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The existence of attractors of Weyl foliations modelled on pseudo-Riemannian manifolds

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Abstract. A foliation that admits a Weyl structure arising from a pseudo-Riemannian metric of any signature as its transverse structure is called a pseudo-Riemannian Weyl foliation or (for short) a Weyl foliation. We investigate codimension $q \geq 2$ Weyl foliations on (not necessarily compact) manifolds. Different interpretations of their holonomy groups are given. We prove a criterion for a Weyl foliation to be pseudo-Riemannian. We find a condition on the holonomy groups which guarantees the existence of a transitive attractor of (M, F). Moreover, if the Weyl foliation is complete, this condition implies the existence of a global transitive attractor. We describe the structure of complete Weyl foliations modelled on Riemannian manifolds.

1. Introduction

Hermann Weyl introduced a new geometry in an attempt to create a unified field theory uniting magnetism and gravitation, which is a generalization of Riemannian geometry [17]. This geometry is named now Weyl geometry.

Weyl geometry is actively used in theoretical physics, in such areas as modern cosmology, gravitation, quantum mechanics, physics of elementary particles, etc. The review of some applications of Weyl geometry in physics was made by Scholz [11].

Advanced achievements in mathematical physics show the importance of pseudo-Riemannian geometry. In this work we consider Weyl geometry arising from pseudo-Riemannian geometry. This geometry is referred to as *pseudo-Riemannian Weyl geometry*, and it includes Weyl geometry arising from Riemannian geometry.

Our purpose is to investigate the influence of the transverse geometrical structure of a foliation on its topological and dynamical properties. In the given work the role of the transverse geometrical structure plays a pseudo-Riemannian Weyl structure.

Everywhere in the given work we assume that M is an n-dimensional manifold and F is a Weyl foliation on M of codimension $q \geq 2$, unless otherwise specified. Compactness of M is not assumed.

Recall that a diffeomorphism f of a pseudo-Riemannian manifold (N, g) is called a conformal transformation if there exists a smooth positive function λ on N such that $f^*g = \lambda g$. If λ is a constant, then f is referred to as a similarity transformation or a similarity of (N, g). Two pseudo-Riemannian metrics g_1 and g_2 on M are said to be conformally equivalent if there exists a smooth positive function λ on M such that $g_2 = \lambda g_1$.

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Let $\Omega^m(N)$ be the space of external forms of degree $m \geq 0$, in particular $\Omega^0(N)$ is the algebra of smooth functions on the manifold N.

Definition 1. A *pseudo-Riemannian Weyl structure* of signature (k,s) on a manifold N is called a pair $([g],f)$, where $[g]$ is a class of conformally equivalent **a conformal equivalence class of** pseudo-Riemannian metrics of signature (k,s) on N, and $f:[g]\to\Omega^1(N)$ is a map satisfying the equality $f(e^\lambda g) = f(g) - d\lambda \quad \forall\lambda\in\Omega^0(N)$.

A manifold N equipped with a pseudo-Riemannian Weyl structure of signature (k,s) is said to be a *pseudo-Riemannian Weyl geometry* or a *pseudo-Riemannian Weyl manifold of signature* (k,s) and is denoted by $(N,[g],f)$.

Weyl manifolds form a category (see Section 2.2).

A foliation (M,F) is called *pseudo-Riemannian Weyl* or (for short) Weyl if its transverse structure is modelled on a pseudo-Riemannian Weyl manifold $(N,[g],f)$. (the exact definition is given in Section 2.3).

A foliation (M,F) is referred to as *a transversally similar foliation* if all holonomy transformations are similarities of respective open subsets of the transverse pseudo-Riemannian manifold (N,g^N). If moreover, the curvature of (N,g^N) is zero, then (M,F) is called a *transversally similar pseudo-Euclidean foliation* (for the details, see Section 2.3).

First we give the following characterization of Weyl foliations.

Theorem 1. *A smooth foliation (M,F) of codimension $q\geq 2$ is a Weyl foliation modelled on a pseudo-Riemannian Weyl geometry $(N,[g],f)$ if and only if there exists $h\in[g]$ such that (M,F) is a transversally similar foliation modelled on (N,h).*

We consider Weyl foliations as a particular case of Cartan foliations in the sense of Blumenthal [2] or, that is equivalent in this case, in the sense of [19]. Denote by $\mathbb{R}^+$ the multiplicative group of positive real numbers. Let $O(k,s)$ be the pseudo-orthogonal matrix group and let $CO(k,s) := \mathbb{R}^+\times O(k,s)$ be the conformal group. Theorem 1 implies that Weyl foliations of signature (k,s) are Cartan of the type (G,H), where $G = CO(k,s)\ltimes\mathbb{R}^q$, $q=k+s$, is the semi-direct product of the conformal group $H := CO(k,s)$ and the abelian group $\mathbb{R}^q$. The consideration of Weyl foliations as Cartan foliations allows us to apply results of the previous works of the author [19] and [22]. In particular, the foliated bundle with the lifted foliation over a Weyl foliation is the basic tool in the given work.

The following theorem contains different interpretations of the germ holonomy groups usually used in foliation theory which follow from [20, Theorem 4] and [22, Proposition 5].

Theorem 2. *Let (M,F) be a pseudo-Riemannian Weyl foliation, $\pi:\mathcal{R}\to M$ be its foliated bundle, $x\in M$, $u\in\pi^{-1}(x)$. Let $\mathcal{L}=\mathcal{L}(u)$ be the leaf of the lifted foliation $(\mathcal{R},\mathcal{F})$ through u. Then:*

I. The holonomy group $\Gamma(L,x)$ of the leaf $L=L(x)$ through x is isomorphic to each of the following three groups:

1) the subgroup $H(\mathcal{L},u) := \{a\in H | R_a(\mathcal{L})=\mathcal{L}\}$ of H;

2) the group of deck transformations of the regular covering map $\pi|_{\mathcal{L}}:\mathcal{L}\to L$;

3) the linear holonomy group $D\Gamma(L,x)$ formed by the differentials of the local holonomy diffeomorphisms along leaf loops of a transversal q-dimensional disc at x.

II. If moreover, (M,F) admits an Ehresmann connection $\mathfrak{M}$, then the holonomy group $\Gamma(L,x)$ is isomorphic to the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L,x)$.

Remark 1. Let u' be another point of the set $\pi^{-1}(x)$ and $\mathcal{L}'=\mathcal{L}'(u')$. Then there exists a unique $a\in H$ such that $u'=ua$, and $H(\mathcal{L}',u') = a^{-1}H(\mathcal{L},u)a$, i.e. $H(\mathcal{L},u)$ is replaced by a conjugated subgroup in H.

Let E_q be the unit $q \times q$-matrix. We identify the group $\mathbb{R}^+$ with the subgroup $\{\lambda E_q | \lambda > 0\}$ of the Lie group $H = CO(k, s)$, where $q = k + s$.

According to Remark 1 the following definition is correct.

Definition 2. The holonomy group $\Gamma(L, x)$ of a leaf $L = L(x)$ of a Weyl foliation (M, F) is called *inessential*, if the group $H(\mathcal{L}, u)$ satisfying Theorem 2 belongs to the subgroup $O(k, s)$ of the group $H = CO(k, s)$. Otherwise the holonomy group of L is called *essential*. The holonomy group of $L = L(x)$ is referred to as *α-essential*, if the group $H(\mathcal{L}, u)$ satisfying Theorem 2 has an element $\lambda A \in CO(k, s) = \mathbb{R}^+ \times O(k, s)$, where $\lambda \in \mathbb{R}^+$, $\lambda \neq 1$ and A belongs to a compact subgroup of $O(k, s)$. Otherwise the holonomy group of L is called *α-inessential*.

Note that any α-essential holonomy group is essential, the opposite is in general not true (see Section 5.5). We want to emphasize that for Weyl foliations of signature $(0, q)$ the concepts of essential and α-essential holonomy group coincide.

We prove the following criterion for a Weyl foliation to be pseudo-Riemannian.

Theorem 3. *Let (M, F) be a Weyl foliation modelled on a transverse pseudo-Riemannian Weyl geometry $(N, [g], f)$. Then there exists a pseudo-Riemannian metric $h \in [g]$ such that (M, F) is a pseudo-Riemannian foliation modelled on (N, h) if and only if every holonomy group of this foliation is inessential.*

Corollary 1. *Any pseudo-Riemannian Weyl foliation is a foliation with transverse linear connection.*

In accordance with [7] the automorphism group of every pseudo-Riemannian Weyl foliation in the foliation category admits a structure of infinite-dimensional Lie group modelled on LF-spaces.

