}{3}$. Thus for such x we have $|h(x)| \le \frac{8}{3}\left|\frac{\sum_{j\neq 1}(x_j - 1)}{n-1}\right|$. Finally we get

$$\mathbb{E}_\mu X^{*n}|h(X)|1_{\bar{Q}} \le \frac{8}{3(n-1)}\mathbb{E}_\mu X^{*n}\left|\sum_{j\neq 1}(X_j - 1)\right| \le \frac{8}{3(n-1)}\sqrt{\mathbb{E}_\mu|X^{*n}|^2}\sqrt{\mathbb{E}\left(\sum_{j\neq 1}(X_j-1)\right)^2}$$

Combining all the estimates together, we see that $|\Delta_1| = O\left(\frac{\sqrt{\mathbb{E}_\mu|X^{*n}|^2}}{\sqrt{n}}\right)$. We will estimate $\mathbb{E}_\mu|X^{*n}|^2$ at the end of the proof.

Step 2. Decoupling X^{*n} and $(2-1/X_1)_+$:

Now we continue with B. We proved that B is close to $\mathbb{E}_\mu X^{*n}(2-1/X_1)_+$. Now we want to decouple the two factors and show that B is close to $\mathbb{E}_\mu X^{*n}\mathbb{E}_\nu(2-1/X_1)_+$. Define $\Delta_2 = \mathbb{E}_\mu X^{*n}\cdot\mathbb{E}_\nu\left(2-\frac{1}{X_1}\right)_+ - \mathbb{E}_\mu X^{*n}\left(2-\frac{1}{X_1}\right)_+$. The random variable $\xi = \max_{i=2\dots n} X_i$ is independent from $\left(2-\frac{1}{X_1}\right)_+$. Therefore

$$\Delta_2 = \mathbb{E}_\mu(X^{*n}-\xi)\cdot\mathbb{E}_\nu\left(2-\frac{1}{X_1}\right)_+ - \mathbb{E}_\mu(X^{*n}-\xi)\left(2-\frac{1}{X_1}\right)_+$$

By the definition X^{*n} is greater than ξ. Hence $|\Delta_2| \le 2\mathbb{E}_\mu(X^{*n}-\xi)$. To estimate the difference of expectations define X^{*n}_{-j} as $\max_{k=1\dots n,\ j\neq i} X_k$. Then $\mathbb{E}X^{*}*n_{-j} = \mathbb{E}\xi$ for all j. If X_i is the maximal over $i = 1,\cdots,n$, then all X^{*n}_{-j} except the one with $j = i$ coincide and are equal to X^{*n}. Thus $n\mathbb{E}\xi = \mathbb{E}\sum_{j=1..n} X^{*n}_{-j} \ge \mathbb{E}(n-1)X^{*n}$ and $\mathbb{E}X^{*n} - \mathbb{E}\xi \le \frac{\mathbb{E}X^{*n}}{n}$. Finally $|\Delta_2| = O\left(\frac{\mathbb{E}_\mu X^{*n}}{n}\right)$.

To estimate $\mathbb{E}_m u\,(X^{*n})^\alpha$ we use the standard approach. For $\alpha > 0$ we have $\mathbb{E}_\mu\,(X^{*n})^\alpha = -\int_0^\infty t^\alpha d\mathbb{P}_\mu(X^{*n}\ge t)$ and integration by part gives

$$\alpha\int_0^\infty t^{\alpha-1}\mathbb{P}_\mu(X^{*n}\ge t)dt = \int_0^T + \int_T^\infty.$$

The first integral does not exceed T^α. To estimate the second one we combine the union bound with Markov inequality: $\mathbb{P}_\mu(X^{*n}\ge t) \le n\mathbb{P}_\nu(X_1\ge t) \le n\frac{\mathbb{E}_\nu(X_1)^\beta}{t^\beta}$. Therefore

$$\alpha\int_T^\infty t^{\alpha-1}\mathbb{P}_\mu(X^{*n}\ge t)dt \le \alpha n\mathbb{E}_\nu(X_1)^\beta\int_T^\infty t^{\alpha-\beta-1}dt = \frac{\alpha}{\beta-\alpha}n\mathbb{E}_\nu(X_1)^\beta\frac{1}{T^{\beta-\alpha}}$$

for $\beta > \alpha$. Optimizing over T we get $\mathbb{E}_\mu\,(X^{*n})^\alpha \le \left(\frac{\beta}{\beta-\alpha}\right)\left(n\mathbb{E}_\nu(X_1)^\beta\right)^{\frac{\alpha}{\beta}} = O\left(n^{\frac{\alpha}{\beta}}\right)$.