Recall that a subset of a foliated manifold (M, F) is called saturated, if it is a union of some leaves of the foliation. A minimal set of (M, F) is a nonempty closed saturated subset of M that does not have proper subsets possessing these properties.

The study of minimal sets is one of fundamental problems in qualitative theory of foliations, as well as in theory of dynamical systems [1].

Let (M, F) be a foliated manifold. A nonempty closed saturated subset $\mathcal{M}$ of M for which there exists an open saturated neighbourhood $\mathcal{U}$ such that the closure of any leaf from $\mathcal{U}$ contains $\mathcal{M}$ is called an *attractor* of the foliation. The neighborhood $\mathcal{U}$ is uniquely determined and called the basin of $\mathcal{M}$. We use notation $\mathcal{U} = Attr(\mathcal{M})$. An attractor $\mathcal{M}$ is called transitive if $\mathcal{M}$ is a minimal set of (M, F), i.e., if each leaf from $\mathcal{M}$ is dense in $\mathcal{M}$. If $Attr(\mathcal{M}) = M$ the attractor $\mathcal{M}$ is said to be *global* [19].

Attractors of foliations formed by trajectories of dynamic systems are attractors in sense of [13].

A leaf of a foliation (M, F) is proper [15] if it is an embedded submanifold of M. A leaf L is referred to as closed, if L is a closed subset of M. It is known that any closed leaf is proper. A foliation (M, F) is called proper if all its leaves are proper.

Theorem 4. *Let a pseudo-Riemannian Weyl foliation (M, F) of signature (k, s) have a leaf L with an α-essential holonomy group. Then the closure $\mathcal{M} := \overline{L}$ of the leaf L is a transitive attractor.*

For a proper Weyl foliation (M, F) the attractor $\mathcal{M}$ is a closed leaf.

A Weyl foliation is called complete if it is complete as a Cartan foliation or, equivalently, as a foliation with transverse linear connection (for the exact definition, see Section 5.2).

Theorem 5. *If a complete pseudo-Riemannian Weyl foliation (M, F) has a leaf L with α-essential holonomy group, then the closure $\mathcal{M} := \overline{L}$ of L is a global transitive attractor of the foliation.*

In Section 5.5, we construct an example of a complete Weyl foliation (M, F) that admits leaves with essential holonomy groups, but has no leaves with α-essential holonomy group and no attractors.

A regular covering map $f : L_0 \to L$ onto a leaf L of a foliation is called holonomic if the group of deck transformations of f is isomorphic to the holonomy group of this leaf.

In Section 6.1 we recall the definitions of (G, X)-manifolds and (G, X)-foliations. As the application of Theorem 5 we describe the structure of complete Riemannian Weyl foliations as follows.

Theorem 6. *Let (M, F) be a complete Riemannian Weyl foliation which is not a Riemannian foliation. Then:*

(i) *there exists a leaf L with an essential holonomy group, and $\mathcal{M} = \overline{L}$ is global a transitive attractor;*

(ii) *(M, F) is a complete transversally similar Euclidean foliation, i.e. $(Sim(\mathbb{E}^q), \mathbb{E}^q)$-foliation;*

(iii) *there exists a regular covering map $\kappa : \widetilde{M} \to M$ such that $\widetilde{M}$ coincides with the product of manifolds $L_0 \times \mathbb{E}^q$, where $q = k + s$, and the induced foliation $\widetilde{F} = \kappa^* F$ is formed by the fibers of the canonical projection $pr : L_0 \times \mathbb{E}^q \to \mathbb{E}^q$ and the restriction $\kappa|_{L_0 \times \{b\}}$, $b \in \mathbb{E}^q$, is a holonomic covering map onto the corresponding leaf of (M, F);*

(iv) *there exists an epimorphism*

$$\chi : \pi_1(M, x) \to Sim(\mathbb{E}^q)$$

of the fundamental group $\pi_1(M, x)$ of M onto some subgroup $\Psi = \chi(\pi_1(M, x))$ of $Sim(\mathbb{E}^q)$;

(v) *the holonomy group of any leaf $L = L(x)$, $x \in M$, is isomorphic to the isotropy subgroup Ψ_z of Ψ at $z \in pr(f^{-1}(L)) \subset \mathbb{E}^q_k$.*

In the case when (M, F) is a proper Weyl foliation, the global attractor $\mathcal{M}$ is a unique closed leaf of (M, F).

Definition 3. The group Ψ satisfying Theorem 6 is referred to as the global holonomy group of (M, F).

Other properties and examples of transversally similar Euclidean foliations can be found in [19, Sections 9 and 10].

The structure of complete Riemannian foliations is well known due to works of Molino, Haefliger, Carrier and others.

Weyl manifolds and W-flows on them are studied in [18]. Pseudo-Riemannian Weyl geometry on distributions and foliations was investigated in [6]. The structure of pseudo-Riemannian foliations of signature $(1, 1)$ on closed 3-dimensional manifolds is described in [4]. The work [21] is devoted to the investigation of foliations with transverse Weyl structures arising from Riemannian geometry, i.e. Weyl structures how they were defined by Weyl ([17], see also [8] and [12]).

Notations. Following [10] we denote a principal H-bundle $p : P \to N$ by $P(N, H)$. The module of vector fields on a manifold M is denoted by $\mathfrak{X}(M)$. Let $\mathfrak{X}_{\mathfrak{M}}(M)$ be the set of vector fields tangents to some distribution $\mathfrak{M}$ on M.

2. Weyl foliations as transversally similar pseudo-Riemannian foliations

2.1. The linear connection compatible with a Weyl geometry

Definition 4. Let $(N, [g], f)$ be a Weyl manifold. A torsion free linear connection ∇ on N is called *compatible with the Weyl structure,* if

$$\nabla g + f(g) \otimes g = 0 \quad \forall g \in [g]. \tag{1}$$

We want to emphasize that a Weyl structure $([g], f)$ on a manifold N is determined by a pseudo-Riemannian metric g and 1-form φ. In fact, let $[g]$ be the conformal class of pseudo-Riemannian metrics on N. Define $f : [g] \to \Omega^1(M)$ by the following two equations: $f(g) := \varphi$ and $f(e^\lambda g) = f(g) - d\lambda$ for all $\lambda \in \Omega^0(\mathcal{N})$. In accordance to Definition 1 such pair $([g], f)$ is a Weyl geometry on the manifold N.

Proposition 1. *For the given Weyl manifold $(N, [g], f)$ there exists a unique torsion free linear connection ∇ compatible with Weyl geometry.*

A torsion free linear connection ∇ on a manifold N on which there exist a pseudo-Riemannian metric g and 1-form φ such that $\nabla g + \varphi \otimes g = 0$ is the linear connection compatible with Weyl structure determined by g and φ.

Proof. As it is known, every torsion free linear connection is uniquely determined by the covariant derivative of a symmetric non-degenerate bilinear form. Therefore, on the given Weyl manifold $(N, [g], f)$ there exists a unique torsion free linear connection ∇ compatible with Weyl structure.

The converse, let a pseudo-Riemannian metric g and 1-form φ satisfy the equality $\nabla g + \varphi \otimes g = 0$. It is easy to check that $e^\lambda g$ and $\varphi - d\lambda$ so satisfy this equality for all smooth function λ. This means that ∇ is the unique torsion free linear connection ∇ compatible with this Weyl geometry $(N, [g], f)$ determined by g and φ. □

The statement analogous to Proposition 1 for Weyl geometries modelled on Riemannian manifolds was proved in [8, Theorem 2 and Corollary].

2.2. The category of Weyl manifolds

Let $(N, [g], f)$ and $(\widetilde{N}, [\widetilde{g}], \widetilde{f})$ be two Weyl manifolds. A smooth map $h : N \to \widetilde{N}$ is called *a morphism of Weyl manifolds*, if its codifferential h^* satisfies the following equality

$$h^* \circ \widetilde{f} = f \circ h^*$$

for every pseudo-Riemannian metric from $[\widetilde{g}]$.

Proposition 1 implies that $h : N \to \widetilde{N}$ is a morphism of the Weyl manifolds $(N, [g], f)$ and $(\widetilde{N}, [\widetilde{g}], \widetilde{f})$ if and only if h is a morphism of the manifolds of linear connections (N, ∇) and $(\widetilde{N}, \widetilde{\nabla})$, where ∇ and $\widetilde{\nabla}$ are torsion free linear connections compatible with the Weyl structures $([g], f)$ and $([\widetilde{g}], \widetilde{f})$ respectively, i.e. if $h_*(\nabla_X Y) = \widetilde{\nabla}_{h_* X} h_* Y$ for any $X, Y \in \mathfrak{X}(N)$, where h_* is the differential of h.

The category whose objects are Weyl manifolds, morphisms are morphisms of Weyl manifolds, and the composition of morphisms coincides with the composition of maps is called *a category of Weyl manifolds.*

2.3. Pseudo-Riemannian Weyl foliations

Let N be a q-dimensional manifold, and the topological space of N can be disconnected. Let M be an n-dimensional manifold, and $n > q$, An N-cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in J}$ is given if there are the following:

(i) an open covering $\{U_i \,|\, i \in J\}$ of the manifold M and submersions $f_i : U_i \to N$ to N with connected fibers;

(ii) if $U_i \cap U_j \neq \emptyset$, there exists a diffeomorphism $\gamma_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ satisfying to the equality $f_i = \gamma_{ij} \circ f_j$ on $U_i \cap U_j$;

(iii) $\gamma_{ik} = \gamma_{ij} \circ \gamma_{jk}$ for all $x \in f_k(U_i \cap U_j \cap U_k)$, where $i, j, k \in J$.

The family η is supposed to be maximal, i.e. it contains all U_i, f_i, γ_{ij} having indicated above properties and $N = \bigcup_{i \in J} f_i(U_i)$. The set of fibers of submersions $\{f_i^{-1}(x) \,|\, x \in N, i \in J\}$ forms

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a base of a new topology Υ in M which called the leaf topology. Connected components of the topological space (M, Υ) form a partition $F = \{L_\alpha \,|\, \alpha \in A\}$ of the manifold M which is referred to as *a foliation of codimension q determined by the N-cocycle η*. The foliation is denoted by (M, F).

Let $(N, [g], f)$ be a Weyl manifold of a signature (k, s) determined by the N-cocycle η. If each element γ_{ij} from the cocycle η is an isomorphism of the Weyl manifolds induced on the open subsets $f_j(U_i \cap U_j)$ and $f_i(U_i \cap U_j)$, then (M, F) is called *a foliation with transverse Weyl structure of the signature* (k, s) or (for short) *Weyl foliation of the signature* (k, s). It is said also that the Weyl foliation (M, F) is modelled on the transverse Weyl manifold $(N, [g], f)$.

If an N-cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ determines a foliation (M, F), and on N there exists a pseudo-Riemannian metric h such that any element γ_{ij} is a similarity of the pseudo-Riemannian manifolds induced on open subsets $f_j(U_i \cap U_j)$ and $f_i(U_i \cap U_j)$, then (M, F) is called *a transversally similar foliation* modelled on (N, h). If moreover, (N, h) is the pseudo-Euclidean space E^q_k, then (M, F) is called a *transversally similar pseudo-Euclidean* foliation of the signature $(k, n-k)$. In particular, when $k = 0$ and $E^n_k = E^n$ the foliation (M, F) is called a *transversally similar Euclidean* foliation.

A transversally similar foliation modelled on (N, h) is named *a pseudo-Riemannian foliation* if any element γ_{ij} is a local isometry of (N, h).

As similarity transformations of a pseudo-Riemannian manifold (N, h) preserve the Levi Civita connection ∇^h on N, transversally similar foliations may be considered as Weyl foliations modelled on $(N, [h], f)$ where $f(h)$ is an exact 1-form. In this case the Levi-Civita connection ∇^h is the unique torsion free connection compatible with this Weyl geometry which will denoted by $(N, [h], \nabla^h)$.

2.4. Proof of Theorem 1

Let (M, F) be a Weyl foliation modelled on a Weyl geometry $(N, [g], f)$. If there exists a non simply connected component N_1 of the manifold N, then we change N_1 by its universal covering manifold N_0 with the induced Weyl structure. Hence without loss a generality we assume that each component of N is simply connected. Remark that for any manifold N having only simply connected components the cohomology group $H^1(N)$ is trivial.

Since 1-forms $\varphi := f(g)$ and $f(\widetilde{g})$ for every $\widetilde{g} \in [g]$ differ by an exact form, they have a common exterior derivative. Hence they define an element of the group $H^1(N)$. As $H^1(N)$ equals to zero, there exists $h \in [g]$ such that $f(h) = \varphi_0 = 0$. This implies $\nabla g = 0$ by the definition of the compatible connection. Therefore the Weyl connection $\nabla = \nabla^h$ is the Levi-Civita connection of the pseudo-Riemannian manifold (N, h).

As every γ_{ij} from the N-cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ determining the foliation (M, F) is both the conformal diffeomorphism and the isomorphism of the induced Levi-Civita connections on the corresponding open subsets of the pseudo-Riemannian manifold (N, h), then [14, Chap 1, Sec. 1], γ_{ij} is a local similarity transformation of (N, h).

Thus (M, F) is a transversally similar foliation modelled on (N, h).

The converse is proved by the obvious way. $\square$

Remark 2. On accordance with Theorem 1, without loss a generality we assume further that any Weyl foliation is modelled on a Weyl geometry $(N, [g], \nabla^g)$, where ∇^g is the Levi-Civita connection defined by some pseudo-Riemannian metric g belonging to the class of conformally equivalent metrics $[g]$.

3. A criterion for a Weyl foliation to be pseudo-Riemanniann

3.1. Weyl foliations as Cartan foliations

Recall the definition of a Cartan geometry. More information about Cartan geometries can be found in [9], [12], [5].

Let G be a Lie group with the Lie algebra $\mathfrak{g}$. Let H be a closed subgroup of G with the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Assume that $p : P \to N$ is a principal H-bundle formed by a free right action of the group H on the manifold P. The action of an element $a \in H$ on P is denoted by R_a. A non-degenerated $\mathfrak{g}$-valued 1-form ω_0 on P is called a *Cartan connection*, if the following two conditions are satisfied:

1) $\omega_0(A^*) = A$ for $A \in \mathfrak{h}$, where A^* is the fundamental vector field on P defined by A;

2) the 1-form ω_0 is H-equivariant, i.e. $(R_a)^*\omega_0 = Ad_G(a^{-1})\omega_0 \ \forall a \in H$, where Ad_G is the adjoint representation of G on $\mathfrak{g}$.

The principal H-bundle $P(N, H)$ with the Cartan connection ω_0 is referred to as a *Cartan geometry* of the type (G, H). It is denoted by $\xi = (P(N, H), \omega_0)$.

A Cartan geometry of the type (G, H) is reductive, if the homogeneous space G/H is reductive [12]. If the group G acts effectively at the left on G/H, then the Cartan geometry of the type (G, H) is called effective.

Everywhere further in this work we denote by $G = CO(k, s) \ltimes \mathbb{R}^q$ the semi-direct product of the conformal group $H = CO(k, s)$ and the q-dimensional abelian additive group $\mathbb{R}^q$, and $\mathbb{R}^q$ is a normal subgroup of G. We write any element of G as $\langle \lambda A, a\rangle$, where $\lambda \in \mathbb{R}^+, A \in O(q), a \in \mathbb{R}^q$. The multiplication in the Lie group G is defined by the following equality

$$\langle \lambda A, a\rangle\langle \mu B, b\rangle := \langle \lambda\mu AB, \lambda Ab + a\rangle, \ \ \langle \lambda A, a\rangle, \langle \mu B, b\rangle \in G.$$

Note that the group G is realized as group of all similarity transformations of the pseudo-Euclidean space $\mathbb{E}^q_k$, and H is realized as its stationary group at zero $0 \in \mathbb{E}^q_k$.

Let $\mathfrak{so}(k, s)$ be the Lie algebra of the pseudo-orthogonal group $O(k, s)$ and $\mathbb{R}^1$ be the Lie algebra of $\mathbb{R}^+$. The Lie algebra $\mathfrak{g}$ of G admitted the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{h} = \mathfrak{co}(k, s) = \mathbb{R}^1 \oplus \mathfrak{so}(k, s)$ is the Lie algebra of the conformal Lie group $CO(k, s)$ and $\mathfrak{p} = \mathbb{R}^q$ is the abelian ideal in $\mathfrak{g}$. The vector space $\mathfrak{p}$ is $Ad_G(H)$-invariant with respect to the adjoint representation $Ad_G(H) : H \to GL(\mathfrak{g})$ of the subgroup H of G on $\mathfrak{g}$.

According to Remark 2 in the investigation of Weyl foliations we consider a Weyl geometry $(N, [g], \nabla^g)$ of signature (k, s), $k + s = q$, on q-dimensional manifold N. Such Weyl geometry defines the Cartan connection $\omega_0 = \widetilde{\omega} + \theta$ in H-bundle of conformal frames $P(N, H)$, where $\widetilde{\omega}$ is the $\mathfrak{h}$-valued 1-form on $\mathcal{R}$ and θ is the $\mathbb{R}^q$-valued canonical 1-form on P which are defined by the Levi-Civita connection ∇^g on the pseudo-Riemannian manifold (N, g). We want to emphasize that the Weyl geometry $(N, [g], \nabla^g)$ will be considered as the effective reductive Cartan geometry $\xi = (P(N, H), \omega_0)$ of the type (G, H).