It remains to put all pieces together:

$$\Delta_0+\Delta_1+\Delta_2 = O\left(\frac{1}{\sqrt{n}}\right) + O\left(\frac{\sqrt{E_\mu|X^{*n}|^2}}{\sqrt{n}}\right) + O\left(\frac{\mathbb{E}_\mu X^{*n}}{n}\right) = O\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{\beta}}}\right)$$

for any $\beta > 2$ such that $\mathbb{E}_\nu(X_1)^\beta < \infty$. This implies formula (17) for $\theta = 1$.

C.1.2 Proof of Lemma 2

For unbounded distributions $\mathbb{E}\max_i X_i$ tends to $+\infty$ and thus by Proposition 6 the utilitarian performance of φ^1 converges to $\left(1-\mathbb{E}_\nu\left(2-\frac{1}{X_1}\right)_+\right)^{-1}$. Thus the lower bound immediately follows from the inequality $|x_1-1|\geq x_1-\left(2-\frac{1}{x_1}\right)_+$.

For the upper bound we have

$$\left(R(\varphi^1,\mathcal{P}_\infty(\nu))\right)^{-1}\geq\mathbb{E}_\nu\left(X_1-\left(2-\frac{1}{X_1}\right)_+\right)\geq\mathbb{E}_\nu\left(X_1-\left(2-\frac{1}{X_1}\right)_+\right)1_{X_1\geq 1}=$$

$$=\mathbb{E}_\nu\left(X_1+\frac{1}{X_1}-2\right)1_{X_1\geq 1}=\mathbb{E}_\nu\left(\frac{(X_1-1)^2}{X_1}\right)1_{X_1\geq 1}=\mathbb{E}_\nu g(X_1)1_{X_1\geq 1},$$

where 1_A stands for the indicator of the event A. In order to relate the expected value of $g(X_1)$ to D we apply the Cauchy inequality

$$\frac{D}{2}=\mathbb{E}_\nu|X_1-1|1_{X_1\geq 1}=\mathbb{E}_\nu\left(\sqrt{g(X_1)}1_{X_1\geq 1}\cdot\frac{|X_1-1|1_{X_1\geq 1}}{\sqrt{g(X_1)}}\right)\leq$$

$$\leq\sqrt{\mathbb{E}_\nu g(X_1)1_{X_1\geq 1}}\sqrt{\mathbb{E}_\nu\frac{(X_1-1)^2}{g(X_1)}1_{X_1\geq 1}}.$$

The second factor on the right hand side can be estimated as follows

$$\mathbb{E}_\nu\frac{(X_1-1)^2}{g(X_1)}1_{X_1\geq 1}=\mathbb{E}_\nu X_1 1_{X_1\geq 1}=\mathbb{E}_\nu|X_1-1|1_{X_1\geq 1}+\mathbb{E}_\nu 1_{X_1\geq 1}\leq\frac{D}{2}+1,$$

which completes the proof.

C.2 Bads

C.2.1 Not much weight around zero

Proposition 7 *Consider a distribution ν such that $E_\nu X_1=1$ and $E_\nu\frac{1}{X_1}<\infty$. Then the utilitarian performance of the BH rule can be represented as*

$$R(\varphi^1,\mathcal{P}_n(\nu))=\frac{\mathbb{P}_\nu(X_1<T)+\gamma\mathbb{P}_\nu(X_1=T)}{\mathbb{E}_\mu\min_i X_i}(1+o(1)),\quad n\to\infty,\tag{18}$$

where $T>0$ and $\gamma, 0\leq\gamma<1$ are defined by the following condition[17]

$$\mathbb{E}_\nu\frac{1_{\{X_1<T\}}}{X_1}+\gamma\mathbb{P}(X_1=T)\frac{1}{T}=1.$$

For the proportional rule

$$R(\varphi^{pro},\mathcal{P}_n(\nu))=\frac{1}{\mathbb{E}_\mu\min_i X_i\cdot\mathbb{E}_\nu\frac{1}{X_1}}(1+o(1)).\tag{19}$$

[17] Formulas simplify for continuous distribution because $\mathbb{P}(X_i=T)=0$ for all T and thus we can always pick $\gamma=0$.