Applying [19, Proposition 2] we get the following statement.

Proposition 2. *A Weyl foliation (M, F) codimension q modelled on a Weyl geometry $\xi = (P(N, H), \omega_0)$ of signature (k, s), $q = k + s \geq 2$, is a Cartan foliation of the type (G, H), where $G = CO(k, s) \ltimes \mathbb{R}^q$ and $H = CO(k, s)$, and there exist the principal H-bundle $\pi : \mathcal{R} \to M$, the H-invariant foliation $(\mathcal{R}, \mathcal{F})$ and the $\mathfrak{g}$-valued H-equivariant 1-form ω on $\mathcal{R}$ satisfying the following conditions:*

1) *$\omega(A^*) = A$ for a fundamental vector field A^* corresponding to $A \in \mathfrak{h}$ for every $A \in \mathfrak{h}$;*

2) *the map $\omega_u : T_u\mathcal{R} \to \mathfrak{g}$, for any $u \in \mathcal{R}$ is surjective, and $ker(\omega) = TF$, where TF is the distribution tangent to the foliation $(\mathcal{R}, F)$;*

3) *the Lie derivative $L_X\omega$ is equal to zero for each vector field X tangent to the foliation $(\mathcal{R}, \mathcal{F})$.*

The principal H-bundle $\pi : \mathcal{R} \to M$ is said to be *the foliated bundle* and $(\mathcal{R}, \mathcal{F})$ is said to be *the lifted foliation* for the Weyl foliation (M, F).

3.2. Proof of Theorem 3

Assume that all holonomy groups of a Weyl foliation (M, F) are inessential. Let $\pi : \mathcal{R} \to M$ be the projection of the foliated H-bundle over (M, F) satisfying Proposition 2. The smooth right free action of the Lie group $H = CO(k, s)$ is determined on the manifold $\mathcal{R}$, and the orbit space $\mathcal{R}/H$ coincides with the manifold M.

The smooth right free action of the normal subgroup R^+ of H on $\mathcal{R}$ is defined, and the orbit space is a smooth manifold $\widehat{\mathcal{R}} = \mathcal{R}/\mathbb{R}^+$. A smooth right free action of the factor group $H/\mathbb{R}^+ = O(k, s)$ is induced on $\widehat{\mathcal{R}}$, and the orbit space $\widehat{\mathcal{R}}/O(k, s)$ coincides with M. Thus, the canonical projections onto the orbit spaces $\alpha : \mathcal{R} \to \widehat{\mathcal{R}}$ and $\widehat{\pi} : \widehat{\mathcal{R}} \to M$ satisfy the equality $\pi = \widehat{\pi} \circ \alpha$. Since the lifted foliation $(\mathcal{R}, \mathcal{F})$ is H-invariant, it is invariant with respect to the subgroup R^+. Hence $O(k, s)$-invariant foliation $(\widehat{\mathcal{R}}, \widehat{F})$ is induced, and its leaves are images of leaves of the foliation $(\mathcal{R}, \mathcal{F})$ with respect to the map α.

By the condition of Theorem 3, any leaf of (M, F) has an inessential holonomy group. Hence the restriction α on each leaf $\mathcal{L}$ of the lifted foliation $(\mathcal{R}, \mathcal{F})$ is a diffeimorphism onto the corresponding leaf $\widehat{L}$ of the foliation $(\widehat{\mathcal{R}}, \widehat{\mathcal{F}})$.

$Ad_G(H)$-invariance of the subspace $\mathfrak{p}$ of the vector space of the Lie algebra $\mathfrak{g} = \mathfrak{co}(q) \oplus \mathfrak{p}$ implies the existence of a smooth H-invariant distribution $\mathfrak{N}$ on $\mathcal{R}$, where $\mathfrak{N}_u = \{X \in T_u(\mathcal{R}) | \omega(X) \in \mathfrak{p}\}$ for any $u \in \mathcal{R}$. Hence, $\mathfrak{N}$ is a projectable connection on the H-bundle $\mathcal{R}(M, H)$ with respect to the foliation $(\mathcal{R}, \mathcal{F})$. Therefore the distribution $\widehat{\mathfrak{N}} := \alpha_* \mathfrak{N}$ is a projectable connection on the $O(k, s)$-bundle $\widehat{\mathcal{R}}(M, O(k, s))$ with respect to $(\widehat{\mathcal{R}}, \widehat{\mathcal{F}})$.

The bundle $\mathcal{R}(\widehat{\mathcal{R}}, \mathbb{R}^1)$ has the contractible fiber, and for each leaf $\mathcal{L}$ of the foliation $(\mathcal{R}, \mathcal{F})$ the restriction $\alpha_{|\mathcal{L}}$ is a diffeomorphism of $\mathcal{L}$ onto the corresponding leaf of $(\widehat{\mathcal{R}}, \widehat{\mathcal{F}})$. According to [22, Proposition 4] in this case there exists a foliated section $\sigma : \widehat{\mathcal{R}} \to \mathcal{R}$. Thus, $\sigma(\widehat{\mathcal{R}})$ is a reduction of the bundle $\mathcal{R}(M, H)$ to the closed subgroup $O(k, s)$, and $\sigma(\widehat{\mathcal{R}})$ is a foliated $O(k, s)$-bundle for (M, F) such that $\sigma(\widehat{F}) = \mathcal{F}|_{\sigma(\widehat{\mathcal{R}})}$. It is not difficult to show that σ satisfies the equation $R_a \circ \sigma = \sigma \circ R_a$ for every $a \in O(k, s)$ and $\sigma_* \widetilde{\mathfrak{N}} = \mathfrak{N}|_{\sigma(\widehat{\mathcal{R}})}$. This means that the original Weyl foliation (M, F) with the transverse Weyl geometry $(N, [g], \nabla^g)$ is a pseudo-Riemannian foliation modelled on a pseudo Riemannian manifold (N, h) where $h \in [g]$.

The converse is fulfilled by obvious a way. $\square$

4. A Riemannian metric and a connection adapted to the lifted foliation

Let (M, F) be a Weyl foliation of codimension q. Let $\mathcal{R}(M, H)$ be its foliated bundle with the lifted foliation $(\mathcal{R}, \mathcal{F})$ satisfying Proposition 2.

Consider a smooth q-dimensional distribution $\mathfrak{M}$ transverse to leaves of (M, F), i.e. $T_x M = \mathfrak{M}_x \oplus T_x F$ for any point $x \in M$. Let $\widetilde{\mathfrak{M}} := \pi^* \mathfrak{M}$, i.e. $\widetilde{\mathfrak{M}}_u := \{X \in T_u \mathcal{R} | \pi_{*u} X \in \mathfrak{M}_x, x = \pi(u)\}$ for all $u \in \mathcal{R}$. Denote by $\mathfrak{P}$ the smooth q-dimensional distribution on $\mathcal{R}$ which is equal to the intersection $\mathfrak{N}$ and $\widetilde{\mathfrak{M}}$, i.e. $\mathfrak{P}_u = \{X \in \widetilde{\mathfrak{M}}_u \mid \omega(X) \in \mathfrak{p}\}$ for all $u \in \mathcal{R}$. H-invariance of the distributions $\mathfrak{N}$ and $\widetilde{\mathfrak{M}}$ implies H-invariance of the distribution $\mathfrak{P}$.

Definition 5. A smooth vector field $X \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$, for which $\omega(X) = c = const$ is said to be a $\mathfrak{g}$-*field.* If moreover $c \in \mathfrak{p}$, then X is said to be a $\mathfrak{p}$-*field,*

A piecewise smooth curve in $\mathcal{R}$ is called a $\mathfrak{g}$-*curve* (accordingly a $\mathfrak{p}$-*curve*), if each its smooth piece is an integral curve of some vector $\mathfrak{g}$-field (accordingly a $\mathfrak{p}$-*field*).

Remark, that locally anyone smooth $\mathfrak{g}$-curve σ can be represented as $\sigma(t) = \varphi_t^X(v), t \in (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$, φ_t^X is the 1-parametric group of local diffeomorphisms of the manifold $\mathcal{R}$ generated by the $\mathfrak{g}$-field X for which $\sigma(t)$ is an integral curve, and $v = \sigma(0) = \varphi_0^X(v)$.

Lemma 1. *Let $g_{\mathcal{R}}$ be an arbitrary Riemannian metric on the space of the foliated bundle $\mathcal{R}$. Let d_0 be the Euclidean metric on the vector space of $\mathfrak{g}$ invariant with respect to the action of* *the compact group $Ad_G(O(k)\times O(s))$. Let $Z = Z_{\mathcal{F}} \oplus Z_{\widetilde{\mathfrak{M}}}$ be the decomposition of any vector field $Z \in \mathfrak{X}(\mathcal{R})$ corresponding to the decomposition of the tangent vector space to $\mathcal{R}$ in the direct sum of the vector subspaces $T_u\mathcal{R} = T_u\mathcal{F} \oplus \widetilde{\mathfrak{M}}_u, u \in \mathcal{R}$.*

Then the equality

$$d(X,Y) := g_{\mathcal{R}}(X_{\mathcal{F}}, Y_{\mathcal{F}}) + d_0(\omega(X), \omega(Y)) \ \ \forall X, Y \in \mathfrak{X}(\mathcal{R}),$$

defines a Riemannian metric d on $\mathcal{R}$ transversally projectable with respect to the foliation $(\mathcal{R},\mathcal{F})$ satisfying the following properties:

1) the length $l(\sigma)$ of any smooth $\mathfrak{g}$-curve σ, $\sigma(t) = \varphi_t^X(v)$, where $t \in [0, t_1], v = \sigma(0)$, is equal to $\parallel \omega(X) \parallel_{d_0} \cdot t_1$, where $\parallel \omega(X) \parallel_{d_0}^2 = d_0(\omega(X), \omega(X))$;

2) $l(\varphi_t^X(v)) = l(\varphi_t^X(v'))$, where X is a $\mathfrak{g}$-field, $t \in [0, t_1]$, for all $v, v' \in \mathcal{R}$, if $\varphi_t^X(v)$ and $\varphi_t^X(v')$ are defined at $t \in [0, t_1]$;

3) for any element $a = \lambda^{-1} \cdot A \in R^+ \times O(k) \times O(s)$, where $\lambda > 0$, and for any $\mathfrak{p}$-curve σ the curve $\widetilde{\sigma} := R_a \circ \sigma$ is a $\mathfrak{p}$-curve, with $l(\widetilde{\sigma}) = \lambda \cdot l(\sigma)$.

Proof. As $\mathfrak{g}$-valued 1-form ω is projectable, so the Riemannian metric d is also projectable with respect to the foliation $(\mathcal{R},\mathcal{F})$. As $\sigma(t) = \varphi_t^X(v)$, where $t \in [0, t_1]$, is an integral curve of some $\mathfrak{p}$-field X, that $d(X,X) = \parallel \omega(X) \parallel_{d_0}^2$ and $d(\sigma)/dt = X_{\sigma(t)}$. Therefore the length $l(\sigma)$ is calculated by the formula specified in 1).

The relation 2) is followed from 1).

Let us verify 3). First we assume that $\sigma(t) = \varphi_t^X(v), t \in [0, t_1]$, is a $\mathfrak{p}$-curve, where $\sigma(0) = v$, and $\widetilde{\sigma} := R_a \circ \sigma$. As $\widetilde{\sigma}(t) = \varphi_t^Y(v \cdot a)$, where $Y = R_{a*}(X)$, the $Ad_G(O(k) \times O(s))$-invariance of $\mathfrak{p}$ implies that Y is a $\mathfrak{p}$-field. Hence $\widetilde{\sigma}$ is a $\mathfrak{p}$-curve.

According to 1), its length is calculated by the formula $l(\widetilde{\sigma}) = \parallel \omega(X) \parallel_{d_0} \cdot t_1$. H-equivariance of the form ω implies the equality $\omega(Y) = Ad_G(a^{-1})\omega(X)$. Therefore $l(\widetilde{\sigma}) = \parallel Ad_G(a^{-1})\omega(X) \parallel_{d_0} \cdot t_1$. Direct calculations show that for any element $a = \lambda^{-1} \cdot A \in R^+ \times O(k) \times O(s)$ the following equality $\parallel Ad_G(a^{-1})\omega(X) \parallel_{d_0} = \lambda \cdot \parallel \omega(X) \parallel_{d_0}$ is valid, hence $l(\widetilde{\sigma}) = \lambda \cdot l(\sigma)$. Let now σ be a piecewise smooth $\mathfrak{p}$-curve. Then it is devised into finite number of smooth pieces $\sigma \mid_{I_i}, i = 1, ..., m$, for each of which, as it was proved above, there is the equality $l(\widetilde{\sigma} \mid_{I_i}) = \lambda \cdot l(\sigma|_{I_i})$, hence $l(\widetilde{\sigma}) = \lambda \cdot l(\sigma)$. ☐

The following easily proved lemma takes place.

Lemma 2. *Let $E_i, i = \overline{1, dim\mathfrak{g}}$ be a basis of the Lie algebra $\mathfrak{g}$. Let X_i be such $\mathfrak{g}$-field, that $\omega(X_i) = E_i$. We shall denote by $\widetilde{\nabla}$ the Levi-Civita connection of the Riemannian manifold $(\mathcal{R}, d)$. Then the equality*

$$\nabla_Y^0 Z := Y(Z^i)X_i + \widetilde{\nabla}_Y Z_F, \qquad (*)$$

where $Z = Z_F \oplus Z_{\widetilde{\mathfrak{M}}}$, $Z_{\widetilde{\mathfrak{M}}} = Z^i X_i \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$, $Z_F \in \mathfrak{X}_{T\mathcal{F}}(\mathcal{R})$, $Y \in \mathfrak{X}(\mathcal{R})$, defines a linear connection ∇^0 in $\mathcal{R}$, generally speaking with torsion, with respect to which all $\mathfrak{g}$-fields are parallel. Besides the parallel transfer keeps the scalar product of $\mathfrak{g}$-fields inducted by d, and integrated curves $\mathfrak{g}$-fields are geodesic lines of the connection ∇^0.

5. The existence of attractors

5.1. Proof the of Theorem 4

We use notations from Section 4. Let (M,F) be a Weyl foliation modelled on a Weyl geometry $(N, [g], \nabla^g)$.

Let $(\mathcal{R}, \mathbb{F})$ be the foliation formed by connected components of submanifolds $\pi^{-1}(L_\alpha)$ of $\mathcal{R}$, where L_α are leaves of (M,F).

Suppose that the foliation (M,F) has a leaf $L = L(x)$, $x \in M$ with an α-essential holonomy group. Pick $v \in \pi^{-1}(x)$. Let $\mathcal{L} = \mathcal{L}(v)$ be the leaf of the lifted foliation $(\mathcal{R},\mathcal{F})$.

We emphasize that geodesics of the linear connection ∇^0 with a torsion having the Christoffel symbols Γ^k_{ij} in some coordinate neighborhood coincide with the geodesics of the torsion free linear connection $\widehat{\nabla}$ having the Christoffel symbols $\widehat{\Gamma}^k_{ij} = \frac{1}{2}(\Gamma^k_{ij} + \Gamma^k_{ji})$. Hence geodesics of a linear connection ∇^0 with a torsion has the same properties as geodesics of torsion free linear connections.

Therefore for the point v there exist $\varepsilon > 0$ and an embedded submanifold $V := D_{\widetilde{\mathfrak{M}}}(v, \varepsilon)$ formed by $\mathfrak{g}$-curves of the length $< \varepsilon$ with the origin at v. Similarly, there exists $V_1 := D_{\mathfrak{P}}(v, \varepsilon) \subset V$ formed by $\mathfrak{p}$-curves of length less than ε with the origin at v.

Since the leaf L has α-essential holonomy group, the group $H(\mathcal{L}, v)$ contains the element $a = \lambda^{-1} \cdot A$, where $\lambda \in (0, 1)$ and A belongs to a compact subgroup of $O(k, s)$. Since $O(k) \times O(s)$ is the maximal compact subgroup of $O(k, s)$, another maximal subgroup of $O(k, s)$ is conjugated with $O(k) \times O(s)$. According to Remark 1 we can take such point $v \in \pi^{-1}(x)$ that $A \in O(k) \times O(s)$.

As points v and $u := v \cdot a$ belong to the same leaf $\mathcal{L}$, we can connect them by a smooth curve $h : [0, 1] \to \mathcal{L}$, where $v = h(0), u = h(1)$. Without loss a generality we believe that the holonomy diffeomorphism Φ_h along the path h ([15]) with respect to $(\mathcal{R}, \mathcal{F})$ is defined on V.

We emphasize that the normal neighborhoods respectively to ∇^0 form a base of neighborhoods at each point of $\mathcal{R}$. Therefore for some $\varepsilon > 0$ there is a neighborhood $\mathcal{W} = \mathcal{W}(\varepsilon)$ of the point v satisfying the following conditions: 1) $\mathcal{W}$ is adapted to the foliation $(\mathcal{R}, \mathbb{F})$ and 2) the submanifold $V_1 := D_{\mathfrak{P}}(v, \varepsilon)$ is transversal to leaves of $(\mathcal{W}, \mathbb{F}_{\mathcal{W}})$, and V_1 intersects each its leaf exactly once.