As in the proof of Proposition 6, symmetry of the problem implies $\pi(\varphi^1, \mathcal{P}_n(\nu)) = n\mathbb{E}_\mu X_1 \varphi_1^1(X)$ and hence it is enough to estimate the contribution of one agent. We will calculate this expectation in two steps: assuming first that $X_1 = z$ is fixed and averaging over X_j, $j \geq 2$ and then averaging over z.

Consider $\mathbb{E}_\mu \left(nX_1\varphi_1^1(X) \mid X_1 = z\right)$. By the definition of the BH rule we get

$$n \cdot X_1\varphi_1(X)\big|_{X_1=z} = \frac{X_{N\setminus 1}}{(n-1)} \cdot 1_Q + z \cdot \frac{1 - \sum_{j: X_j < z} \frac{1}{n} \frac{X_{N\setminus j}}{(n-1)X_j}}{|j \in N: \ X_j = z|/n} \cdot 1_{Q'}, \tag{20}$$

where Q is the event that $\sum_{j:X_j \leq z} \frac{X_{N\setminus j}}{n(n-1)X_j} \leq 1$ (in other words i belongs to the group of agents whose share is given by equation (10)) and the event Q' tells that the share of agent 1 comes from equation (11), i.e, $\sum_{j:X_j<z} \frac{X_{N\setminus j}}{n(n-1)X_j} < 1 < \sum_{j:X_j\leq z} \frac{X_{N\setminus j}}{n(n-1)X_j}$.

Let us apply the strong law of large numbers to (20). Hence $\frac{X_{N\setminus 1}}{n-1}$ converges to 1 almost surely, the sum $\sum_{j:X_j\leq z} \frac{X_{N\setminus j}}{n(n-1)X_j}$ from the definition of Q converges to $\mathbb{E}_\nu\left(\frac{1}{X_j} \cdot 1_{\{X_j \leq z\}}\right)$. Therefore the first summand of (20) tends to $1_{z<T}$, where T is defined as $\inf\left\{T' \mid \mathbb{E}_\nu\left(\frac{1_{X_j \leq T'}}{X_j}\right) \geq 1\right\}$. Thus the contribution of the first term to $\pi(\varphi^1, \mathcal{P})$ is $\mathbb{P}_\nu(X_1 < T)$.

Similar application of the law of large numbers allows to compute the contribution of the second summand. We omit these computations.

C.2.2 Singularity at zero

Lemma 5: *If a distribution ν has an atom at zero, then the BH and the Proportional rules collect the optimal relative gain in the limit:*

$$R(\varphi^1, \mathcal{P}_\infty(\nu)) = R(\varphi^{pro}, \mathcal{P}_\infty(\nu)) = 1.$$

If there is no atom and ν has a continuous density f on $(0, a]$, but this density is unbounded, $f(x) = \frac{\lambda}{x^\alpha}(1 + o(1))$ as $x \to +0$ for some $\lambda > 0$ and $\alpha \in (0, 1)$, then

$$R(\varphi^1, \mathcal{P}_\infty(\nu)) = 1, \quad \text{however, } R(\varphi^{pro}, \mathcal{P}_n(\nu)) = \Omega\left(n\right).$$

In case of an atom, there is an agent i having $X_i = 0$ with high probability for large n. In such situation both rules φ^1 and φ^{pro} coincide with the utilitarian optimum and therefore their performance is 1.

The second statement is proved similarly to Lemma 4. For such ν the expected value of the order statistic X^{*k} for small k equals $\left(\frac{1-\alpha}{\lambda}\frac{k}{n}\right)^{\frac{1}{1-\alpha}} \cdot (1 + o(1))$. Therefore only the agent i with $X_i = \min_j X_j$ receives a bad under the BH rule with high probability, which gives $R(\varphi^1, \mathcal{P}_\infty(\nu)) = 1$. We omit a similar argument for the Proportional rule.