Therefore any two leaves of $(\mathcal{W}, \mathbb{F}_{\mathcal{W}})$ can be connected by some smooth $\mathfrak{p}$-curve of length $< \varepsilon$. Note that $U := \pi(\mathcal{W})$ is an adapted neighborhood with respect to (M, F) and $D := \pi(V_1)$ is a transversal q-dimensional manifold in U.

Show that for the closure $\overline{L}_\alpha$ of any leaf L_α of a foliation (M, F) intersecting U we have the inclusion $\overline{L}_\alpha \supset L$.

Let $L_\alpha \cap U \neq \emptyset$, then there exist points $x_0 \in L_\alpha \cap D$ and $v_0 \in \pi^{-1}(x_0) \cap V_1$. Therefore there is a $\mathfrak{p}$-curve $\varphi^X_t(v), t \in [0, t_1]$ in V_1 connecting $v = \varphi^X_0(v)$ with $v_0 = \varphi^X_{t_1}(v)$. This means that $v_0 \in \mathbb{L} \cap W$ for the leaf $\mathbb{L} \subset \pi^{-1}(L_\alpha)$ of the foliation $(\mathcal{R}, \mathbb{F})$.

Introduce the notations $\sigma(t) := \varphi^X_t(v), t \in [0, t_1], \widetilde{\sigma} := R_a \circ \sigma$, where $a = \lambda^{-1} \cdot A$, $\lambda \in (0, 1)$, $A \in O(k) \times O(s)$. Then $\widetilde{\sigma}(t) = \varphi^Y_t(v \cdot a)$ where $Y = R_{a*}(X)$. In accordance with the statement 3) of Lemma 1 $\widetilde{\sigma}(t)$ is a $\mathfrak{p}$-curve, and $l(\widetilde{\sigma}) = \lambda \cdot l(\sigma) < \lambda \cdot \varepsilon$.

According to the statement 2) of Lemma 1, if there exists a $\mathfrak{p}$-curve $\sigma_1(t) := \varphi^Y_t(v), t \in [0, t_1]$, then it has the length equal to the length of the curve $\widetilde{\sigma}$, and $l(\sigma_1) = \lambda \cdot l(\sigma) < \lambda \cdot \varepsilon < \varepsilon$. Hence, such $\mathfrak{p}$-curve exists in the neighborhood of V_1, i.e. $\sigma_1(t) \in V_1$ for all $t \in [0, t_1]$. Therefore the holonomy diffeomorphism Φ_h is defined at all points of this curve.

Since $\mathfrak{p}$-curves are transversally projectable, by the definition of the holonomy diffeomorphism Φ_h, the curve $\varphi^Y_{t_1}(h(\tau)), \tau \in [0, 1]$, lies in the leaf $\mathcal{L}' = \mathcal{L}'(v_0)$ of the foliation $(\mathcal{R}, \mathcal{F})$. This implies that points $v_1 := \sigma_1(t_1)$ and $\widetilde{v}_1 := \widetilde{\sigma}(t_1)$ of this curve are projected to the same leaf $L_\alpha = \pi(\mathcal{L}')$ of (M, F). The point v is connected to the point of v_1 by the curve σ_1 of length $l(\sigma_1) < \lambda \cdot \varepsilon$, where $\lambda \in (0, 1)$.

Repeating the above argument with $(m-1)$ times we get a point v_m which is connected with v by the $\mathfrak{p}$-curve σ_m of length $l(\sigma_m) = \lambda^m \cdot l(\sigma) < \lambda^m \cdot \varepsilon$, and points v_m and v_0 are projected to the same leaf L_α. Since $v_m \to v$ as $m \to \infty$, $x_m := \pi(v_m) \to x = \pi(v)$ as $m \to \infty$. Since $x_m \in L_\alpha \ \ \forall m$ and $L = L(x)$, this means what $L \subset \overline{L}_\alpha$. This implies that $\mathcal{M} := \overline{L} \subset \overline{L}_\alpha$ for each L_α intersecting U.

The union $\mathcal{U} := \cup L_\alpha$ of leaves $L_\alpha \in F$ such that $L_\alpha \cap U \neq \emptyset$ is an open saturated neighborhood of $\mathcal{M}$ in M. Thus, $\mathcal{M}$ is an attractor of (M, F), and $\mathcal{U} = Attr\mathcal{M}$.

Consider an arbitrary leaf $L' \subset \mathcal{M}$, then $\overline{L}' \subset \mathcal{M} = \overline{L}$. According to the definition of the closure $\overline{L}$, we have $L' \cap U \neq \emptyset$. By the definition of $\mathcal{U}$ we have $L' \subset \mathcal{U}$, therefore the shown above inclusion $L \subset \overline{L}'$ implies $\mathcal{M} = \overline{L} \subset \overline{L}'$. Thus $\overline{L}' = \mathcal{M}$ and $\mathcal{M}$ is a minimal set of the foliation (M, F). Therefore $\mathcal{M}$ is a transitive global attractor of $(\mathcal{U}, F_{\mathcal{U}})$. $\square$

5.2. Completeness of Weyl foliations

Recall that a smooth vector field X on $\mathcal{R}$ is *complete*, if X generates a global 1-parametric group of diffeomorphisms of $\mathcal{R}$.

Definition 6. Weyl foliation (M, F) is called *complete,* if every $\mathfrak{g}$-field $X \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$ is complete.

Thus, by the definition, the completeness of a Weyl foliation (M, F) is equivalent to the completeness of (M, F) considered as a Cartan foliation.

By Corollary 1 a Weyl foliation (M, F) can be considered as a foliation with transverse linear connection. Therefore in accordance with [7, Theorem 4.1] on M there exists a transversally projectable with respect to (M, F) linear connection ∇ and a geodesic invariant transverse q-dimensional distribution $\mathfrak{M}$. It is not difficult to show that a Weyl foliation (M, F) is *complete* if and only if the canonical parameter on every maximal geodesic tangent to $\mathfrak{M}$ of the connection ∇ is defined on $(-\infty, +\infty)$.

5.3. Lemma

The following lemma will be used in the proof of the existence of a global attractor.

Lemma 3. *Let L and L' be any two leaves of a complete Weyl foliation (M, F). Then subsets $\pi^{-1}(L)$ and $\pi^{-1}(L')$ of $\mathcal{R}$ may be connected by some piecewise smooth $\mathfrak{p}$-curve.*

Proof. Let $f : M \to M/F$ be a quotient map onto the leaf space. We denote $[L]$ the leaf L of (M, F) considered as a point of M/F. We say that $[L]$ is equivalent to $[L']$ if there exists a piecewise smooth $\mathfrak{p}$-curve $\sigma : [0, 1] \to \mathcal{R}$ connected $\pi^{-1}(L)$ with $\pi^{-1}(L')$, i.e., $\sigma(0) \in \pi^{-1}(L)$ and $\sigma(1) \in \pi^{-1}(L')$.

Let us show that this relation is indeed an equivalence relation in M/F. The reflexivity and symmetry are obvious. Let us check transitivity of the introduced relation. Let $[L_0] \sim [L_1]$ and $[L_1] \sim [L_2]$, and a $\mathfrak{p}$-curve σ connects $\pi^{-1}(L_0)$ with $\pi^{-1}(L_1)$, a $\mathfrak{p}$-curve σ_1 connects $\pi^{-1}(L_1)$ with $\pi^{-1}(L_2)$. Let $v_0 = \sigma(0) \in \pi^{-1}(L_0)$, $v_1 = \sigma(1) \in \pi^{-1}(L_1)$, $v_2 = \sigma_1(0) \in \pi^{-1}(L_1)$, $v_3 = \sigma_1(1) \in \pi^{-1}(L_2)$. Let $x_i = \pi(v_i), i = 0, ..., 3$. Then $x_1 \cup x_2 \subset L_2$, hence there exists a point $u_0 \in \mathcal{L}(v_2) \cap \pi^{-1}(x_1)$, where $\mathcal{L}(v_2)$ is the leaf of the lifted foliation containing v_2.

Case I: the curve σ_1 is smooth. In this case $\sigma_1(t) = \varphi_t^X(v_2), t \in [0, 1]$, where X is a $\mathfrak{p}$-field. The completeness of the Weyl foliation (M, F) implies that $\mathfrak{p}$-field X is complete. Therefore for any point $v \in \mathcal{R}$ the integral curve $\varphi_t^X(v)$ is defined for all $t \in (-\infty, +\infty)$. Then the $\mathfrak{p}$-curve $\widehat{\sigma}_1(t) := \varphi_t^X(u_0)$ is defined. Since for each fixed t the diffeomorphism φ_t^X is an automorphism of the foliation $(\mathcal{R}, \mathcal{F})$, it is necessary $\widehat{\sigma}_1(1) = \varphi_1^X(u_0) \in \mathcal{L}(v_3) \subset \pi^{-1}(L_2)$.

Case II: the $\mathfrak{p}$-curve σ_1 is piecewise smooth. In this case we consistently apply the previous reasoning to each smooth piece in order to obtain $\mathfrak{p}$-curve $\widehat{\sigma}_1$ with the starting at u_0 and the end at $\widehat{\sigma}_1(1) \in \pi^{-1}(L_2)$.

As points v_1 and u_0 belong to the same fiber $\pi^{-1}(x_1)$, there exists an element $a \in H$ such that $v_1 = u_0 \cdot a$. According to Lemma 1 $\sigma^* := R_a \circ \widehat{\sigma}_1$ is a $\mathfrak{p}$-curve with the origin at v_1. Note that $\sigma^*(0) = v_1 = \sigma(1)$ and $\sigma^*(1) \in \pi^{-1}(L_2)$. Therefore the product of paths $\delta = \sigma \cdot \sigma^*$ is defined. Thus, δ is a $\mathfrak{p}$-curve connecting $\pi^{-1}(L_0)$ with $\pi^{-1}(L_2)$. This means that $[L_0] \sim [L_2]$, i.e. the relation $\sim$ is transitive, hence the introduced relation is an equivalence relation.

We now show that each equivalence class is an open subset of M/F. Consider a point $[L] \in M/F$. Let $A([L])$ be the equivalence class containing $[L]$. In the proof of Lemma 1 for any $x \in L$ and $v \in \pi^{-1}(x)$ we constructed the neighborhood $\mathcal{W}$ at the point v which is adapted with respect to $(\mathcal{R}, \mathbb{F})$. The point v can be connected with any local leaf of $(\mathcal{W}, \mathbb{F}|_{\mathcal{W}})$ by a certain $\mathfrak{p}$-curve. This implies that any two leaves of (M, F) intersected the neighborhood $U = \pi(\mathcal{W})$

are equivalent. Since the projection $f : M \to M/F$ onto the leaf space is open mapping, $f(U)$ is an open subset of M/F contains $[L]$ and $f(U) \subset A([L])$. Thus, the equivalence class $A([L])$ is an open subset of M/F.

Since the complement of $A([L])$ is formed by the union of the remaining equivalence classes each of which is open, then $A([L])$ is a closed subset of M/F. Due to the connectivity of the topological space of M, the leaf space M/F is also connected. Hence a non-empty open-closed subset $A([L])$ coincides with M/F. $\square$

5.4. Proof of Theorem 5

Let (M, F) be a complete Weyl foliation of a signature (k, s) and codimension $q = k + s \geq 2$. Assume that (M, F) admits a leaf $L = L(x)$ with an α-essential holonomy group. Let L' be any other leaf of this foliation. By Lemma 3 there exists a $\mathfrak{p}$-curve σ connecting $\pi^{-1}(L)$ and $\pi^{-1}(L')$. Let $v = \sigma(0) \in \pi^{-1}(x)$, $x \in L$, $v_0 := \sigma(1) \in \pi^{-1}(x_0)$, $x_0 \in L'$.

According to Theorem 4 $\mathcal{M} = \overline{L}$ is a transitive attractor. Let $\mathcal{U} := \mathcal{A}ttr(\mathcal{M})$ and $\mathcal{V} = \pi^{-1}(\mathcal{U})$. Since the leaf $L = L(x)$ has an α-essential holonomy group, without loss a generality we assume that there exist a point $v \in \pi^{-1}(x)$ and an element $b = \lambda^{-1}A \in H(\mathcal{L}, v)$, where $\lambda \in (0, 1)$. Then there is the ball B of the radius ε with the center in v of the Riemannian manifold $(\mathcal{R}, d)$ such that $B \subset \mathcal{V}$. Therefore there is a natural number k, for which $\lambda^k \cdot l(\sigma) < \varepsilon$. According to the statement 3) of Lemma 1 the length of the curve $\widetilde{\sigma} := R_a \circ \sigma$, where $a = b^k \in H(\mathcal{L}, v) \subset H$, satisfies to the relations $l(\widetilde{\sigma}) = \lambda^k \cdot l(\sigma) < \varepsilon$. Connect the points $u = R_a(v)$ and v by a smooth path h in the leaf $\mathcal{L} = \mathcal{L}(v)$, $h(0) = u$, $h(1) = v$.

As it is known [19, Proposition 3], $\mathfrak{M}$-completeness of the Cartan foliation (M, F) implies that $\mathfrak{M}$ is an Ehresmann connection in sense of Blumenthal and Hebda [3] for this foliation. Here $\mathfrak{M}$ is a q-dimensional distribution on M which is transversal to (M, F). Therefore the induced distribution $\widetilde{\mathfrak{M}} = \pi^*\mathfrak{M}$ is an Ehresmann connection for the foliation $(\mathcal{R}, \mathcal{F})$. Hence there exists the transfer of the $\mathfrak{p}$-curve $\widetilde{\sigma}$ along the leaf path h with respect to the Ehresmann connection $\widetilde{\mathfrak{M}}$. Let $\widehat{\sigma}$ be the result of this transfer. We observe that $\widehat{\sigma}$ is a $\mathfrak{p}$-curve. Then $\widehat{\sigma}(0) = v$ and $l(\widehat{\sigma}) = l(\widetilde{\sigma}) < \varepsilon$ in $(\mathcal{R}, d)$ (see Lemma 1), therefore $\widehat{\sigma}(1) = v_1 \in B$ and $\pi(v_1) \in \mathcal{U}$.

By the property of an Ehresmann connection $\widetilde{\mathfrak{M}}$, the points $v_1 = \widehat{\sigma}(1)$ and $\widetilde{\sigma}(1)$ belong to the same leaf $\mathcal{L}'$ of $(\mathcal{R}, \mathcal{F})$. Since $\widetilde{\sigma} = R_a \circ \sigma$, the points $v_0 = \sigma(1)$ and $\widetilde{\sigma}(1)$ belong to $\pi^{-1}(x_0)$. Therefore the points $\pi(v_1)$ and $\pi(v_0) = x_0$ belong to the leaf $\pi(\mathcal{L}') = L' = L'(x_0)$ and $\pi(v_1) \in L' \cap \mathcal{U}$. Hence the closure $\overline{L'}$ of the leaf L' satisfies to the inclusion $\overline{L'} \supset L$ and $\mathcal{M} = \overline{L}$ is a global transitive attractor of (M, F).

It is necessary to note, that for a proper foliation any minimal set is a closed leaf. Therefore for a proper foliation (M, F) the global attractor $\mathcal{M}$ is a unique closed leaf. $\square$

5.5. Example

Take any pair of natural numbers (k, s). Let $\mathbb{E}^q_k$ be the pseudo-Euclidean space of the signature (k, s), where $q = k + s$, where g_0 is its pseudo-Euclidean metric. Then $g_0(x, x) = -x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_q^2$ for $x = (x_1, \ldots, x_q) \in \mathbb{E}^q_k$. The linear transformation ψ defined by the following block matrix

$$\frac{1}{e}A = \frac{1}{e}\begin{pmatrix} E_{k-1} & 0 & 0 \\ 0 & A_e & 0 \\ 0 & 0 & E_{s-1} \end{pmatrix}, \ A_e = \begin{pmatrix} ch1 & sh1 \\ sh1 & ch1 \end{pmatrix},$$

where $\mathbb{E}_{k-1}$ and $\mathbb{E}_{s-1}$ are the unit matrices $(k-1) \times (k-1)$ and $(s-1) \times (s-1)$ accordingly, and ψ is a similarity with the coefficient $\dfrac{1}{e}$ of the pseudo-Euclidean space $\mathbb{E}^q_k$. The equality

$$n(t, z) := (t - n, \psi^n(z)), \ n \in \mathbb{Z}, (t, z) \in \mathbb{R}^1 \times \mathbb{E}^q_k,$$

defines the free proper discontinues action of the group $\mathbb{Z}$ on the product of manifolds $\mathbb{R}^1 \times \mathbb{E}^q_k$. One-dimensional foliation $F = \{f(\mathbb{R}^1 \times \{z\}) | z \in \mathbb{E}^q_k\}$ is induced on the factor-manifold $M := \mathbb{R}^1 \times_{\mathbb{Z}} \mathbb{E}^q_k$. It is a complete Weyl foliation of the signature (k, s), and (M, F) is $(Sim(\mathbb{E}^q_k), \mathbb{E}^q_k)$-foliation.

According to Theorem 3 this foliation is not a pseudo-Riemannian for any metric conformally equivalent to g_0.

The foliation (M, F) has the family of compact leaves continuously depending on $q - 1$ parameters, and each compact leaf is diffeomorphic to the circle. Other leaves are diffeomorphic to the real line $\mathbb{R}^1$ and have an essential holonomy groups.

We want to emphasize that there are no leaves with an α-essential holonomy group. It is not difficult to see that (M, F) has not an attractor. $\square$

6. Riemannian Weyl foliations

6.1. (G, X)-manifolds and (G, X)-foliations

Let X be a connected manifold and G be a group of diffeomorphisms of X. The group G is referred to as act quasi-analytically on X if, for any open subset U in X and an element $g \in G$, the condition $g|_U = id_U$ implies g is the identity transformation of X. We assume that the group G of diffeomorphisms of a manifold X acts on X quasi-analytically.

Definition 7. A foliation (M, F) determined by an X-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in J}$ is called a (G, X)-foliation if for any $U_i \cap U_j \neq \emptyset$, $i, j \in J$, there exists an element $g \in G$ such that $\gamma_{ij} = g|_{f_j(U_i \cap U_j)}$.

Definition 8. A manifold B is called a (G, X)-manifold if its zero-dimensional foliation is a (G, X)-foliation.

6.2. Proof of Theorem 6

(i). Let (M, F) be a complete Riemannian Weyl foliation of codimension q, $q \geq 2$, which is not Riemannian. According to [21, Corolarry 5.1] there exists a leaf L with an essential holonomy group. In this case the holonomy group of L is α-essential. Therefore Theorem 5 implies that (M, F) is a transverse similar foliation of the signature $(0, q)$ having a global attractor $\mathcal{M}$.

(ii), (iv). Assume, that (M, F) is modelled on a transverse Riemannian Weyl geometry $(N, [g], \nabla^g)$, where (N, g) is a Riemannian manifold.

Let $q \geq 4$ and W be the Weyl tensor of the type $(1, 3)$ of the conformal curvature for Riemannian manifold (N, g). Considering W as a polylinear map $W : \mathfrak{X}N \times \mathfrak{X}N \times \mathfrak{X}N \to \mathfrak{X}N$, we define a norm $\parallel W \parallel (x)$, $x \in N$ by the following a way. Let $\parallel X \parallel (x) := \sqrt{g_x(X, X)}$ for any vector field $X \in \mathfrak{X}(N)$. We put $\parallel W \parallel (x) := \sup\limits_{\|X_i\|(x) \leq 1} \parallel W(X_1, X_2, X_3) \parallel (x)$ $\forall x \in N$, $i = 1, 2, 3$.

As any Weyl foliation is a conformal foliation, then the holonomy pseudogroup of (M, F) consists from local conformal diffeomorphisms of the Riemannian manifold (N, g). It is well known that the Weyl tensor W is a conformal invariant. Let $\mathfrak{M}$ be a q-dimensional distribution on M transversal to (M, F), i.e. $T_x M = \mathfrak{M}_x \oplus T_x F$ for every $x \in M$. Therefore, on the distribution $\mathfrak{M}$ the transversally projectable Weyl tensor $\widetilde{W}$ is induced. Thus $\widetilde{f}(x) :=\parallel \widetilde{W} \parallel (x)$, $x \in M$, is the base function with respect to (M, F), i.e. a function which is constant on leaves of this foliation. The existence of a global attractor and the continuity of this function imply $\parallel \widetilde{W} \parallel = const$, i.e. the constancy of the function $f(z) :=\parallel W \parallel (z)$, $z \in N$. Let $c =\parallel W \parallel$.

Assume, that $c \neq 0$. As all transformations from the holonomy pseudogroup $\mathcal{H}$ of the foliation (M, F) are local conformal diffeomorphisms, it is not difficult to check up that each transformation from $\mathcal{H}$ preserves the Riemannian metric cg on N, i.e. $\mathcal{H}$ is a pseudogroup of local isometries of the Riemannian manifold (N, cg). Hence (M, F) is a Riemannian foliation. This contradicts the condition of Theorem 6.

Therefore it is necessary that $W \equiv 0$.

If $q = 3$, then $W \equiv 0$ and the replacing W by the Schouten tensor V of the type $(1, 2)$ [16], similarly to the previous case we show the invariance of the Riemannian metric $\parallel V \parallel^{\frac{2}{3}} \cdot g$ with respect to the holonomy pseudogroup $\mathcal{H}$ of (M, F) and get the contradiction. Hence $V \equiv 0$.

According to the Weyl-Schouten theorem [16], a q-dimension, $q \geq 3$, the Riemannian manifold (N, g) is conformally flat, i.e. it is locally conformally equivalent to the Euclidean space $\mathbb{E}^q$ and the standard sphere $\mathbb{S}^q$, if and only if, for $q = 3$ the Schouten tensor V is equal to zero and for $q \geq 4$ that the Weyl tensor of conformal curvature W is equal to zero.

It is known that any two-dimensional Riemannian manifold is locally conformally flat.

Thus for $q \geq 2$ the Riemannian manifold (N, g) is conformally flat.

Therefore (N, g) is locally conformal to $\mathbb{S}^q$. Denote by $Conf(\mathbb{S}^q)$ the Lie group of all conformal transformations of $\mathbb{S}^q$. According to the Liouville theorem, for any connected open subsets U, V of $\mathbb{S}^q$ and for each conformal transformation $f : U \to V$ there exists a unique element $\widetilde{f} \in Conf(\mathbb{S}^q)$ such that $f = \widetilde{f}|_U$. Therefore we can consider (M, F) as a conformal foliation modelled on the conformal geometry of the sphere $\mathbb{S}^q$. This means that (M, F) is a transversally homogenious $(Conf(\mathbb{S}^q), \mathbb{S}^q)$-foliation.

The completeness of the Weyl foliation (M, F) implies the completeness of (M, F) considered as $(Conf(\mathbb{S}^q), \mathbb{S}^q)$-foliation. Therefore, by [19, Proposition 3] there exists an Ehresmann connection for (M, F) and we may apply [22, Theorem 2]. According to this theorem there exists a regular covering map $\kappa : \widetilde{M} \to M$ and a simply connected $(Conf(\mathbb{S}^q), \mathbb{S}^q)$-manifold B such that the induced foliation $\widetilde{F} := \kappa^* F$ is formed by fibers of a submersion $r : \widetilde{M} \to M$. Moreover, a group homomorphism

$$\chi : \pi_1(M, x) \to Sim(B)$$

is defined, and the global holonomy group $\Psi := \chi(\pi_1(M, x))$ is isomorphic to the group of covering transformations of the map $\kappa : \widetilde{M} \to M$.

Since (M, F) is a complete conformal foliation, the induced conformal foliation $(\widetilde{M}, \widetilde{F})$ is also complete. This implies completeness of the conformally flat Riemannian manifold B. Therefore B is conformal to the standard sphere $\mathbb{S}^q$ or the Euclidean space $\mathbb{E}^q$.

Observe that the global holonomy group group Ψ has an essential transformation ψ with the fixed point $b = r(\kappa^{-1})(x)$. We emphasize that ψ is the similarity of B with the fixed point b. According to [10, Lemma 2, Chap VI] this is only possible when B is the Euclidean space $\mathbb{E}^q$.

Thus, (M, F) is a a $(Sim(\mathbb{E}^q), \mathbb{E}^q)$-foliation and the statements (ii) and (iv) are proved.

(v). Since the holonomy pseudogroup $\mathcal{H}(M, F)$ is generated by the group Ψ, then $\mathcal{H}(M, F)$ is quasi-analytical. Therefore Theorem 2 implies that the holonomy group of the leaf L is isomorphic to the isotropy subgroup Ψ_z of Ψ at point $z \in pr(\kappa^{-1}(L))$. Hence (v) is proved.

(iii). Since $\mathbb{E}^q$ is contractible, the locally trivial bundle $r : \widetilde{M} \to \mathbb{E}^q$ is trivial, hence $\widetilde{M} = L_0 \times \mathbb{E}^q$ and $r = pr : L_0 \times \mathbb{E}^q \to \mathbb{E}^q$ is the canonical projection.

It is easy to see the restriction $\kappa|_{L_0 \times \{b\}}$, $b \in \mathbb{E}^q$ is the regular covering map onto the corresponding leaf L of (M, F), and the deck transformation group is isomorphic to the isotropy subgroup Ψ_b of Ψ at point b. According to the proved above statement (v) the group Ψ_b is isomorphic to the holonomy group of L. This completes the proof of (iii). $\square$

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